Maximum First Open Numbers and Goldbach's Conjecture

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Abstract. For a fixed last prime, sieve the positive integers as follows. For every prime up to and including that last prime, choose one arbitrary remainder and its negative. Sieve the positive integers by eliminating all numbers congruent to the chosen remainders modulo their prime. Consider the maximum of the first open numbers left by all such sieves for a particular last prime. Computations for small last primes support a conjecture that the maximum first open number is less than (last prime) $^{1.75}$. If this conjecture could be proved, it would imply Goldbach's Theorem is true.

1. FIRST OPEN NUMBER – DEFINITION AND AN EXAMPLE. Fix a particular prime called the Last Prime, LP. For each prime up to and including LP, choose one arbitrary remainder and its negative. Sieve the positive integers by eliminating numbers congruent to the chosen remainders modulo their primes. The sieve has a smallest number that is left open (not eliminated). Call that number the First Open Number (FON) of the sieve.

A smallest number (FON) always exists. Numbers less than or equal to the product of the primes up to LP have every combination of remainders modulo these primes. Some combinations of remainders do not include any of the remainders which have been selected to be eliminated. Therefore, the numbers which correspond to those combinations of remainders are not eliminated by the sieve.

Example 1. Let LP = 7. For this example, choose remainders 1 mod 2, 0 mod 3, 1 and -1 mod 5, 2 and -2 mod 7. The calculation of the first open number, the value of the sieve, is as follows.

From 1, 2, 3, 4, 5,…
Eliminate numbers = 1 mod 2 (odds) leaving: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, …
Then eliminate numbers = 0 mod 3 leaving: 2, 4, 8, 10, 14, 16, 20, …
Then eliminate numbers = $\pm$1 mod 5 leaving: 2, 8, 10, 20, …
Also eliminate numbers = $\pm$2 mod 7 leaving: 8, 10, 20, …
So 8 is the FON, the first number not eliminated, for this particular sieve, that is, for LP 7, and these remainders.

2. MAXIMUM FIRST OPEN NUMBER PROBLEM. Consider all possible such sieves for a particular last prime. There are a finite number of ways of choosing the remainders that define the sieve. How big is the maximum first open number (MFON) of any such sieve? How is the MFON related to LP? This is likely a hard problem, because it is related to Goldbach's Theorem.

To show that MFON $\geq$ LP, choose the sieve with remainders 0 for all primes up to LP, and 1 and –1 for LP. LP is the FON for this sieve, which is like the Sieve of Eratosthenes, but eliminates primes in addition to numbers divisible by primes.

Example 1 Continued. For LP 7 there are 2x2x3x4 = 48 choices of remainders (1 or 0 for 2; ± 1 or 0 for 3; ± 1, ± 2 or 0 for 5; ± 1, ± 2, ± 3 or 0 for 7). The sieve for any of the 48 could potentially produce the MFON.

3. ALTERNATIVE DEFINITION OF THE MAXIMUM FIRST OPEN NUMBER. Fix a prime LP. Let N be any positive integer. Let *a* be the smallest positive integer which has no remainder which is the same as, or the negative of, any remainder of N modulo every prime up to and including LP. The integer *a* is the first open number for N. The largest *a* for any N is the maximum first open number for LP. (Only numbers N from one up to the product of primes from 2 to LP need to be considered.)

Example 2. Let LP = 5. For integers 1 to 30 (30 = 2x3x5), their FON's are 12, 9, 4, 3, 6, 5, 6, 9, 2, 3, 12, 1, 6, 3, 2, 3, 6, 1, 12, 3, 2, 9, 6, 5, 6, 3, 4, 9, 12, 1. Therefore, the MFON for LP = 5 is 12.

4. COMPUTATIONS OF THE MAXIMUM FIRST OPEN NUMBER FOR A FEW LAST PRIMES, AND A CONJECTURE. For very small last primes, the MFON can be calculated without difficulty. For larger primes, the number of sieves to be checked increases rapidly.

Computed results for LP up to 43 are shown in Table 1. From these few results, it seems that MFON increases faster than LP, and also faster than LP times the natural logarithm of LP. The last column shows the exponent that applied to LP produces MFON. The exponent appears to be roughly constant, between 1.6 and 1.7. These few examples lead to a conjecture that MFON $\leq$ $LP^{1.75}$.

Table 1. Ratios for maximum first open number and last prime

Last Prime	MFON	MFON/LP	MFON/(ln(LP)*LP)	ln(MFON)/ln(LP)
2	2	1	1.4	1
3	6	2	1.8	1.63
5	12	2.4	1.5	1.54
7	24	3.4	1.8	1.63
11	42	3.8	1.6	1.56
13	75	5.8	2.2	1.68
17	90	5.3	1.9	1.59
19	150	7.9	2.7	1.70
23	180	7.8	2.5	1.66
29	216	7.4	2.2	1.60
31	312	10.1	2.9	1.67
37	339	9.2	2.5	1.61
41	447	10.9	2.9	1.64
43	519	12.1	3.2	1.66

5. THE MFON CONJECTURE WOULD PROVE GOLDBACH'S THEOREM. One way to state Goldbach's Theorem (which famously has not been proved) is that every even number greater than six is the sum of two different primes. For example 38 = 7+31.

