Via Geometric Algebra: Direction and Distance between Two Points on a Spherical Earth

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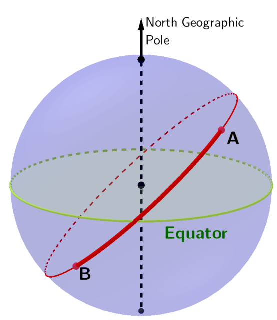
Abstract

As a high-school-level example of solving a problem via Geometric (Clifford) Algebra, we show how to calculate the distance and direction between two points on Earth, given the locations' latitudes and longitudes. We validate the results by comparing them to those obtained from online calculators. This example invites a discussion of the benefits of teaching spherical trigonometry (the usual way of solving such problems) at the high-school level versus teaching how to use Geometric Algebra for the same purpose.



"Given the latitudes and longitudes of the starting point A and the destination B, calculate the distance between these points, and the bearing of B from A."

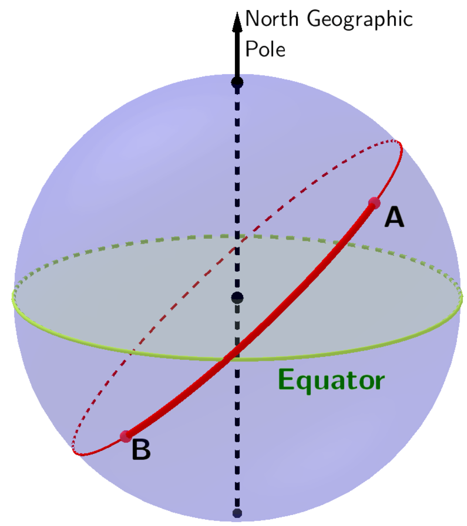


Figure 1: Points A and B are at longitudes and longitudes (λ_A, ℓ_A) and (λ_B, ℓ_B), respectively.

1 Problem Statement

> *"Given the latitudes and longitudes of the starting point A and the destination B (Fig. 1), calculate the great-circle distance between these points, and the bearing of B from A along the same great circle."*

2 Background

2.1 Geography

2.1.1 Latitude and longitude

The latitude of a location is its angle above the plane of the equator (Fig. 2). Latitudes of locations in the Northern Hemisphere are defined as positive, and those of locations in the Southern Hemisphere are designed as negative. Values of longitudes range from $-90°$ to $+90°$. To understand longitudes, we need to review the concept of meridians, which are half-circles that pass through both poles (Fig. 3). The longitude of a location is the angle between the plane that contains the Prime Meridian, and that which contains the meridian of the location of interest (Fig. 4). Longitudes of locations east of the Prime Meridian are defined as positive, and longitudes of locations west of the Prime Meridian are defined as negative. Thus, the values of longitudes range from $-180°$ to $+180°$.

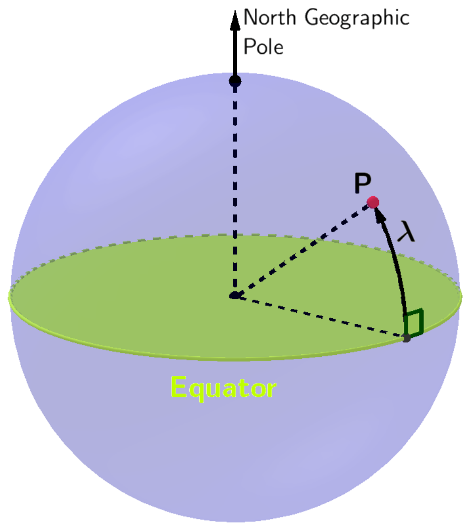


Figure 2: The latitude of a location is its angle above the plane of the equator. Latitudes of locations in the Northern Hemisphere are defined as positive, and those of locations in the Southern Hemisphere are designed as negative. Therefore, the latitude λ of location P is positive.

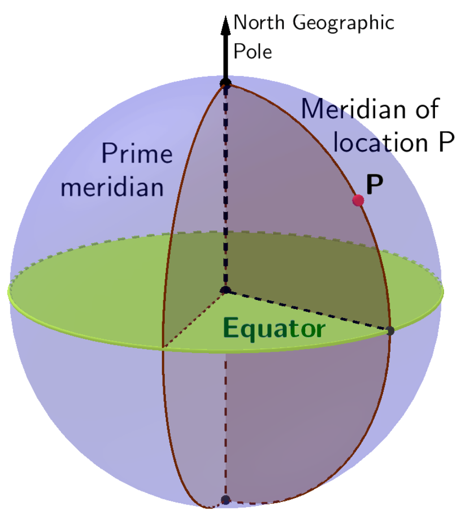


Figure 3: Meridians are half-circles that pass through both poles.

