

# A Reformulation of Classical Mechanics

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This paper presents a reformulation of classical mechanics which is invariant under transformations between inertial and non-inertial reference frames and which can be applied in any reference frame without introducing fictitious forces.

## Introduction

The reformulation of classical mechanics presented in this paper is obtained starting from an auxiliary force of interaction ( called kinetic force, since this auxiliary force of interaction is directly related to kinetic energy )

The kinetic force  $\mathbf{K}_{ij}$  exerted on a particle  $i$  of mass  $m_i$  by another particle  $j$  of mass  $m_j$ , caused by the interaction between particle  $i$  and particle  $j$ , is given by:

$$\mathbf{K}_{ij} = - \frac{m_i m_j}{M} \left[ (\vec{a}_i - \vec{a}_j) - 2 \vec{\omega} \times (\vec{v}_i - \vec{v}_j) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{r}_j)] - \vec{\alpha} \times (\vec{r}_i - \vec{r}_j) \right]$$

where  $\vec{a}_i$ ,  $\vec{v}_i$ ,  $\vec{r}_i$  are the acceleration, the velocity and the position of particle  $i$ ,  $\vec{a}_j$ ,  $\vec{v}_j$ ,  $\vec{r}_j$  are the acceleration, the velocity and the position of particle  $j$  ( belonging to an auxiliary system of  $N$  particles, called Systema ) and finally  $M$ ,  $\vec{\omega}$ ,  $\vec{\alpha}$  are the mass, the angular velocity and the angular acceleration of the Systema ( see Annex I )

From the above equation it follows that the net kinetic force  $\mathbf{K}_i$  ( =  $\sum_j^N \mathbf{K}_{ij}$  ) acting on a particle  $i$  of mass  $m_i$  is given by:

$$\mathbf{K}_i = - m_i \left[ (\vec{a}_i - \vec{A}) - 2 \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R})] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \right] \quad [\text{Eq. 2}]$$

where  $\vec{R}$ ,  $\vec{V}$  and  $\vec{A}$  are the position, the velocity and the acceleration of the center of mass of the Systema.

The magnitudes  $[m_i, m_j, M, \mathbf{K}_{ij}, \mathbf{K}_i]$  are invariant under transformations between inertial and non-inertial reference frames.

Any reference frame  $S$  is an inertial reference frame when the angular velocity  $\vec{\omega}$  of the Systema and the acceleration  $\vec{A}$  of the center of mass of the Systema are equal to zero (  $\vec{\omega} = 0$  and  $\vec{A} = 0$  ) relative to  $S$ . Therefore, the reference frame  $S$  is a non-inertial reference frame when the angular velocity  $\vec{\omega}$  of the Systema or the acceleration  $\vec{A}$  of the center of mass of the Systema are not equal to zero (  $\vec{\omega} \neq 0$  or  $\vec{A} \neq 0$  ) relative to  $S$ .

# Equation of Motion

The total force  $\mathbf{T}_i$  acting on a particle  $i$  is always zero.

$$\mathbf{T}_i = 0$$

If the total force  $\mathbf{T}_i$  is divided into the following two parts: the net kinetic force  $\mathbf{K}_i$  and the net dynamic force  $\mathbf{F}_i$  ( $\sum$  of gravitational forces, electrostatic forces, etc.) then we have:

$$\mathbf{K}_i + \mathbf{F}_i = 0$$

Now, substituting  $\mathbf{K}_i$  by [Eq. 2] dividing by  $m_i$  and rearranging, we obtain:

$$\vec{a}_i = \mathbf{F}_i/m_i + \vec{A} + 2\vec{\omega} \times (\vec{v}_i - \vec{V}) - \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R})] + \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

From the above equation it follows that particle  $i$  can have a non-zero acceleration even if there is no dynamic force acting on particle  $i$ , and also that particle  $i$  can have zero acceleration (state of rest or of uniform linear motion) even if there is an unbalanced net dynamic force acting on particle  $i$ .