Let 2N be an even positive number, where 2N > 6. The pairs of numbers N±1, N±2, …, N±*a*, …, N±(N-3), each are two different numbers that sum to 2N. The last sum to consider is N±(N-3), or (2N-3) + 3, because 3 is the smallest odd prime. Thus, the numbers added to/subtracted from N to get summands to 2N are 1,2,3,…,N-3. These are the positive integers that were sieved above, but with an upper limit of N-3.

For Goldbach's Theorem, one must test whether any N+*a* and N-*a* are both prime. If no prime smaller than or equal to the square root of a number divides it evenly, the number is prime. For summands to 2N, only primes up to the square root of 2N-3 need to be used to check primality of the summands. This defines a last prime to use, LP, where $LP^2 \leq 2N-3 = 2(N-3) + 3$. Therefore LP satisfies the inequality $(LP^2 - 3)/2 \leq N-3$

If *a* and N have no remainders that are equal modulo any prime ≤ LP, then N-*a*, if it is positive, is not divisible by any such prime and so is prime. Also, if *a* and -N have no remainders that are equal mod any prime ≤ LP, then N+*a* is not divisible by any such prime and so is prime. Thus, if *a* has no remainder in common with either N or –N, modulo the primes up to and including LP, both N-*a* and N+*a* are prime.

List the remainders of N mod each prime up to LP, and form a sieve with these remainders and their negatives. The FON for this sieve is ≤ the MFON for LP.

If the MFON for LP is ≤ N-3, then the FON for these remainders is a number *a*, $1 \leq a \leq N-3$. Moreover, by the construction of the sieve, neither N nor -N has a remainder in common with *a* = FON for any prime ≤ LP. Therefore, N-*a* and N+*a* would be prime.

It remains to show that the MFON for LP is ≤ N-3.

The MFON conjecture, which is based only on the few primes in Table 1, is that $MFON \leq LP^{1.75}$. But $LP^{1.75} \leq (LP^2 - 3)/2$ for LP large enough. (Specifically, for LP ≥ 17, which corresponds to N≥146 or 2N≥292. Smaller values of 2N are known to be sums of two different primes.)

Thus, if the MFON conjecture were true, and N≥146, then $MFON \leq LP^{1.75} \leq (LP^2 - 3)/2 \leq N-3$, which would prove Goldbach's theorem.

6. MORE ON CALCULATING THE MAXIMUM FIRST OPEN NUMBER. There are four different sequences of open numbers for the four sieves for LP=3. These four sequences are also used to start the calculation of the MFON for larger last primes. In general, if there are k primes from 5 to LP, there are "only" 4 k! cases to calculate. This will be illustrated by completing the MFON calculation for LP=7, by examining only 8 instead of 48 cases.

The calculation of the four sieve cases for LP = 3 follows. Note that the 4 open number sequences left by these sieves are sequences with difference 6.

A. Eliminate positive integers = 0 mod 2, (evens), and = ±1 mod 3, leaving the sequence of numbers 3, 9, 15, 21, 27, 33, 39, … open

B. Eliminate positive integers = 1 mod 2, (odds), and = 0 mod 3, leaving 2, 4, 8, 10, 14, 16, 20, 22, … If the underlined number sequence is separated, reversed, and given negative signs, the combination is a sequence with difference 6.

C. Eliminate positive integers = 0 mod 2, (evens), and = 0 mod 3, leaving 1, 5, 7, 11, 13, 17, 19, 23, … Again, using negative signs for the separated and reversed underlined number sequence, the combined sequence has difference 6.

D. Eliminate positive integers = 1 mod 2, (odds), and = ±1 mod 3, leaving 6, 12, 18, 24, 30, 36, 42, …

The respective FON's for these four sequences are 3, 2, 1, 6 so the MFON for last prime 3 is 6.

These four cases A, B, C, D of disjoint sequences with difference 6, left by the choices of remainders for 2 and 3, are used for the calculation of the MFON for larger last primes. Only sieves for some of the possible choices of remainders have to be checked to find the MFON, that is, choices of remainders that match the next open number.

Example 1 for LP = 7 continued. Compute the maximum first open number for LP = 7.

For prime 5, the remainder could be ±1, ±2, or 0. For 7, the remainder could be ±1, ±2, ±3, or 0. To find the largest possible FON's for each of the cases A, B, C, or D, (for any remainders for primes 2 and 3) either 5 first and then 7 can have its remainder chosen to match the next open number in the sequence, or 7 first and then 5 can have remainders chosen to do so. The MFON is the largest FON from these 4 times 2 = 8 cases. Other sieves would have smaller (or equal) FON's.

