A SIMPLE PROOF FOR CATALAN'S CONJECTURE

Abstract. In 2002 Preda Mihailescu used the theory of cyclotomic fields and Galois modules to prove Catalan's Conjecture. In this short paper, we offer a simpler proof. We first prove that when $x = 2$ the only possible value for a is 3 and the only possible value for y is 3. Then we prove that no solutions exist for $a^2 - b^y = 1^z$ for $a, b, y, z > 1$.

Introduction Catalan's Conjecture was first made by Belgian mathematician Eugne Charles Catalan in 1844, and states that 8 and 9 (2^3 and 3^2) are the only consecutive powers, excluding 0 and 1. That is to say, that the only solution in the natural numbers of $a^x - b^y = 1$ for $a, b, x, y > 1$ is $a = 3$, $x = 2$, $b = 2$, $y = 3$. In other words, Catalan conjectured that $3^2 - 2^3 = 1$ is the only nontrivial solution. It was finally proved in 2002 by number theorist Preda Mihailescu making extensive use of the theory of cyclotomic fields and Galois modules.

Theorem 0.1. *To demonstrate that the only solution in the natural numbers of $a^x - b^y = 1^z$ for $a, b, x, y, z > 1$ is $a = 3$, $x = 2$, $b = 2$, $y = 3$.*

Proof. In Lemma (0.2) we will first prove that when $x = 2$ the only solutions for $a, b, y, z > 1$ are $a = 3$, $b = 2$, and $y = 3$. We will follow this in Lemma (0.3) with a proof that for the equation $a^x - b^y = 1^z$ no solutions exist when $a, b > 1$ and $x, y, z > 2$.

Lemma 0.2. *To demonstrate that for $a^x - b^y = 1$ when $x = 2$ the only solutions for $a, b, y > 1$ in the natural numbers are $a = 3$, $b = 2$, $y = 3$.*

We first rearrange the equation as follows:

$$a^2 - 1 = b^y \tag{0.1}$$

$$\Rightarrow (a + 1)(a - 1) = b^y \tag{0.2}$$

Assume first that both factors on the left are power to y. Then, where p and q are positive integers, let $(a + 1) = p^y$ and $(a - 1) = q^y$ and rearrange such that:

$$\Rightarrow p^y - q^y = 2. \tag{0.3}$$

Since $p - q$ divides 2, $p - q = 2$ from which it follows that the only solution to this is $p = 4$, $q = 2$, $y = 1$. But since $y > 1$ this assumption is false.

So let $(a + 1) = b^{y-k}$ and $(a - 1) = b^k$, where $y - k > k$, such that:

$$(a + 1)(a - 1) = b^{y-k} b^k. \tag{0.4}$$

Date: Aug 2018.

2010 *Mathematics Subject Classification.* Primary 11D41.

Key words and phrases. Number Theory, Catalan's Conjecture, Mihailescu's Theorem.

Now, since b divides both $(a+1)$ and $(a-1)$ and is the only common factor, it follows that a cannot be greater than 3. And since $a > 1$, it follows that either $a = 3$ or $a = 2$.

Let us take each in turn, first when $a = 2$.

So if $b^{y-k} = (a+1)$ and $b^k = (a-1)$ and $a = 2$, it follows that:

$$b^{y-k} = 3, \tag{0.5}$$

and

$$b^k = 1. \tag{0.6}$$

Then from (0.6), $b = 1$. But since $b > 1$, it follows that $b \neq 1$. So no solutions when $a = 2$.

So let $a = 3$. In this case, if $b^{y-k} = (a+1)$ and $b^k = (a-1)$ if follows that:

$$b^{y-k} = 4, \tag{0.7}$$

and

$$b^k = 2, \tag{0.8}$$

$$\Rightarrow b = \sqrt[k]{2}. \tag{0.9}$$

But from (0.9), solutions only exist if $k = 1$, when $b = 2$. From (0.10) when $k = 1$, then $2^{y-1} = 4$, from which $y = 3$. Thus there are solutions for this option.