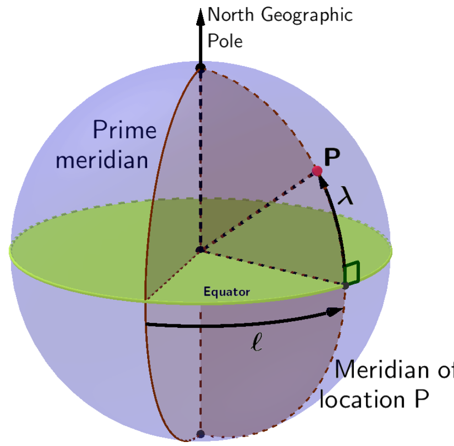


Figure 4: The longitude of a location is the angle between the plane that contains the Prime Meridian, and that which contains the meridian of the location of interest. Longitudes of locations east of the Prime Meridian are defined as positive, and longitudes of locations west of the Prime Meridian are defined as negative. Thus, the longitude l of location P is positive.

2.1.2 The bearing, and local north

We'll start by imagining that we are on a flat, horizontal plaza at location P in Fig. 5. Now, we draw two arrows on the plaza, starting from the point on which we're standing. First, we draw an arrow in the direction 'north" for location P, and then we draw an arrow pointing in the direction that we would take to follow a great-circle course to a second point, Q. The bearing for that course is the angle between the "local north" arrow and the arrow that shows the direction to Q.[1]

Now, let's pull back to look at the whole globe, taking note of the positions and orientations of same plaza and those same arrows (Fig. 6). We can see that the "plaza" is a plane that is tangent, at point P, to the globe (which we´ll assume to be a perfect sphere). The "local north" arrow is tangent to P's meridian, and the "direction to Q" arrow is tangent to the great circle that passes through P and Q.

Before leaving our review of geography, we should follow some advice that is useful in almost any geometry problem, and extend the radius that passes through P (Fig. 7). Recalling previous lessons in geometry, we see that the extension of that radius is perpendicular to the directions "local north" and the "direction to Q" arrow (Fig. 8). In addition, that radius is perpendicular to the plaza (Fig 9). This configuration is a striking one that Geometric Algebra (GA) is particularly suitable for handling.

[1]To be more correct, the angle that we've just described is the *initial bearing* from P to Q. The so-called *final bearing* is the angle between local north at Q, and the direction from which we would approach Q along the great circle that connects P and Q.

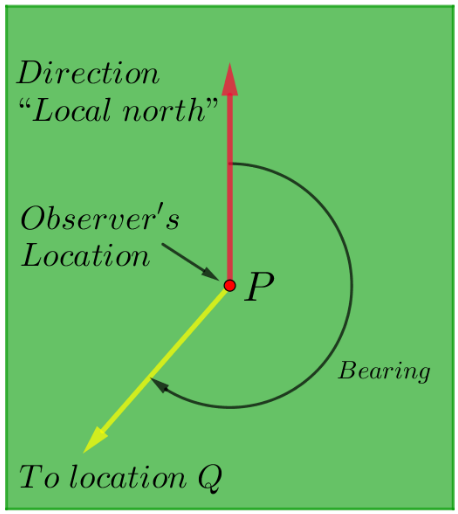


Figure 5: The bearing for the course from P tp Q is the angle between the "local north" arrow and the arrow that shows the direction to Q.

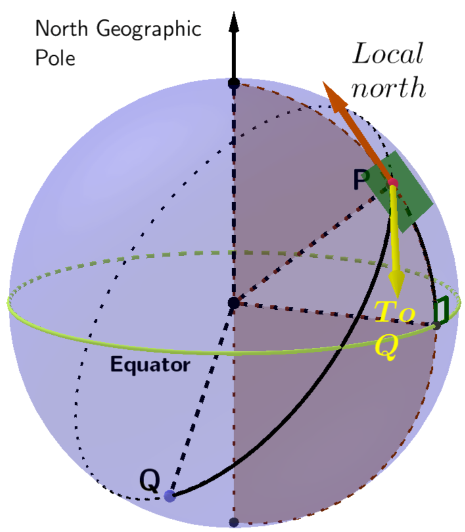


Figure 6: The position and orientation, on the globe, of the plaza and directions shown in Fig. 5. The "plaza" is a plane that is tangent, at point P, to the globe (which we assume to be a perfect sphere). The "local north" arrow is tangent to P's meridian, and the "direction to Q" arrow is tangent to the great circle that passes through P and Q.

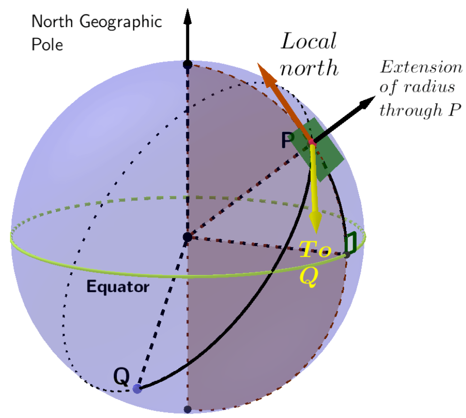


Figure 7: Following a good piece of typical advice, we've extended the radius through P. (Compare to Fig. 6.)