A, 3, 9, 15, 21, 27, 33, 39, … 1) remainders ±2 mod 5 match 3, leaving 9, 15, 21, 39, …, then ±2 mod 7 match 9 leaving 15, 21, 39, … FON is 15.
A, 3, 9, 15, 21, 27, 33, 39, … 2) remainders ±3 mod 7 match 3, leaving 9, 15, 21, 27, 33, …, then ±1 mod 5 match 9 leaving 15, 27, 33, … FON is 15.

B, 2, 4, 8, 10, 14, 16, 20, 22, …1) remainders ±2 mod 5 match 2, leaving 4, 10, 14, 16, 20, …, then ±3 mod 7 match 4, leaving 14, 16, 20, … FON is 14.
B, 2, 4, 8, 10, 14, 16, 20, 22, …2) remainders ±2 mod 7 match 2, leaving 4, 8, 10, 14, 20, 22, …, then ±1 mod 5 match 4, leaving 8, 10, 20, 22, … FON is 8.

C, 1, 5, 7, 11, 13, 17, 19, 23, … 1) remainders ±1 mod 5, match 1, leaving 5, 7, 13, 17, 23, …, then ±2 mod 7 match 5, leaving 7, 13, 17, … FON is 7.
C, 1, 5, 7, 11, 13, 17, 19, 23, … 2) remainders ±1 mod 7, match 1, leaving 5, 7, 11, 17, 19, 23, …, then 0 mod 5 matches 5, leaving 7, 11, 17, 19, 23, … FON is 7.

D, 6, 12, 18, 24, 30, 36, 42, …1) remainders ±1 mod 5, match 6, leaving 12, 18, 30, 42, …, then ±2 mod 7 match 12 leaving 18, 42, … FON is 18.
D, 6, 12, 18, 24, 30, 36, 42, …2) remainders ±1 mod 7, match 6, leaving 12, 18, 24, 30, 42, …, then ±2 mod 5 match 12, leaving 24, 30, … FON is 24.

The FON's are 15, 15, 14, 8, 7, 7, 18, and 24 so the MFON for LP = 7 is 24.

In summary, to calculate the maximum first open number for a last prime 5 or larger, for each sequence A, B, C, D and each order of primes from 5 to LP, choose remainders of primes successively to eliminate the next open number of the sequence. The MFON is the largest FON of all these cases. The number of orderings of the k primes from 5 to LP is k!. Therefore there are 4 k! cases to calculate. This "small" number of sieve cases to compute allowed the construction of Table 1 above. (When k is large, doing all the remainder cases for some smaller primes can save a few more calculations.)

7. RANDOM CASES FOR LP – "M"FON. The number of cases needed to compute the MFON for larger LP's is extremely large. For each LP, some cases, that is, permutations of primes leading to sieves, were selected at random. From these, the maximum FON for the random cases, called an "M"FON, is a lower bound for the MFON. About 30,000 x 4 cases were calculated for each LP up to 2753. The "M"FON should be smaller relative to MFON as LP increases.

Table 2. Results for Random Cases for Selected LP's

LP	"M"FON	ln("M"FON)/ln(LP)	"M"FON/LP$^{1.75}$
181	2094	1.47	.23
421	5205	1.42	.13
673	8589	1.39	.10
953	12801	1.37	.08
1231	17232	1.37	.07
1531	21858	1.36	.06
1831	25839	1.35	.05
2137	31065	1.35	.05
2441	36027	1.35	.04
2753	41112	1.34	.04

Table 2 gives some values of "M"FON. The third column, which shows the exponent of LP that produces the "M"FON, gives numbers that would possibly be about 1.65 for the real MFON, based on the pattern of exponents in the last column of Table 1.

“M”FON is a decreasing fraction of the conjectured limit for MFON, as shown in the last column.

8. A RELATED FIRST OPEN NUMBER PROBLEM.

Choose a positive integer N.

For each prime (called last prime), LP, use the remainders of N for each prime up to and including LP, and their negatives to define a sieve of the positive integers. That is, eliminate integers congruent to each remainder and its negative mod the respective primes. Consider the sequence of the first numbers left open (not eliminated) by the sieve for each LP.

How is the growth of the sequence related to LP and to N?

Note that if the MFON conjecture is true, FON is less than or equal to $LP^{1.75}$ for every N.

9. LITERATURE. A 1948 paper by R.D. James describes work on Goldbach’s Conjecture by G.H. Hardy and J.E. Littlewood, and by others up to that time. T. Tao, O. Ramere and J.R. Chen, among others, more recently obtained results on what sums of primes can represent integers. Some recent papers, for example one by Hashem Sazegar, claim to have proved Goldbach’s Conjecture.

The sequence of maximum first open numbers is oeis.org/A307211 in the Online Encyclopedia of Integer Sequences, where it was extended by Bert Dobbelaere.