A trigonometric proof of Oppenheim's Inequality

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Abstract. This problem first appeared in the American Mathematical Monthly in 1965, proposed by Sir Alexander Oppenheim. As a matter of curiosity, the American Mathematical Monthly is the most widely read mathematics journal in the world. On the other hand, Oppenheim was a brilliant mathematician, and for the excellence of his work in mathematics, obtained the title of " Sir ", given by the English to English citizens who stand out in the national and international scenario.Oppenheim is better known in the academic world for his contribution to the field of Number Theory, known as the Oppenheim Conjecture.

Theorem 1. *Let x, y, z real positive numbers and ΔABC a triangle. $[ABC]$ denotes the triangle area and a, b, c the sides of the triangle. The inequality below is true:*

$$a^2x + b^2y + c^2z \geq 4[ABC]\sqrt{xy + xz + yz}$$

Proof. Let α, β, γ denote the opposite angles to the sides a, b, c, respectively. R is the circumradius of ΔABC. Observe that:

$$a^2x + b^2y + c^2z \geq 4[ABC]\sqrt{xy + xz + yz}$$

$$a^2x + b^2y + c^2z \geq \frac{abc}{R}\sqrt{xy + xz + yz}$$

$$\frac{aRx}{bc} + \frac{bRy}{ac} + \frac{cRz}{ab} \geq \sqrt{xy + xz + yz}$$

$$\frac{1}{2}\left(\frac{4aR^2x}{2Rbc} + \frac{4bR^2y}{2Rac} + \frac{4cR^2z}{2Rab}\right) \geq \sqrt{xy + xz + yz}$$

$$\frac{\sin\alpha x}{\sin\beta\sin\gamma} + \frac{\sin\beta y}{\sin\alpha\sin\gamma} + \frac{\sin\gamma z}{\sin\alpha\sin\beta} \geq 2\sqrt{xy + xz + yz}$$

$$\frac{\sin(\pi-\alpha)x}{\sin\beta\sin\gamma} + \frac{\sin(\pi-\beta)y}{\sin\alpha\sin\gamma} + \frac{\sin(\pi-\gamma)z}{\sin\alpha\sin\beta} \geq 2\sqrt{xy + xz + yz}$$

$$\frac{\sin(\alpha+\beta+\gamma-\alpha)x}{\sin\beta\sin\gamma} + \frac{\sin(\alpha+\beta+\gamma-\beta)y}{\sin\alpha\sin\gamma} + \frac{\sin(\alpha+\beta+\gamma-\gamma)z}{\sin\alpha\sin\beta} \geq 2\sqrt{xy + xz + yz}$$

$$\frac{\sin(\beta+\gamma)x}{\sin\beta\sin\gamma} + \frac{\sin(\alpha+\gamma)y}{\sin\alpha\sin\gamma} + \frac{\sin(\alpha+\beta)z}{\sin\alpha\sin\beta} \geq 2\sqrt{xy + xz + yz}$$

$$\frac{(\sin\beta\cos\gamma+\sin\gamma\cos\beta)x}{\sin\beta\sin\gamma}+\frac{(\sin\alpha\cos\gamma+\sin\gamma\cos\alpha)y}{\sin\alpha\sin\gamma}+\frac{(\sin\alpha\cos\beta+\sin\beta\cos\alpha)z}{\sin\alpha\sin\beta}\geq 2\sqrt{xy+xz+yz}$$

$$(\cot\beta+\cot\gamma)x+(\cot\alpha+\cot\gamma)y+(\cot\alpha+\cot\beta)z\geq 2\sqrt{xy+xz+yz} \tag{1}$$

Since inequality is homogeneous in the variables x, y, z, do it $xy + xz + yz = 1$ and take the substitution $x = \cot\alpha', y = \beta', z = \gamma'$, we have que α', β', γ' are angles of a triangle, and our inequality will be equivalent to the inequality below:

$$(\cot\beta+\cot\gamma)\cot\alpha'+(\cot\alpha+\cot\gamma)\cot\beta'+(\cot\alpha+\cot\beta)\cot\gamma'\geq 2 \tag{2}$$