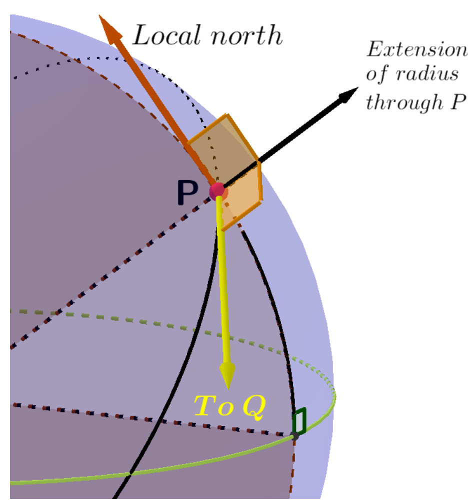


Figure 8: The extension of the radius through P is perpendicular to the directions "local north" and the "direction to Q" arrow shown in Fig. 7.

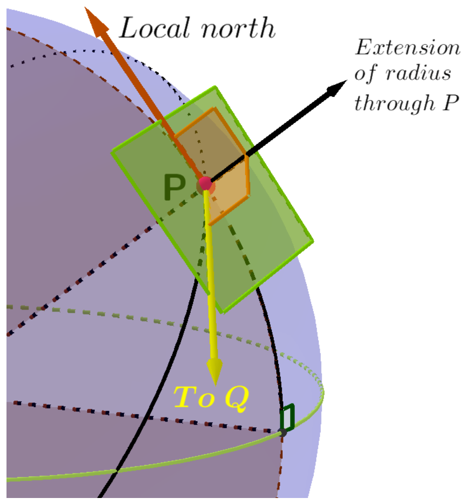


Figure 9: The extension of the radius through P is perpendicular to the plaza at P.

2.2 Geometric Algebra

We'll present only the aspects of GA that are relevant to this problem. For more-complete treatments, see References [1] and [2] .

2.2.1 Rotation of a vector through a bivector angle

Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ be two unit vectors (Fig. 10). Their outer product $\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}$ is the bivector parallel to the plane that contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and whose orientation is that which rotates $\hat{\mathbf{u}}$ into $\hat{\mathbf{v}}$. To make a unit bivector out of $\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}$, we divide that bivector by its norm $(\|\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}\|)$. Now, consider a third vector, $\mathbf{z}$, which is parallel to the plane that contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ (Fig. 11). Then, the product $\mathbf{z}\left[(\hat{\mathbf{u}} \wedge \hat{\mathbf{v}})/\|\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}\|\right]$ is a vector. Specifically, the vector that results from a 90° rotation of $\mathbf{z}$ in a plane parallel to that which contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and whose orientation is that which rotates $\hat{\mathbf{u}}$ into $\hat{\mathbf{v}}$ (Fig.11)).

In contrast, the product $\left[(\hat{\mathbf{u}} \wedge \hat{\mathbf{v}})/\|\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}\|\right]\mathbf{z}$ is the vector that results from the rotation of $\mathbf{z}$ in the opposite direction. This result is an example of a useful rule: if a vector $\mathbf{z}$ is parallel to bivector $\mathbf{M}$, then $\mathbf{Mz} = -\mathbf{zM}$.

2.2.2 The unit bivector perpendicular to a given unit vector

The vector known as the dual of a given bivector $\mathbf{B}$ is given by $\mathbf{B}I_3^{-1}$, where I_3 is the unit trivector for three-dimensional GA. The direction of that vector is as given by the right-hand rule. Thus, unit bivector $\hat{\mathbf{B}}$ perpendicular to the unit vector $\hat{\mathbf{w}}$ is $\hat{\mathbf{w}}I_3$ (Fig. 12).

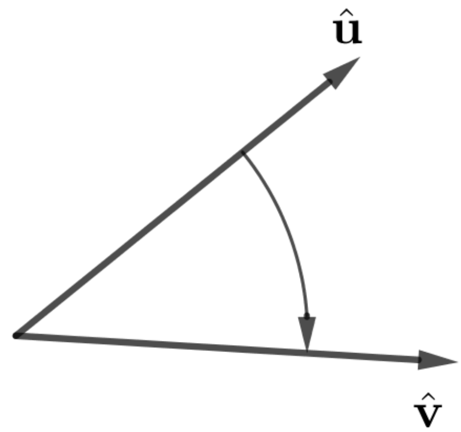


Figure 10: Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ be two unit vectors (Fig. 10). Their outer product $\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}$ is the bivector parallel to the plane that contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and whose orientation is that which rotates $\hat{\mathbf{u}}$ into $\hat{\mathbf{v}}$.

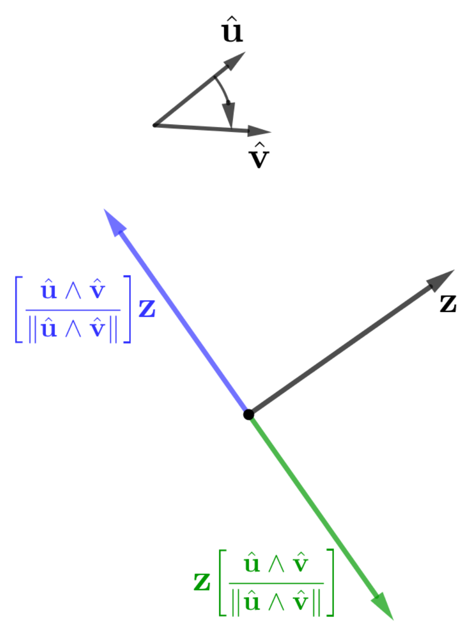


Figure 11: If vector $\mathbf{z}$ is parallel to the plane that contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ (Fig. 11), then the product $\mathbf{z}\left[(\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}) / \|\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}\|\right]$ is a vector. Specifically, the vector that results from a 90° rotation of $\mathbf{z}$ in a plane parallel to that which contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and whose orientation is that which rotates $\hat{\mathbf{u}}$ into $\hat{\mathbf{v}}$ (Fig.11)). In contrast, the product $\left[(\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}) / \|\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}\|\right]\mathbf{z}$ is the vector that results from the rotation of $\mathbf{z}$ in the opposite direction.