So now, when $b = 2$ and $y = 3$. From this:

$$a^2 - 1 = 2^3$$

$$\Rightarrow a^2 = 2^3 + 1 \tag{0.10}$$

$$\Rightarrow a^2 = 9 \tag{0.11}$$

$$\Rightarrow a = 3. \tag{0.12}$$

Thus for this option, whole number solutions exist.

We have shown that when $x = 2$ the only solutions in the natural numbers for $a^x - b^y = 1$ for $a, b, y > 1$ are $a = 3$, $b = 2$, $y = 3$.

We will now turn to Lemma (0.3) to prove that for the equation $a^x - b^y = 1^z$ where $a, b > 1$ and $x, y, z > 2$, no integer solutions exist.

Lemma 0.3. *To prove that for the equation $a^x - b^y = 1^z$ no solutions exist for values of $x, y, z > 2$.*

We first rearrange the equation as follows:

$$b^y + 1^z = a^x.$$

Using a standard proof by contradiction, we first assume that solutions exist to this equations under those conditions. We now observe the following identity for $b^y + 1^z$ as a binomial expansion (where the upper index n is an indeterminate integer):

$$b^y + 1^z = \sum_{k=0}^{n} \binom{n}{k} (b+1)^{n-k}(-b)^k(b^{y-n-k} + 1^{z-n-k}). \tag{0.13}$$

Regardless of the value of n, the right hand side always equals $b^y - 1^z$. So we fix n as x, such that:

$$b^y + 1^z = \sum_{k=0}^{x} \binom{x}{k} (b+1)^{x-k}(-b)^k(b^{y-x-k} + 1^{z-x-k}). \tag{0.14}$$

Still assuming that a solution exists for the equation $b^y + 1^z = a^2$ for values of $x, y, z > 2$. From (0.2), if $b^y + 1^z = a^2$, it follows that:

$$\sum_{k=0}^{x} \binom{x}{k} (b+1)^{x-k}(-b)^k(b^{y-x-k} + 1^{z-x-k}) = [(b+1)|s| - b|t|]^x. \tag{0.15}$$

for all $s, t \in \mathbb{Q}$, where $gcd(s, b) = 1$, such that $[(b+1)|s| - b|t|] = a$. This expands to:

(0.16)

$$\sum_{k=0}^{x} \binom{x}{k} (b+1)^{x-k}(-b)^k(b^{y-x-k} + 1^{z-x-k}) = \sum_{k=0}^{x} \binom{x}{k} (b+1)^{x-k}(-b)^k|s|^{x-k}|t|^k.$$

Comment: We know that the right hand side is a perfect power, i.e. a^x; we are assuming the left hand side is but it may not be. In our proof we are hoping that it is not, and that there can be no equality for $x > 2$.

Since the factors $(b+1)^{x-k}$ and $(-b)^k$ are constant on both sides, and standard in exponential form for a perfect power, we need only test whether the associated factor on the left, i.e. $((b^{y-x-k} + 1^{z-x-k})$, is the same in exponential form as the associated factors on the right, i.e. $|s|^{x-k}|t|^k$. For if it cannot even be represented in the correct standard *form* for a binomial expansion of a perfect power, then there can be no equality in value either (while remaining a perfect power).

It is pointless to test for inequality of *value* when the associated factors, $|s|^{x-k}$ and $|t|^k$, are variable (whose values depend on each other), since other values exist (e.g. v, w) even when $s \neq v$ and $t \neq w$, such that:

$$[(b+1)|s| - b|t|]^x = [(b+1)|v| - b|w|]^x$$

i.e. when $b = 5$, $s = 15$, $t = 12$, $v = 20$ and $w = 18$. In the binomial expansion of this equation, the exponential *forms* of each counterpart term may be the same even while their *values* differ. So trying to prove inequality of counterpart terms (in terms of value) will pose problems.