Suppose without loss of generality that (the reverse case is analogous) :

$$\cot\alpha\geq\cot\alpha' \tag{3}$$

$$\cot\beta\geq\cot\beta' \tag{4}$$

$$\cot\gamma'\geq\cot\gamma \tag{5}$$

Because these variables are angles of a triangle, we can not have $\alpha\geq\alpha', \beta\geq\beta', \gamma\geq\gamma'$.In fact, this can not occur, since it supposes without loss of generality that $\alpha'\geq\alpha$ and $\beta'\geq\beta$(as the angent is decreasing, this implies that $\cot\alpha\geq\cot\alpha'$ and $\cot\beta\geq\cot\beta'$), summing up these first two inequalities we have:

$\alpha'+\beta'\geq\alpha+\beta\Rightarrow\cot(\alpha+\beta)\geq\cot(\alpha'+\beta')\Rightarrow-\cot(\pi-\alpha+\beta)\geq-\cot(\pi-\alpha'+\beta')\Rightarrow-\cot(\alpha+\beta+\gamma-(\alpha+\beta))\geq-\cot(\alpha'+\beta'+\gamma'-(\alpha'+\beta'))\Rightarrow-\cot(\gamma)\geq-\cot(\gamma')\Rightarrow\cot(\gamma')\geq\cot(\gamma)$

Now set the $f_1(\alpha,\beta,\gamma,\alpha',\beta',\gamma'):\mathbb{R}^6\to\mathbb{R}$ and $f_2(\alpha,\beta,\gamma,\alpha',\beta',\gamma'):\mathbb{R}^6\to\mathbb{R}$ such that:

$$f_1(\alpha,\beta,\gamma,\alpha',\beta',\gamma')=$$

$$(\cot\beta+\cot\gamma)(\cot\alpha'-\cot\alpha)+(\cot\alpha+\cot\gamma)(\cot\beta'-\cot\beta)+(\cot\alpha+\cot\beta)(\cot\gamma'-\cot\gamma) \tag{6}$$

$$f_2(\alpha,\beta,\gamma,\alpha',\beta',\gamma')=$$

$$(\cot\beta'+\cot\gamma')(\cot\alpha-\cot\alpha')+(\cot\alpha'+\cot\gamma')(\cot\beta-\cot\beta')+(\cot\alpha'+\cot\beta')(\cot\gamma-\cot\gamma') \tag{7}$$

Note now that by inequalities (3), (4) and (5) it follows that:

$$0 \geq \cot \alpha' - \cot \alpha \tag{8}$$

$$0 \geq \cot \beta' - \cot \beta \tag{9}$$

$$\cot \gamma' - \cot \gamma \geq 0 \tag{10}$$

We know that α', β', γ' are angles of a triangle, so there exists a', b', c' such that $a'^2 = b'^2 + c'^2 - 2b'c' \cos \alpha', b'^2 = a'^2 + c'^2 - 2a'c' \cos \beta', c'^2 = a'^2 + b'^2 - 2a'b' \cos \gamma'$.Let R' the radius of the circumference inscribed to the triangle of sides a ', b' and c '.Look to the below inequality:

$$(\cot \beta + \cot \gamma) \cot \alpha' + (\cot \alpha + \cot \gamma) \cot \beta' + (\cot \alpha + \cot \beta) \cot \gamma' \geq 2 \tag{11}$$

Let $a = \cot(\alpha), b = \cot \beta, c = \cot \gamma, x = \cot(\alpha'), y = \cot \beta', z = \cot \gamma'$ and the inequality above will be:

$$x(b + c) + y(a + c) + z(a + b) \geq 2 \tag{12}$$

See that

$$x\left(\frac{1}{1/b} + \frac{1}{1/c}\right) + y\left(\frac{1}{1/a} + \frac{1}{1/c}\right) + z\left(\frac{1}{1/a} + \frac{1}{1/b}\right) \geq 2 \tag{13}$$

Making $1/a = a', 1/b = b', 1/c' = c$ we get x, y, z, a', b', c' wil be homogeneous in inequality below:

$$x\left(\frac{1}{b'} + \frac{1}{c'}\right) + y\left(\frac{1}{a'} + \frac{1}{c'}\right) + z\left(\frac{1}{a'} + \frac{1}{b'}\right) \geq 2 \tag{14}$$

Really, if we change a', b', c', x, y, z by $a't, b't, c't, xt, yt, zt$ that we can see it's true.So, supose WLOG:

$$\frac{1}{a'} + \frac{1}{b'} + \frac{1}{c'} = x + y + z \tag{15}$$

And

$$\cot \alpha + \cot \beta + \cot \gamma = \cot \alpha' + \cot \beta' + \cot \gamma' \tag{16}$$