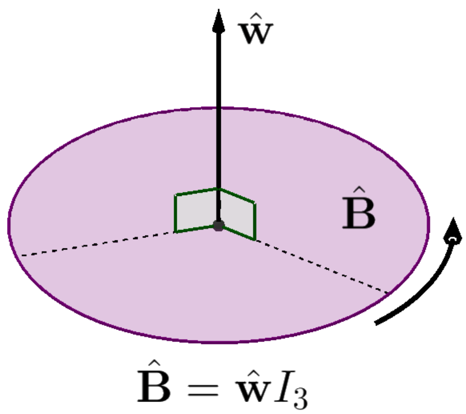


Figure 12: The unit vector $\hat{\mathbf{w}}$ in this figure is the dual of the unit bivector $\hat{\mathbf{B}}$. Because $\hat{\mathbf{w}} = \hat{\mathbf{B}} I_3^{-1}$, where I_3 is the unit trivector for three-dimensional GA, $\hat{\mathbf{B}} = \hat{\mathbf{w}} I_3$. The curved arrow shows the rotation of $\hat{\mathbf{B}}$ according to the right-hand rule.

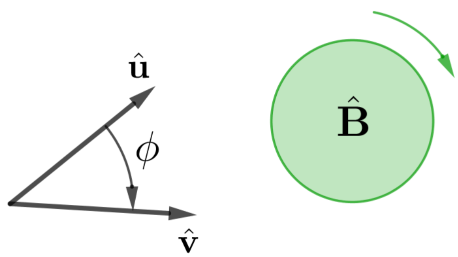


Figure 13: The geometric product of two unit bivectors, $\hat{\mathbf{u}}\hat{\mathbf{v}}$, can be written in terms of the angle of rotation ϕ from $\hat{\mathbf{u}}$ to $\hat{\mathbf{v}}$: $\hat{\mathbf{u}}\hat{\mathbf{v}} = \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + \hat{\mathbf{u}} \wedge \hat{\mathbf{v}} = \cos\phi + \hat{\mathbf{B}} \sin\phi$, where the unit bivector $\hat{\mathbf{B}}$ is parallel to the plane that contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and whose orientation is that which rotates $\hat{\mathbf{u}}$ into $\hat{\mathbf{v}}$.

2.2.3 Finding the angle between two vectors via the geometric product

The geometric product of two unit bivectors, $\hat{\mathbf{u}}\hat{\mathbf{v}}$, can be written in terms of the angle of rotation ϕ from $\hat{\mathbf{u}}$ to $\hat{\mathbf{v}}$:

$$\begin{aligned} \hat{\mathbf{u}}\hat{\mathbf{v}} &= \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} + \hat{\mathbf{u}} \wedge \hat{\mathbf{v}} \\ &= \cos\phi + \hat{\mathbf{B}} \sin\phi, \end{aligned}$$

where $\hat{\mathbf{B}}$ is parallel to the plane that contains $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, and whose orientation is that which rotates $\hat{\mathbf{u}}$ into $\hat{\mathbf{v}}$ (Fig. 13). From the previous equation,

$$\begin{aligned} \cos\phi &= \hat{\mathbf{u}} \cdot \hat{\mathbf{v}}, \\ \hat{\mathbf{B}} \sin\phi &= \hat{\mathbf{u}} \wedge \hat{\mathbf{v}}. \end{aligned} \tag{2.1}$$

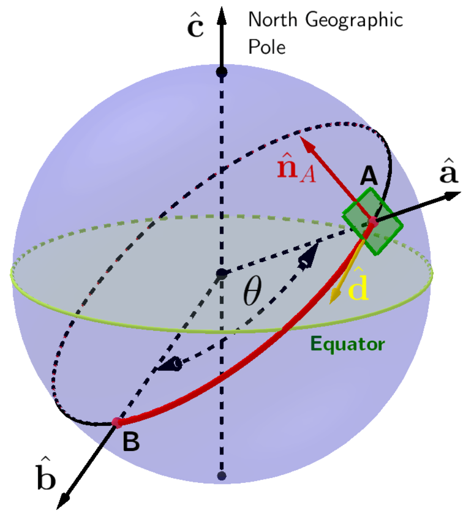


Figure 14: We model the Earth as a perfect sphere of radius $\mathcal{R}$.The directions of the radii from the Earth's center to the points A and B are $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, respectively, the direction from the center to the North Geographic pole is $\hat{\mathbf{c}}$ (Fig. 14). "Local North" at A is $\hat{\mathbf{n}}_A$, and the initial direction from A to B along the great-circle route that connects those points is $\hat{\mathbf{d}}$. Angle θ is that between the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