Therefore, we will seek to prove only inequality in form, and simplify the problem as follows:

$$\sum_{k=0}^{x} |s|^{x-k}|t|^k = \sum_{k=0}^{x} (b^{y-x-k} + 1^{z-x-k}) \tag{0.17}$$

For this equation to have solutions the associated factor, $(b^{y-x-k} + 1^{z-x-k})$, must exactly correspond in form with $|s|^{x-k}|t|^k$ in each counterpart (k^{th}) term, for any given value of x. If it does, then the whole of the left hand side of (0.4) will be a power to x (as we know the right hand side is), and the initial equation will have solutions. But if just one term of the corresponding binomials exists where $(b^{y-x-k}+1^{z-x-k})$ does not equal $|s|^{x-k}|t|^k$ in form, then not only will the integrity of that particular k^{th} term be compromised as a valid binomial term, but also the whole expression as an expansion of a power to x.

To do this we can assume that all the counterpart (k^{th}) terms in (0.16) are equal and in particular the first and last terms. [Then we will show that when these are equal there is inequality in the second and penultimate terms. This circumvents the need to demonstrate inequality in any further terms however large x becomes.]

So, we can deduce the second term from the equation in (0.16). If the first term is $|s|^x = \pm(b^{y-x}+1^{z-x})$, and the last term is $|t|^x = \pm(b^{y-2x}+1^{z-2x})$, we can raise the powers accordingly and multiply together to get:

$$|s|^{x-1}|t| = \pm(b^{y-x} + 1^{z-x})^{(x-1)/x}(b^{y-2x} + 1^{z-2x})^{1/x}. \tag{0.18}$$

We can also calculate the second term directly from the right hand side of (0.5). So when $k = 1$, the second term is:

$$\pm(b^{y-x-1} + 1^{z-x-1}). \tag{0.19}$$

Putting (0.18) and (0.19) together, we get:

$$\pm(b^{y-x} + 1^{z-x})^{(x-1)/x}(b^{y-2x} + 1^{z-2x})^{1/x} = \pm(b^{y-x-1} + 1^{z-x-1}). \tag{0.20}$$

Now we raise both sides by x and divide both sides by $(b^{y-x} + 1^{z-x})^{(x-2)}$ and rearrange to get:

$$\pm(b^{y-x} + 1^{z-x})(b^{y-2x} + 1^{z-2x}) = \pm\frac{(b^{y-x-1} + 1^{z-x-1})^x}{(b^{y-x} + 1^{z-x})^{(x-2)}}. \tag{0.21}$$

The procedure for the penultimate term is exactly the same. So, again using $|s|^{z-k}$ and $|t|^k$ as our point of reference, we raise the powers accordingly and multiplying together to get the penultimate term:

$$|s||t|^{z-1} = \pm(b^{y-x} + 1^{z-x})^{1/x}(b^{y-2x} + 1^{z-2x})^{(x-1)/x}. \tag{0.22}$$

And *directly* from the binomial formula, when $k = x - 1$, the penultimate term is:

$$\pm(b^{y-2x+1} - 1^{z-2x+1}) \tag{0.23}$$

Putting (0.22) and (0.23) together, we get:

$$\pm(b^{y-x} + 1^{z-x})^{1/x}(a^{x-2z} - b^{y-2z})^{(x-1)/x} = \pm(b^{y-2x+1} - 1^{z-2x+1}). \tag{0.24}$$

This time, we raise both sides by x and divide both sides of by $(a^{x-2z} - b^{y-2z})^{(x-2)}$ and rearrange to get:

$$\pm(b^{y-x} + 1^{z-x})(b^{y-2x} + 1^{z-2x}) = \pm\frac{(b^{y-2x+1} - 1^{z-2x+1})^x}{(a^{x-2z} - b^{y-2z})^{(x-2)}}. \tag{0.25}$$

At this point we can ignore the $\pm$ sign. This was introduced by the absolute values of s and t, which are now no longer necessary. It is self-evident that inequality exists when there is opposite polarity. The harder task is to prove inequality when polarity is the same. So by ignoring the signs, we are not making the proof easier. So we will remove the $\pm$ sign and focus on circumstances where polarity is the same.