And them:

$$\frac{aR}{bc} + \frac{bR}{ac} + \frac{cR}{ab} = \frac{a'R'}{b'c'} + \frac{b'R'}{a'c'} + \frac{c'R'}{a'b'} \tag{17}$$

Take the inequality (3) and consider the development (applying the law of cosines and law of sines):
$\cot\alpha \geq \cot\alpha' \Rightarrow \dfrac{(b^2+c^2-a^2)R}{abc} \geq \dfrac{(b'^2+c'^2-a'^2)R'}{a'b'c'} \Rightarrow \left(\dfrac{a}{bc}+\dfrac{b}{ac}+\dfrac{c}{ab}-2\dfrac{a}{bc}\right)R \geq \left(\dfrac{a'}{b'c'}+\dfrac{b'}{a'c'}+\dfrac{c'}{a'b'}-2\dfrac{a'}{b'c'}\right)R' \Rightarrow -2\dfrac{aR}{bc} \geq -2\dfrac{a'R'}{b'c'} \Rightarrow \dfrac{a'R'}{b'c'} \geq \dfrac{aR}{bc} \Rightarrow$

$$\frac{a'R'}{b'c'} \geq \frac{aR}{bc} \tag{18}$$

Applying the same rationale for inequality (4), we conclude:

$$\frac{b'R'}{a'c'} \geq \frac{bR}{ac} \tag{19}$$

Suppose by contradiction that it occurs:

$$\cot\alpha + \cot\gamma > \cot\alpha' + \cot\gamma' \tag{20}$$

See that:

$\cot\alpha + \cot\gamma > \cot\alpha' + \cot\gamma' \Rightarrow \dfrac{(b^2+c^2-a^2)R}{abc} + \dfrac{(a^2+b^2-c^2)R}{abc} > \dfrac{(b'^2+c'^2-a'^2)R'}{a'b'c'} + \dfrac{(a'^2+b'^2-c'^2)R'}{a'b'c'} \Rightarrow \dfrac{bR}{ac} > \dfrac{b'R'}{a'c'}$

This contradicts the inequality (14).On the oder hand, suppose by contradiction that it occurs:

$$\cot\beta + \cot\gamma > \cot\beta' + \cot\gamma' \tag{21}$$

See that:

$\cot\beta + \cot\gamma > \cot\beta' + \cot\gamma' \Rightarrow \dfrac{(a^2+c^2-b^2)R}{abc} + \dfrac{(a^2+b^2-c^2)R}{abc} > \dfrac{(a'^2+c'^2-b'^2)R'}{a'b'c'} + \dfrac{(a'^2+b'^2-c'^2)R'}{a'b'c'} \Rightarrow \dfrac{aR}{bc} > \dfrac{a'R'}{b'c'}$

This contradicts the inequality (13).Therefore:

$$\cot\alpha + \cot\gamma \leq \cot\alpha' + \cot\gamma' \tag{22}$$

$$\cot\beta + \cot\gamma \leq \cot\beta' + \cot\gamma' \tag{23}$$

Multiplying (17) by $\cot\beta' - \cot\beta$ and multiplying (18) by $\cot\alpha' - \cot\alpha$, note that these inequalities will reverse, since we are multiplying by non-positive quantities, we will have, respectively:

$$(\cot\alpha + \cot\gamma)(\cot\beta' - \cot\beta) \geq (\cot\alpha' + \cot\gamma')(\cot\beta' - \cot\beta) \tag{24}$$

$$(\cot\beta + \cot\gamma)(\cot\alpha' - \cot\alpha) \geq (\cot\beta' + \cot\gamma')(\cot\alpha' - \cot\alpha) \tag{25}$$

On the other hand of inequalities (3) and (4) we know that:

$$\cot\alpha + \cot\beta \geq \cot\alpha' + \cot\beta' \tag{26}$$

Multiplying the above inequality by $\cot\gamma' - \cot\gamma$, that by the inequality (10) we know to be greater than or equal to zero, we will have:

$$(\cot\alpha + \cot\beta)(\cot\gamma' - \cot\gamma) \geq (\cot\alpha' + \cot\beta')(\cot\gamma' - \cot\gamma) \tag{27}$$