2.2.4 The definition of the inner and outer products that we will use Here

Macdonald ([1], p. 101) provides general formulas that reduce, for inner and outer products of vectors, to the following:

$$\begin{aligned} \mathbf{w} \cdot \mathbf{z} &= \langle \mathbf{wz} \rangle_0 \\ \mathbf{w} \wedge \mathbf{z} &= \langle \mathbf{wz} \rangle_2, \end{aligned} \tag{2.2}$$

where $\langle \mathbf{wz} \rangle_0$ and $\langle \mathbf{wz} \rangle_2$ are, respectively, the scalar and bivector parts of the product $\mathbf{wz}$.

3 Our Model, and Its Expression in GA Terms

We model the Earth as a perfect sphere of radius $\mathcal{R}$. The directions of the radii from the Earth's center to the points A and B are $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, respectively; the direction from the center to the North Geographic pole is $\hat{\mathbf{c}}$ (Fig. 14). "Local North" at A is $\hat{\mathbf{n}}_A$, and the initial direction from A to B along the great-circle route that connects those points is $\hat{\mathbf{d}}$. Angle θ is that between the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

Because the direction of rotation from $\hat{\mathbf{a}}$ to $\hat{\mathbf{n}}_A$ is the same as that from $\hat{\mathbf{a}}$ to $\hat{\mathbf{c}}$, and because the angle between $\hat{\mathbf{a}}$ to $\hat{\mathbf{n}}_A$ is 90° (Fig. 15), we may write

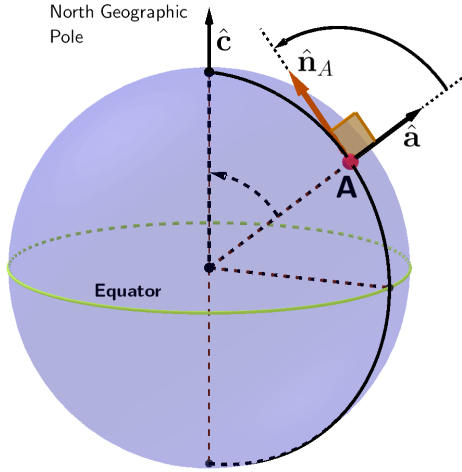


Figure 15: Because the direction of rotation from $\hat{\mathbf{a}}$ to $\hat{\mathbf{n}}_A$ is the same as that from $\hat{\mathbf{a}}$ to $\hat{\mathbf{c}}$, and because the angle between $\hat{\mathbf{a}}$ to $\hat{\mathbf{n}}_A$ is 90°, we may write $\hat{\mathbf{n}}_A = \hat{\mathbf{a}} \left[\dfrac{\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\|} \right]$.

(based upon Section 2.2.1),

$$\hat{\mathbf{n}}_A = \hat{\mathbf{a}} \left[\frac{\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\|} \right]. \tag{3.1}$$

Similarly, based upon Section 2.2.1 and Fig. 16,

$$\hat{\mathbf{d}} = \hat{\mathbf{a}} \left[\frac{\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\|} \right]. \tag{3.2}$$

The bivector parallel to the “plaza” at A is $\hat{\mathbf{a}}I_3$ (Fig. 17).

In our solution, we will use the angles θ and δ, shown in Figs. 14 and 17, respectively. The angle θ will be used only to find the distance between A and B, so we will not define its algebraic sign. Thus, it will range from 0 to π radians. In contrast, the angle δ is signed, and ranges from 0° to 180°.

4 Solution

We will begin by solving for θ, δ, the initial bearing, and the distance in terms of GA operations, after which we will define a coordinate system so that we may express our solution in terms of the latitudes and longitudes of the points A and B.

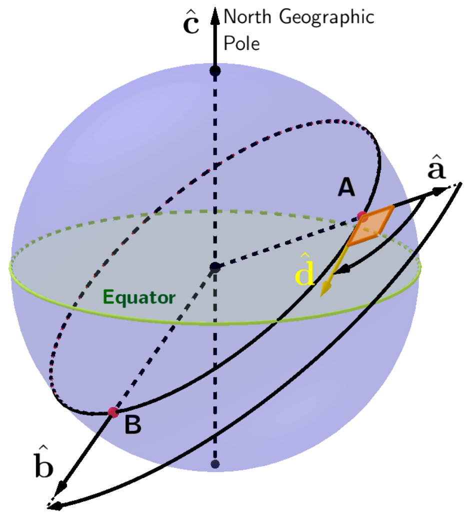


Figure 16: Because the direction of rotation from $\hat{\mathbf{a}}$ to $\hat{\mathbf{d}}$ is the same as that from $\hat{\mathbf{a}}$ to $\hat{\mathbf{b}}$, and because the angle between $\hat{\mathbf{a}}$ to $\hat{\mathbf{d}}$ is 90°, we may write $\hat{\mathbf{d}} = \hat{\mathbf{a}} \left[\dfrac{\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\|} \right]$.

