Zero Points of Riemann Zeta Function

Xuan Zhong Ni, Santa Rosa, CA, USA

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Abstract

In this article, we assume that the Riemann Zeta function equals the Euler product at the non zero points of the Riemann Zeta function. From this assumption we can prove that there are no zero points of Riemann Zeta Function $\varsigma(s)$ in $Re(s) > 1/2$. We applied proof by contradiction.

Some useful formulas and definition

Riemann Zeta Function,

$$\varsigma(s) = \Sigma_{1 \leq n < \infty} 1/n^s \tag{1}$$
$$s = \sigma + i\tau$$

the infinite sum "converges" absolutely for $Re(s) > 1$,

Euler product formula

$$E(s) = \prod_p (1 - 1/p^s)^{-1} \tag{2}$$

here we use p for all the primes.

When $Re(s) > 1$,

$$\varsigma(s) = E(s) \tag{3}$$

In general, we define a complex infinite series as following, $\Sigma_{n=1}^{\infty} a_n = \Sigma_{n=1}^{\infty} Re(a_n) + i\Sigma_{n=1}^{\infty} Im(a_n)$

The left side is defined when both $\Sigma_{n=1}^{\infty} Re(a_n)$ and $\Sigma_{n=1}^{\infty} Im(a_n)$ converge.

Define a complex infinite product $\prod_{n=1}^{\infty} c_n$, if $ln(\prod_{n=1}^{\infty} c_n) = \Sigma_{n=1}^{\infty} ln(c_n)$ is defined as a complex infinite series.

By the above definition we extend the definition of both functions, $\varsigma(s)$ and E(s) to $Re(s) > 1/2$.

$\varsigma(s)$ and E(s)can be defined as following;

$$\varsigma(s) = \Sigma_{1 \leq n < \infty} cos(\tau ln(n))/n^{\sigma} \ + i\Sigma_{1 \leq n < \infty} sin(\tau ln(n))/n^{\sigma} \tag{4}$$

$$ln(E(s)) = ln(\prod_{p}(1 - 1/p^s)^{-1}) = \Sigma_p 1/p^s + \frac{1}{2}\Sigma_p 1/p^{2s} + \frac{1}{3}\Sigma_p 1/p^{3s} + \dots \tag{5}$$

Hypothesis I: Riemann Zeta Function equals Euler product at non zero points of Riemann Zeta function.

The hypothesis I is obviously correct for $Re(s) > 1$ as Eq.3.

Hypothesis II: If a real infinite series $\Sigma_{n=1}^{\infty} a_n$ converges and its infinite sub series $\Sigma_{j=1}^{\infty}|a_{n_j}|$ absolutely converges, then $\Sigma_{n=1}^{\infty} a_n = \Sigma_{n=1, n \neq n_j}^{\infty} a_n + \Sigma_{j=1}^{\infty} a_{n_j}$

Hypothesis III: If $\varsigma(s)$ is defined as Eq.4, $\varsigma(s)$ should have the same value as the analytical continuation of $\varsigma(s)$ from $Re(s) > 1$.

Hypothesis IV: If ln(E(s)) is defined as Eq.5, ln(E(s)) should have the same value as the analytical continuation of ln(E(s)) from $Re(s) > 1$.

All the Hypothesis are used later to proove the Eq.7 and Eq.10.

Hadamard had proved that there are no zeroes of $\varsigma(s)$ for $Re(s) = 1$. In his proof he used Eq.3 and the result of $\Sigma 1/p^{\sigma} \sim -\ln(\sigma - 1) \sim +\infty$, as $\sigma \to 1^+$, made the analytical continuation along the line of $Im(s) = t$ for $Re(s) \to 1^+$, and used the fact that there are infinite of primes p for which $t \ln p \sim (2n+1)\pi$ have to be true and the corresponding summation of these p, $\Sigma' 1/p^{\sigma}$ is infinitely large. Such primes p would be even "overwhelming majority", if he assumed there was a zero of $\varsigma(s)$ at $s = 1 + it$.

Following his proof, we have proved that there are no zeroes of $\varsigma(s)$ for $Re(s) > 1/2$.

Assume that the first zero of $\varsigma(s)$ is on the line of $Im(s = \alpha + it) = t$, where $1/2 < \alpha < 1$, and $\varsigma(s) = \prod_p (1 - 1/p^s)^{-1}$, by Eq.3.

Equation

$$\ln \varsigma(s) = \Sigma_p 1/p^s + \frac{1}{2}\Sigma_p 1/p^{2s} + \frac{1}{3}\Sigma_p 1/p^{3s} + \ldots \tag{6}$$

is valid for $\alpha < Re(s) < 1$, along the line of $Ims = t$. The zero of $\varsigma(s)$ at $s = \alpha + it$ implies $Re(\ln \varsigma(s))$ near $-\infty$ when $\sigma = \mathrm{Re(s)}$ is close to α.

The sum of the terms after the first in Eq.6 is bounded by the number $B = \frac{1}{2}\Sigma_p 1/p^{2\alpha} + \frac{1}{3}\Sigma_p 1/p^{3\alpha} + \ldots$,

hence $Re(\ln \varsigma(s)) \geq \Sigma_p \cos(t \ln p)/p^\sigma - B$

approaches $-\infty$ as $\sigma \to \alpha$, only if the first term approaches $-\infty$. Then it followes that

$$\Sigma_p \cos(t \ln p)/p^\sigma = -\infty \quad as \quad \sigma \to \alpha \tag{7}$$

Since we assume $s = \alpha + it$ is a zero of $\varsigma(s)$, $\varsigma(\alpha + it) = 0$, meaning $Re(\varsigma(\alpha + it)) = 0$ and $Im(\varsigma(\alpha + it)) = 0$, which gives the following,

$$\Sigma_m \cos(t \ln m)/m^\alpha = -1 \tag{8}$$

and

$$\Sigma_m \sin(t \ln m)/m^\alpha = 0 \tag{9}$$

In the summation of $\Sigma_m\{\ldots\}$,

we separate out only those m of prime numbers p with p, and put the rest terms into the remainder, $C(s)$.

So $\Sigma_m\{\ldots\} = \Sigma_p\{\ldots\} + C(s)$

$C(s) = \Sigma_m\{\ldots\}$ is for all the terms of m which have at least a factor of two prime numbers.

It is easy to prove that the $|C(s)|$ satisfies the following;

$|C(s)| < C$,

here
$C = \Sigma_{m,m\ is\ not\ a\ prime} 1/m^{\alpha}$
$< \Sigma_{m,m\ is\ not\ a\ prime} 1/([m^{\frac{1}{2}}])^{2\alpha}$.

We use [b] to indicate the largest integer $\leq b$, for a real number b. Obviously C is finite and less than $\varsigma(2\alpha)$.

From Eq.8 and Eq.9 we have the following

$$|\Sigma_p \cos(t \ln p)/p^{\alpha}| < 1 + C \tag{10}$$
$$|\Sigma_p \sin(t \ln p)/p^{\alpha}| < C \tag{11}$$

Eq.10 contradicts Eq.7. Thus our assumption of zero of $\varsigma(\alpha + it)$, when $\alpha > 1/2$, is wrong.

References

[1] S. J. Patterson, An Introduction to the Theory of the Riemann Zeta-Function Cambridge University Press, 1988.

[2] H. M. Edwards, Riemann's Zeta Function, Academic Press, 1974.