However, from the above equation it also follows that Newton's first and second laws are valid in any inertial reference frame, since the angular velocity  $\vec{\omega}$  of the Systema and the acceleration  $\vec{A}$  of the center of mass of the Systema are equal to zero relative to any inertial reference frame.

## General Observations

All the equations presented in this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Additionally, inertial and non-inertial observers must not introduce fictitious forces into  $\mathbf{F}_i$ .

In this paper, the following magnitudes [ $m$ ,  $\mathbf{r}$ ,  $\mathbf{v}$ ,  $\mathbf{a}$ ,  $M$ ,  $K$ ,  $\mathbf{T}$ ,  $\mathbf{K}$ ,  $\mathbf{F}$ ] are invariant under transformations between inertial and non-inertial reference frames.

The kinetic forces are caused by the interactions between the particles and the net kinetic force is the force that balances the net dynamic force in each particle of the Universe.

In addition, the kinetic forces remain invariant under transformations between inertial and non-inertial reference frames ( as all dynamic forces do )

In this paper, the kinetic forces and the dynamic forces can obey or disobey Newton's third law in its weak form or in its strong form ( this is one of the main goals of this paper )

On the other hand, this paper does not contradict Newton's first and second laws since these two laws are valid in any inertial reference frame ( in Newtonian mechanics the kinetic forces are completely excluded )

Finally, the reformulation of classical mechanics presented in this paper is observationally equivalent to Newtonian mechanics. However, non-inertial observers can only use Newtonian mechanics if they introduce fictitious forces into  $\mathbf{F}_i$ .

# Annexes

## Relational Systema

In classical mechanics, the Systema is an auxiliary system of  $N$  particles that must always be free of internal and external dynamic forces, that must be three-dimensional, and that the relative distances between the  $N$  particles must be constant.

The position  $\vec{R}$ , the velocity  $\vec{V}$  and the acceleration  $\vec{A}$  of the center of mass of the Systema relative to a reference frame  $S$  (and the angular velocity  $\vec{\omega}$  and the angular acceleration  $\vec{\alpha}$  of the Systema relative to the reference frame  $S$ ) are given by:

$$M \doteq \sum_i^N m_i$$

$$\vec{R} \doteq M^{-1} \sum_i^N m_i \vec{r}_i$$

$$\vec{V} \doteq M^{-1} \sum_i^N m_i \vec{v}_i$$

$$\vec{A} \doteq M^{-1} \sum_i^N m_i \vec{a}_i$$

$$\vec{\omega} \doteq \vec{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\vec{I} \doteq \sum_i^N m_i [ |\vec{r}_i - \vec{R}|^2 \vec{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R}) ]$$

$$\vec{L} \doteq \sum_i^N m_i (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where  $M$  is the mass of the Systema,  $\vec{I}$  is the inertia tensor of the Systema (relative to  $\vec{R}$ ) and  $\vec{L}$  is the angular momentum of the Systema relative to the reference frame  $S$ .

## Invariant Magnitudes

$$(\vec{r}_i - \vec{R}) \doteq \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}'_i - \vec{R}') \doteq \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}'_i - \vec{V}') - \vec{\omega}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2 \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R})] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}'_i - \vec{A}') - 2 \vec{\omega}' \times (\vec{v}'_i - \vec{V}') + \vec{\omega}' \times [\vec{\omega}' \times (\vec{r}'_i - \vec{R}')] - \vec{\alpha}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{a}'_i = \mathbf{a}_i$$