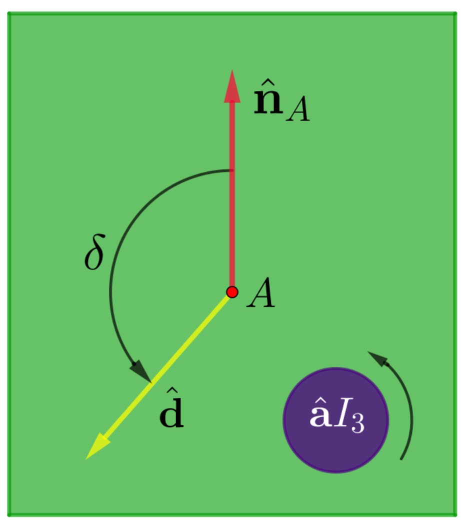


Figure 17: The bivector parallel to the "plaza" at A is $\hat{\mathbf{a}}I_3$. The direction of rotation shown for $\hat{\mathbf{a}}I_3$ is defined as the positive direction for δ.

4.1 Solution in Terms of GA Operations

4.1.1 Distance between A and B

The distance between A and B is $\mathcal{R}\theta$. Because θ is between 0 and π radians, we can determine θ from its cosine alone —we needn't find its sine. Therefore, using $\cos\theta = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$,

$$Distance = \mathcal{R} \arccos\left(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}\right). \tag{4.1}$$

4.1.2 The initial bearing

From Figs. 5 and 17, we can deduce that

$$\text{Bearing} = \begin{cases} 360^\circ - \delta & 0^\circ \leq \delta \leq 180^\circ, \\ -\delta & -180^\circ < \delta < 0. \end{cases} \tag{4.2}$$

Therefore, we need to know δ's algebraic sign as well as its magnitude. Thus, we need to know the sine and cosine of δ.

The cosine of δ is equal to the inner product of $\hat{\mathbf{n}}_A$ and $\hat{\mathbf{d}}$:

$$\cos\delta = \hat{\mathbf{n}}_A \cdot \hat{\mathbf{d}}.$$

Therefore, using Eqs. (3.1), (3.2)(2.2)

$$\cos\delta = \langle \left\{ \hat{\mathbf{a}} \left[\frac{\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\|} \right] \right\} \left\{ \hat{\mathbf{a}} \left[\frac{\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\|} \right] \right\} \rangle_0$$

To simplify that result, we note that because the vector $\hat{\mathbf{a}}$ is parallel to the bivector $\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}$, $\hat{\mathbf{a}}\left(\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\right) = -\left(\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\right)\hat{\mathbf{a}}$:

$$\begin{aligned} \cos\delta &= \langle \left\{ -\left[\frac{\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\|} \right] \hat{\mathbf{a}} \right\} \left\{ \hat{\mathbf{a}} \left[\frac{\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\|} \right] \right\} \rangle_0 \\ &= -\langle \frac{\left(\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\right)\left(\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\right)}{\|\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}\|\|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\|} \rangle_0. \end{aligned} \tag{4.3}$$

Here, we might with benefit write $\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}$ and $\hat{\mathbf{a}} \wedge \hat{\mathbf{c}}$ as $\langle \hat{\mathbf{a}}\hat{\mathbf{c}} \rangle_2$ and $\langle \hat{\mathbf{a}}\hat{\mathbf{b}} \rangle_2$, respectively, to obtain

$$\cos\delta = -\langle \frac{\langle \hat{\mathbf{a}}\hat{\mathbf{c}} \rangle_2 \langle \hat{\mathbf{a}}\hat{\mathbf{b}} \rangle_2}{\|\langle \hat{\mathbf{a}}\hat{\mathbf{c}} \rangle_2\|\|\langle \hat{\mathbf{a}}\hat{\mathbf{b}} \rangle_2} \rangle_0. \tag{4.4}$$

To find $\sin\delta$, we use Eq. (2.1), recognizing that the unit bivector parallel to the plane that contains $\hat{\mathbf{n}}_A$ and $\hat{\mathbf{d}}$ is $\hat{\mathbf{a}}I_3$ (Fig. 17). Therefore,

$$\hat{\mathbf{a}}I_3 \sin\delta = \hat{\mathbf{n}}_A \wedge \hat{\mathbf{d}},$$

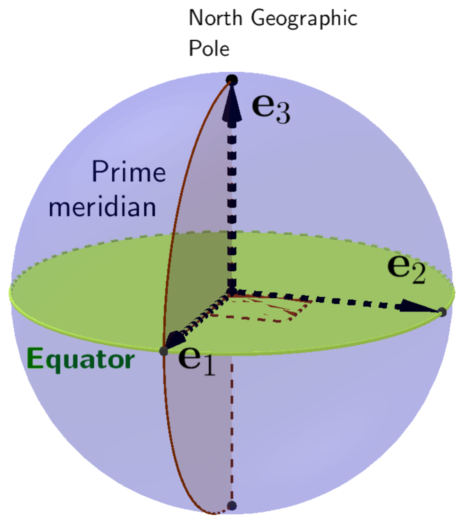


Figure 18: Our right-handed, orthonormal coordinate system. The endpoint of vector $\mathbf{e}_2$ is the intersection of the Equator with the meridian 90° E. Using this system, $I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

from which

$$\begin{aligned}\sin\delta &= (\hat{\mathbf{a}}I_3)^{-1}\left(\hat{\mathbf{n}}_A \wedge \hat{\mathbf{d}}\right)\\ &= I_3^{-1}\hat{\mathbf{a}}\left(\hat{\mathbf{n}}_A \wedge \hat{\mathbf{d}}\right)\\ &= -I_3\hat{\mathbf{a}}\left(\hat{\mathbf{n}}_A \wedge \hat{\mathbf{d}}\right).\end{aligned}$$

