On the coloring of graphs formed by cliques sharing at most one common vertex

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Abstract

In this work we give a short probable proof for the fact that the chromatic number of the graph formed by the adjoining of k cliques such that any two cliques share at most one vertex is k.

1 Introduction

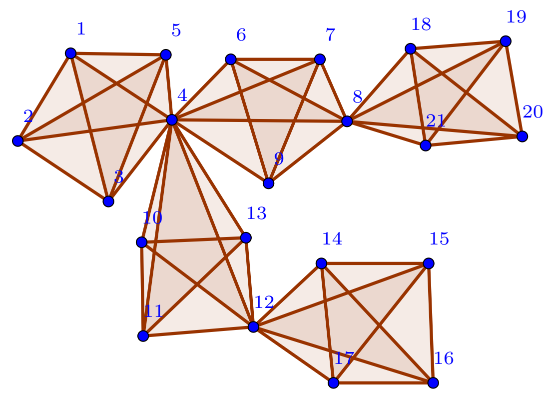
In graph theory, one of the famous conjectures is Erdos-Faber-Lovasz (EFL) conjecture. It states that, if $\boldsymbol{H}$ is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices of $\boldsymbol{H}$ with n colors so that no two vertices with same color are in the same edge. Reducing to the case of simple graphs, it is equivalent to the fact that the union of n pairwise edge-disjoint complete graphs with n vertices is n-colorable[8].

Infact in 1975, Erdos offered 50 USD and in 1981, offered 500 USD for the proof or disproof of the conjecture. Many people worked on this and have some results. Chang et al.[4] showed that for any simple hypergraph $\mathbf{H}$ on n vertices, the chromatic index of H is at most [1.5n-2]. Kahn [3] proved that the chromatic number of $\boldsymbol{H}$ is at most $n + o(n)$. Jackson et al. [1] showed that the conjecture is true when the partial hypergraph S of $\boldsymbol{H}$ determined by the edges of size at least three can be Δ_S-edge-colored and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$. Viji Paul and Germina [6] established the truth of the conjecture for all linear hypergraphs on n vertices and $\Delta(\mathbf{H}) \leq \sqrt{n + \sqrt{n} + 1}$. Sanchez-Arroyo [7] proved the conjecture is true for dense hypergraphs. Faber [5] proved that for fixed degree, there can be only finitely many counterexamples to EFL on both regular and uniform hypergraphs. Hegde et al. [2] gave a method for assigning colors to the graphs which satisfies the hypothesis of the EFL and every complete graphs has at most $\frac{n}{2}$ vertices of clique degree greater than one using symmetric latin squares and clique degrees of the vertices of the graph.

2 Coloring Procedure

Theorem 2.1. *Let graph G be a graph with union of k cliques with size k and no two clique share more than one vertex, then $\chi(G) = k$.*

Proof. We consider the graph formed by k cliques of order k sharing at most one vertex in common. Now, to color the graph, we color one of the cliques with k colors. Now, we consider a vertex of the clique, and, if it is shared by m cliques ($m \geq 0$), then we can color one vertex from $k - m$ cliques which is not dominated by the vertex ; with the same color. Thus, all cliques have one of the vertices colored with the same color. Note that the vertices colored need not be distinct, that is, if one of the $k - m$ vertices are being in turn shared by some other cliques, then we are just actually picking one vertex for all those cliques. This means we are picking each *representative* vertex from each distinct clique. This can always be done, as every vertex has one *undominated* (non-adjacent) (in fact $k - 1$) vertex from each of the cliques that do not share that vertex (otherwise those cliques would share that vertex). Since the cliques may share a single vertex, therefore a single vertex would represent the cliques that share it. Similarly, we pick all the other vertices of the clique we have already colored and do a similar procedure of coloring other non-adjacent vertices (representative vertices from each clique). This gives a $k-$ coloring of the graph.



In the figure above, we first color the clique $1 - 2 - 3 - 4 - 5$, to give us 5 color classes. Now, the vertex 1 dominates $2 - 3 - 4 - 5$ but has all other vertices in the graph undominated. Now, we choose the vertices $6, 18, 10, 14$ and color them with the same color as vertex 1. Again, we choose the vertex 2 and see that all other vertices except $1 - 3 - 4 - 5$ are undominated. Now, we choose the vertices $7, 19, 12$ and color them with the same color as the vertex 2. We see that even though we are choosing 3 distinct vertices, but we are actually choosing 4 representatives from each of the other cliques of which the vertex 2 is not a part of. Next, we choose the vertex 3 and choose the undominated vertices $8, 11, 16$ and give them the same color as the vertex 3. Similarly, the vertex 4 is chosen and the undominated vertices $21, 13, 17$ and put them in the same color class as the vertex 4. Lastly, we choose the vertex 5 and give the undominated vertices $9, 20, 11, 15$ the same color. Thus, the final coloring of the vertices of the graph is $[1, 6, 18, 10, 14]$; $[2, 7, 12, 19]$; $[3, 8, 11, 16]$; $[4, 13, 17, 21]$, $[5, 9, 11, 15, 20]$, which is a 5 coloring of the whole graph.

□

This coloring can be also extended to any n cliques of order k sharing a common vertex. The coloring is done in a similar manner, that is, coloring fully one clique, and putting a representative vertex of each clique in each previous color class.

This can be extended to n-coloring of the vertices of n linear hypergraphs, as the graph version is equivalent to the linear hypergraph verison [8].

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