## Appendix A

### Fields and Potentials I

The net kinetic force  $\mathbf{K}_i$  acting on a particle  $i$  of mass  $m_i$  can also be expressed as follows:

$$\mathbf{K}_i = + m_i \left[ \mathbf{E} + (\vec{v}_i - \vec{V}) \times \mathbf{B} \right]$$

$$\mathbf{K}_i = + m_i \left[ -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\vec{v}_i - \vec{V}) \times (\nabla \times \mathbf{A}) \right]$$

$$\mathbf{K}_i = + m_i \left[ -(\vec{a}_i - \vec{A}) + 2 \vec{\omega} \times (\vec{v}_i - \vec{V}) - \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R})] + \vec{\alpha} \times (\vec{r}_i - \vec{R}) \right]$$

where:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\phi = -1/2 [\vec{\omega} \times (\vec{r}_i - \vec{R})]^2 + 1/2 (\vec{v}_i - \vec{V})^2$$

$$\mathbf{A} = -[\vec{\omega} \times (\vec{r}_i - \vec{R})] + (\vec{v}_i - \vec{V})$$

$$\frac{\partial \mathbf{A}}{\partial t} = -\vec{\alpha} \times (\vec{r}_i - \vec{R}) + (\vec{a}_i - \vec{A})$$

$$\nabla\phi = \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R})]$$

$$\nabla \times \mathbf{A} = -2 \vec{\omega}$$

The net kinetic force  $\mathbf{K}_i$  acting on a particle  $i$  of mass  $m_i$  can also be obtained starting from the following kinetic energy:

$$K_i = -m_i \left[ \phi - (\vec{v}_i - \vec{V}) \cdot \mathbf{A} \right]$$

$$K_i = 1/2 m_i \left[ (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \right]^2$$

$$K_i = 1/2 m_i \left[ \mathbf{v}_i \right]^2$$

Since the kinetic energy  $K_i$  must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i = -\frac{d}{dt} \left[ \frac{\partial 1/2 m_i \left[ \mathbf{v}_i \right]^2}{\partial \mathbf{v}_i} \right] + \frac{\partial 1/2 m_i \left[ \mathbf{v}_i \right]^2}{\partial \mathbf{r}_i} = -m_i \mathbf{a}_i$$

where  $\mathbf{r}_i$ ,  $\mathbf{v}_i$  and  $\mathbf{a}_i$  are the invariant position, the invariant velocity and the invariant acceleration of particle  $i$  (see Annex II)

## Appendix B

### Fields and Potentials II

The net kinetic force  $\mathbf{K}_i$  acting on a particle  $i$  of mass  $m_i$  ( relative to a reference frame  $S$  fixed to a particle  $s$  (  $\vec{r}_s = \vec{v}_s = \vec{a}_s = 0$  ) of mass  $m_s$ , with invariant velocity  $\mathbf{v}_s$  and invariant acceleration  $\mathbf{a}_s$  ) can also be expressed as follows:

$$\mathbf{K}_i = + m_i \left[ \mathbf{E} + \vec{v}_i \times \mathbf{B} \right]$$

$$\mathbf{K}_i = + m_i \left[ -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \vec{v}_i \times (\nabla \times \mathbf{A}) \right]$$

$$\mathbf{K}_i = + m_i \left[ -(\vec{a}_i + \mathbf{a}_s) + 2 \vec{\omega} \times \vec{v}_i - \vec{\omega} \times (\vec{\omega} \times \vec{r}_i) + \vec{\alpha} \times \vec{r}_i \right]$$

where:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\phi = -1/2 (\vec{\omega} \times \vec{r}_i)^2 + 1/2 (\vec{v}_i + \mathbf{v}_s)^2$$

$$\mathbf{A} = -(\vec{\omega} \times \vec{r}_i) + (\vec{v}_i + \mathbf{v}_s)$$

$$\frac{\partial \mathbf{A}}{\partial t} = -\vec{\alpha} \times \vec{r}_i + (\vec{a}_i + \mathbf{a}_s)$$

$$\nabla\phi = \vec{\omega} \times (\vec{\omega} \times \vec{r}_i)$$

$$\nabla \times \mathbf{A} = -2 \vec{\omega}$$

The net kinetic force  $\mathbf{K}_i$  acting on a particle  $i$  of mass  $m_i$  can also be obtained starting from the following kinetic energy:

$$K_i = - m_i \left[ \phi - (\vec{v}_i + \mathbf{v}_s) \cdot \mathbf{A} \right]$$