Now, we note that in (0.21) and (0.25) the left hand sides are exactly the same. This means we can subtract (0.21) from (0.25) and rearrange to get:

$$\left(\frac{b^{y-x-1}+1^{z-x-1}}{b^{y-2x+1}+1^{z-2x+1}}\right)^{x}=\left(\frac{b^{y-x}+1^{z-x}}{b^{y-2x}+1^{z-2x}}\right)^{(x-2)} \tag{0.26}$$

Solutions will exist to this equation
a) *either* if the large bracketed fractions on each side have a value of 1 (since the outer exponents are not equal),
b) *or* if the numerators (to their respective powers) on both sides are equal, *and* simultaneously if the denominators (to their respective powers) on both sides are equal.

Taking these two options in turn (still when $x, y, z > 2$):

a) since $(b^{y-x-1}+1^{z-x-1}) \neq (b^{y-2x+1}+1^{z-2x+1})$, and $(b^{y-x}+1^{z-x}) \neq (b^{y-2x}+1^{z-2x})$, neither side in (0.14) has a value of 1, eliminating this option;

b) even without their respective powers, the base value of the left hand numerator $(b^{y-x+1}+1^{z-x+1})$ is greater than its right hand counterpart, $(b^{y-x}+1^{z-x})$; but when the power is greater, (i.e. $x > (x-2)$), then the inequality is even greater. So it follows that: $(b^{y-x+1}+1^{z-x+1})^{x} \neq (b^{y-x}+1^{z-x})^{(x-2)}$. We do not even need to bother with the denominators.

Having now eliminated both options it follows that, for all values of $x, y, z > 2$:

$$\sum_{k=0}^{x}|s|^{x-k}|t|^{k} \neq \sum_{k=0}^{x}(b^{y-x-k}+1^{z-x-k}). \tag{0.27}$$

However, this contradicts our equation in (0.17). Under these circumstances there is no equality of form, let alone value. Therefore the left hand side of (0.17) cannot be a perfect power (as we assumed it was). And so our initial assumption that, for any value of $x, y, z > 2$, solutions exist for the equation $b^{y}+1^{z}=a^{x}$ is false.

What then happens for the case $x = 1, 2$? Well, from (0.26), when $x = 1$ it follows that:

$$\left(\frac{b^{y-2}+1^{z-2}}{b^{y-1}+1^{z-1}}\right)^{1}=\left(\frac{b^{y-1}+1^{z-1}}{b^{y-2}+1^{z-2}}\right)^{(-1)} \tag{0.28}$$

$$\left(\frac{b^{y-2}+1^{z-2}}{b^{y-1}+1^{z-1}}\right)^{1}=\left(\frac{b^{y-2}+1^{z-2}}{b^{y-1}+1^{z-1}}\right)^{1} \tag{0.29}$$

No contradiction.

And again from (0.26), when $x = 2$, it follows that:

$$\left(\frac{b^{y-3} + 1^{z-3}}{b^{y-3} + 1^{z-3}}\right)^2 = \left(\frac{b^{y-2} + 1^{z-2}}{b^{y-4} + 1^{z-4}}\right)^0 \tag{0.30}$$

$$\Rightarrow 1 = 1. \tag{0.31}$$

Again, no contradiction.

So in both cases, when $x = 1$ and when $x = 2$, the standard rules of binomial expansion can be applied to our non-standard binomial expression without contradiction such that $(b^{y-x-k} - 1^{y-x-k})$ *is* equal to $|s|^{x-k}|t|^k$ in these two cases, and therefore that in these cases solutions to the original equation exist.

For Catalan's Conjecture, this means that $a^x - b^y = 1$ has only 1 solution, namely that $a = 3$, $x = 2$, $b = 2$, and $y = 3$. □

The Rectory, Village Road, Waverton, Chester Ch3 7QN, UK
E-mail address: julianbeauchamp47@gmail.com