Now, we note that $\hat{\mathbf{n}}_A \wedge \hat{\mathbf{d}} = \langle\hat{\mathbf{n}}_A\hat{\mathbf{d}}\rangle_2$ (Eq.(2.2)), and use Eqs. (3.1), (3.2) in a way similar to that which led us to Eq. (4.4), thereby obtaining

$$\sin\delta = I_3\hat{\mathbf{a}}\langle\frac{\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2}{\|\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\|\|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2}\rangle_2. \tag{4.5}$$

4.2 Solution in Terms of Latitudes and Longitudes of A and B

4.2.1 Coordinate system

We will use a right-handed, orthonormal system (Fig. 18). The endpoint of vector $\mathbf{e}_2$ is the intersection of the Equator with the meridian 90° E. Using this system, $I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

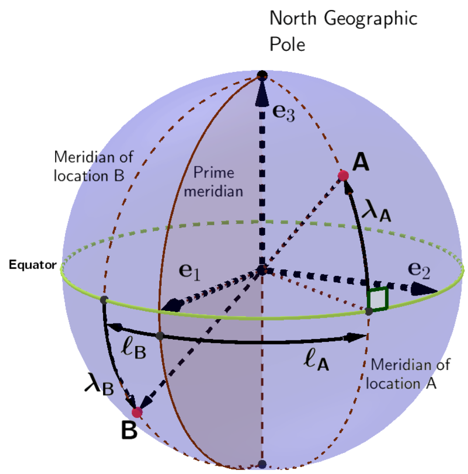


Figure 19: Latitudes and longitudes of locations A and B related to our frame of reference.

4.2.2 Expression of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{n}}_A$ according to our coordinate system

From Fig. 19, we can deduce that

$$\begin{aligned}\hat{\mathbf{a}} &= \cos\lambda_A \cos\ell_A \mathbf{e}_1 + \cos\lambda_A \sin\ell_A \mathbf{e}_2 + \sin\lambda_A \mathbf{e}_3 \\ \hat{\mathbf{b}} &= \cos\lambda_A \cos\ell_B \mathbf{e}_1 + \cos\lambda_A \sin\ell_B \mathbf{e}_2 + \sin\lambda_B \mathbf{e}_3 \\ \hat{\mathbf{n}}_A &= \hat{\mathbf{e}}_3.\end{aligned} \tag{4.6}$$

4.2.3 Distance between A and B

Substituting our expressions for $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ (Eq. (4.6)) in Eq., we find that

$$Distance = \mathcal{R} \arccos\left[\sin\lambda_B \sin\lambda_A + \cos\lambda_B \cos\lambda_A \cos\left(\ell_B - \ell_A\right)\right]. \tag{4.7}$$

4.2.4 The initial bearing

We noted in Section 4.1.2 that we will need to identify both $\cos\delta$ and $\sin\delta$.

Per Eqs. (4.4) and 4.5,

$$\cos\delta = -\langle \frac{\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2 \langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2}{\|\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\| \|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2} \rangle_0$$

and

$$\sin\delta = I_3 \hat{\mathbf{a}} \langle \frac{\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2 \langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2}{\|\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\| \|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2} \rangle_2.$$

Note that for any bivector $\mathbf{M}$ of the form
$\mathbf{M} = m_{12}\mathbf{e}_1\mathbf{e}_2 + m_{13}\mathbf{e}_1\mathbf{e}_3 + m_{23}\mathbf{e}_2\mathbf{e}_3$,
$\|\mathbf{M}\| = \sqrt{m_{12}^2 + m_{13}^2 + m_{23}^2}$.

Regarding Eq. 4.9, $\|\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\| = \|\cos\lambda_A\|$, which is $\cos\lambda_A$ because λ_A is between $-\pi/2$ and $\pi/2$.