$$K_i = 1/2 m_i \left[ (\vec{v}_i + \mathbf{v}_s) - (\vec{\omega} \times \vec{r}_i) \right]^2$$

$$K_i = 1/2 m_i \left[ \mathbf{v}_i \right]^2$$

Since the kinetic energy  $K_i$  must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i = - \frac{d}{dt} \left[ \frac{\partial 1/2 m_i \left[ \mathbf{v}_i \right]^2}{\partial \mathbf{v}_i} \right] + \frac{\partial 1/2 m_i \left[ \mathbf{v}_i \right]^2}{\partial \mathbf{r}_i} = - m_i \mathbf{a}_i$$

where  $\mathbf{r}_i$ ,  $\mathbf{v}_i$  and  $\mathbf{a}_i$  are the invariant position, the invariant velocity and the invariant acceleration of particle  $i$  ( see Annex II )

## Appendix C

### Fields and Potentials III

The kinetic force  $\mathbf{K}_{ij}$  exerted on a particle  $i$  of mass  $m_i$  by another particle  $j$  of mass  $m_j$  can also be expressed as follows:

$$\mathbf{K}_{ij} = + m_i m_j M^{-1} \left[ \mathbf{E} + (\vec{v}_i - \vec{v}_j) \times \mathbf{B} \right]$$

$$\mathbf{K}_{ij} = + m_i m_j M^{-1} \left[ -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\vec{v}_i - \vec{v}_j) \times (\nabla \times \mathbf{A}) \right]$$

$$\mathbf{K}_{ij} = + m_i m_j M^{-1} \left[ -(\vec{a}_i - \vec{a}_j) + 2\vec{\omega} \times (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{r}_j)] + \vec{\alpha} \times (\vec{r}_i - \vec{r}_j) \right]$$

where:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\phi = -1/2 [\vec{\omega} \times (\vec{r}_i - \vec{r}_j)]^2 + 1/2 (\vec{v}_i - \vec{v}_j)^2$$

$$\mathbf{A} = -[\vec{\omega} \times (\vec{r}_i - \vec{r}_j)] + (\vec{v}_i - \vec{v}_j)$$

$$\frac{\partial \mathbf{A}}{\partial t} = -\vec{\alpha} \times (\vec{r}_i - \vec{r}_j) + (\vec{a}_i - \vec{a}_j)$$

$$\nabla\phi = \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{r}_j)]$$

$$\nabla \times \mathbf{A} = -2\vec{\omega}$$

The kinetic force  $\mathbf{K}_{ij}$  exerted on a particle  $i$  of mass  $m_i$  by another particle  $j$  of mass  $m_j$  can also be obtained starting from the following kinetic energy:

$$K_{ij} = -m_i m_j M^{-1} \left[ \phi - (\vec{v}_i - \vec{v}_j) \cdot \mathbf{A} \right]$$

$$K_{ij} = 1/2 m_i m_j M^{-1} \left[ (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times (\vec{r}_i - \vec{r}_j) \right]^2$$

$$K_{ij} = 1/2 m_i m_j M^{-1} \left[ \mathbf{v}_i - \mathbf{v}_j \right]^2$$

Since the kinetic energy  $K_{ij}$  must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_{ij} = -\frac{d}{dt} \left[ \frac{\partial 1/2 \frac{m_i m_j}{M} [\mathbf{v}_i - \mathbf{v}_j]^2}{\partial [\mathbf{v}_i - \mathbf{v}_j]} \right] + \frac{\partial 1/2 \frac{m_i m_j}{M} [\mathbf{v}_i - \mathbf{v}_j]^2}{\partial [\mathbf{r}_i - \mathbf{r}_j]} = -\frac{m_i m_j}{M} [\mathbf{a}_i - \mathbf{a}_j]$$

where  $\mathbf{r}_i, \mathbf{v}_i, \mathbf{a}_i, \mathbf{r}_j, \mathbf{v}_j$  and  $\mathbf{a}_j$  are the invariant positions, the invariant velocities and the invariant accelerations of particles  $i$  and  $j$  (see Annex II)