Adding (19), (20) and (22), we will have:

$$f_1(\alpha, \beta, \gamma, \alpha', \beta', \gamma') \geq$$

$$(\cot\alpha' + \cot\gamma')(\cot\beta' - \cot\beta) + (\cot\beta' + \cot\gamma')(\cot\alpha' - \cot\alpha) + (\cot\alpha' + \cot\beta')(\cot\gamma' - \cot\gamma) \tag{28}$$

Adding the left side of (23) with the left side of (7) and the right side of (23) with the right side of (7), the terms will cancel and we will have:

$$f_1(\alpha, \beta, \gamma, \alpha', \beta', \gamma') + f_2(\alpha, \beta, \gamma, \alpha', \beta', \gamma') \geq 0 \tag{29}$$

And this implies, finally, that:

$$(\cot\beta + \cot\gamma)\cot\alpha' + (\cot\alpha + \cot\gamma)\cot\beta' + (\cot\alpha + \cot\beta)\cot\gamma' \geq 2 \tag{30}$$

That is precisely the inequality (2), which is equivalent to the desired inequality.

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Theorem 2. *Let $\Delta ABC_1, \Delta ABC_2$ a triangle. F_1, F_2 denotes the triangle area and $a_1, b_1, c_1, a_2, b_2, c_2$ the sides of the triangle. The inequality below is true:*

$$a_1^2(b_2^2 + c_2^2 - a_2^2) + b_1^2(a_2^2 + c_2^2 - b_2^2) + c_1^2(a_2^2 + b_2^2 - c_2^2) \geq 16F_1F_2$$

Proof. Make the substitution in the inequality of the previous theorem:

$$x = c^2 + b^2 - a^2 \tag{31}$$

$$y = c^2 + a^2 - b^2 \tag{32}$$

$$z = a^2 + b^2 - c^2 \tag{33}$$

And then on the right side of the inequality of Theorem 1, we have:

$$\sqrt{xy + xz + yz} =$$

$$\sqrt{(c^2 + b^2 - a^2)(c^2 + a^2 - b^2) + (c^2 + b^2 - a^2)(a^2 + b^2 - c^2) + (c^2 + a^2 - b^2)(a^2 + b^2 - c^2)} \tag{34}$$

Replace:

$$x = c^2 + b^2 - a^2 = 2bc \cos\alpha \tag{35}$$

$$y = c^2 + a^2 - b^2 = 2ac \cos\beta \tag{36}$$

$$z = a^2 + b^2 - c^2 = 2ab \cos\gamma \tag{37}$$

Substituting the three equalities above in (34), we have:

$$\sqrt{4a^2bc \cos\gamma \cos\beta + 4b^2ac \cos\gamma \cos\alpha + 4abc^2 \cos\alpha \cos\beta} =$$

$$\sqrt{abc(4a \cos\gamma \cos\beta + 4b \cos\gamma \cos\alpha + 4c \cos\alpha \cos\beta)}$$

By the sine's law, we will have:

$$\sqrt{8Rabc(\sin\alpha \cos\gamma \cos\beta + \sin\beta \cos\gamma \cos\alpha + \sin\gamma \cos\alpha \cos\beta)} =$$

$$\sqrt{8Rabc(cos\gamma(\sin\alpha \cos\beta + \sin\beta \cos\alpha) + \sin\gamma \cos\alpha \cos\beta)} =$$

$$\sqrt{8Rabc(cos\gamma\sin(\alpha+\beta)+\sin\gamma\cos\alpha\cos\beta)} =$$

$$\sqrt{8Rabc(cos\gamma\sin(\gamma)+\sin\gamma\cos\alpha\cos\beta)} =$$

$$\sqrt{8Rabc(\sin(\gamma)(cos\gamma+\cos\alpha\cos\beta)} =$$

$$\sqrt{8Rabc(\sin(\gamma)(cos\gamma+\cos(\alpha+\beta)+\sin(\alpha)\sin(\beta)) =}$$

$$\sqrt{8Rabc(\sin(\gamma)(cos\gamma-\cos(\alpha+\beta+\gamma-(\alpha+\beta))+\sin(\alpha)\sin(\beta) =}$$

$$\sqrt{8Rabc(\sin(\gamma)\sin(\alpha)\sin(\beta) =}$$

$$\sqrt{\frac{a^2b^2c^2}{R^2} =}$$

$$\frac{abc}{R} =$$

$$4F_1$$

Done.

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