Now, we need to express $\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2$, $\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2$, and their respective norms in terms of latitudes and longitudes. Using the expansions for $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ from Eq. (4.6),

$$\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2 = \cos\lambda_A\cos\ell_A\mathbf{e}_1\mathbf{e}_3 + \cos\lambda_A\sin\ell_A\mathbf{e}_2\mathbf{e}_3. \tag{4.8}$$

$$\|\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\| = \cos\lambda_A. \tag{4.9}$$

$$\begin{aligned}\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2 &= \cos\lambda_B\cos\lambda_A\sin(\ell_B-\ell_A)\,\mathbf{e}_1\mathbf{e}_2\\ &\quad+(\sin\lambda_B\cos\lambda_A\cos\ell_A-\cos\lambda_B\sin\lambda_A\cos\ell_B)\,e_1\mathbf{e}_3\\ &\quad+(\sin\lambda_B\cos\lambda_A\sin\ell_A-\cos\lambda_B\sin\lambda_A\sin\ell_B)\,e_2\mathbf{e}_3.\end{aligned} \tag{4.10}$$

$$\begin{aligned}\|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2\| &= \Big\{\cos^2\lambda_B\cos^2\lambda_A\sin^2(\ell_B-\ell_A)+\sin^2(\lambda_B-\lambda_A)\\ &\quad+\sin 2\lambda_B\sin 2\lambda_A\sin^2\left(\frac{\ell_B-\ell_A}{2}\right)\Big\}^{1/2}.\end{aligned} \tag{4.11}$$

Using Eqs. 4.8 and 4.10, then simplifying, we find that

$$\langle\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2\rangle_0 = \cos\lambda_A\left[\cos\lambda_B\sin\lambda_A\cos(\ell_B-\ell_A)-\sin\lambda_B\cos\lambda_A\right]. \tag{4.12}$$

$$\begin{aligned}\langle\langle\hat{\mathbf{a}}\hat{\mathbf{c}}\rangle_2\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2\rangle_2 &= \cos\lambda_B\sin\lambda_A\cos\lambda_A\sin(\ell_B-\ell_A)\,\mathbf{e}_1\mathbf{e}_2\\ &\quad-\cos\lambda_B\cos^2\lambda_A\sin\ell_A\sin(\ell_B-\ell_A)\,\mathbf{e}_1\mathbf{e}_3\\ &\quad+\cos\lambda_B\cos^2\lambda_A\cos\ell_A\sin(\ell_B-\ell_A)\,\mathbf{e}_2\mathbf{e}_3.\end{aligned} \tag{4.13}$$

Now, from Eqs. (4.4), (4.9), and (4.12),

$$\cos\delta = \frac{\sin\lambda_B\cos\lambda_A-\cos\lambda_B\sin\lambda_A\cos(\ell_B-\ell_A)}{\|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2\|}, \tag{4.14}$$

where $\|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2\|$ is as given in Eq. (4.11). Next, we find $\sin\delta$ from Eqs. (4.5), (4.9), and (4.13). The trivector I_3 is $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ for our reference frame, so the final result for $\sin\delta$ is

$$\sin\delta = -\frac{\cos\lambda_B\sin(\ell_B-\ell_A)}{\|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2\|}, \tag{4.15}$$

with $\|\langle\hat{\mathbf{a}}\hat{\mathbf{b}}\rangle_2\|$ as given in Eq. (4.11).

5 Validation of the Formulas

The pairs of cities used in the validation (Fig. 20) were chosen in ways that would test whether our formulas give sign errors when the locations A and B are in different hemispheres. Our calculated distances and bearings agreed well with more-accurate ones from Ref. [4], which correct for the Earth's slight deviation from perfect sphericity .

Locations		Latitudes and longitudes, deg					Distance, km					Bearing, deg	
A	B	λA	$l A$	λB	$l B$	θ, rad	Eq. (4.7)	Ref. [4]	$\cos\delta$	$\sin\delta$	δ, deg	Eq. (4.2)	Ref. [4]
San Cristóbal	Mecca	16.7367	-92.6383	21.4167	39.8167	2.091	13319	13336	0.611	-0.791	-52.312	52.3	52.3
Johannesburg	San Cristóbal	-26.2044	28.0456	16.7367	-92.6383	2.172	13838	13845	0.052	0.999	87.040	273.0	273.0
Port Moresby	Johannesburg	-9.4789	147.1494	-26.2044	28.0456	1.937	12338	12354	-0.543	0.839	122.913	237.1	237.1
San Cristóbal	Port Moresby	16.7367	-92.6383	-9.4789	147.1494	2.121	13512	13523	-0.017	1.000	90.993	269.0	269.0
Port Moresby	Mecca	-9.4789	147.1494	21.4167	39.8167	1.911	12175	12184	0.334	0.943	70.512	289.5	289.5
Mecca	Johannesburg	21.4167	39.8167	-26.2044	28.0456	0.855	5445	5420	-0.970	0.243	165.958	194.0	194.0

Figure 20: Screenshot of the Excel^{TM} worksheet used to validate our formulas. The Earth's radius was taken as 6371 km (Ref. [3]). Latitudes and longitudes are decimal equivalents of the following coordinates, which are from the cities' respective Wikipedia pages: Johannesburg $26°12'16''$S $28°2'44''$E ; Mecca $21°25'$N $39°49'$E; Port Moresby $9°28'44''$S $147°08'58''$E; San Cristóbal de Las Casas $16°44'12''$N $92°38'18''$W.

6 Extension of the Problem

How would you calculate the final bearing (Section 2.1.2)?

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] J. Smith, 2016, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", http://vixra.org/abs/1610.0054.

[3] NASA, "Earth Fact Sheet", https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html, accessed 20 June 2019.

[4] GPS Visualizer, https://www.gpsvisualizer.com/calculators, accessed 20 June 2019.