Speed and Measure Theorems Related to the Lonely Runner Conjecture

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Abstract

Some speed and measure theorems related to the lonely runner conjecture are proven. In particular, we prove that for any sequence of speeds $s_1 < s_2 < \cdots < s_k$ there exists a sequence of speeds $s'_1 < s'_2 < \cdots < s'_k$ that is arbitrarily close to the original set of speeds and for which the LRC is true.

1 Introduction

The lonely runner conjecture (LRC) is a very deep problem in the theory of Diophantine approximations that asks the following question: Given a circular track and k runners starting from a given starting line with speeds $s_1 < s_2 < \ldots s_k$, is it true that each runner becomes lonely at some time, i.e. seperated from the other runners by a distance of at least $\frac{1}{k}$? While easy to state, this conjecture is remarkably difficult and has eluded proof since it was proposed by T.W. Cusick in 1967.

In this paper we prove that the LRC is true for certain classes of speeds. We will also prove that the slowest and fastest runners become lonely for certain speeds. Finally, we will prove some measure theorems concerning these speeds.

2 Speed Theorems

The theorems we prove in this section heavily depend upon Lemma 1 which we state next. This lemma essentially states that the LRC is equivalent to a set of Diophantine approximation problems.

Lemma 1. Let $k \geq 5$. The LRC is equivalent to the following Diophantine approximation problems:

Slowest runner. Let $s_1 < s_2 < s_3 < \ldots < s_k$. Then there exists $n_1, n_2, \ldots n_{k-1} \in \mathbb{N}$ such that

$$\frac{kn_{m-1}+k-1}{kn_{i-1}+1} \geq \frac{s_m - s_1}{s_i - s_1} \geq \frac{kn_{m-1}+1}{kn_{i-1}+k-1} \tag{1}$$

for all $2 \leq i, m \leq k$, $m > i$.

Intermediate runner $j = 2$. There exists $m_{j2}, m_{21} \in \mathbb{N}$ such that

$$\frac{km_{j2}+k-1}{km_{i2}+1} \geq \frac{s_j - s_2}{s_i - s_2} \geq \frac{km_{j2}+1}{km_{i2}+k-1} \tag{2}$$

for $3 \leq j, i \leq k$ and $j > i$ and

$$\frac{km_{i2}+k-1}{km_{21}+1} \geq \frac{s_i - s_2}{s_2 - s_1} \geq \frac{km_{i2}+1}{km_{21}+k-1} \tag{3}$$

for $3 \leq i \leq k$.

Intermediate runner $3 \leq j \leq k-2$. There exists $q_{jm}, q_{ji}, q_{mj}, q_{ij}$ and $q'_{bj}, q'_{ja} \in \mathbb{N}$ such that

$$\frac{kq_{jm}+k-1}{kq_{ji}+1} \geq \frac{s_j - s_m}{s_j - s_i} \geq \frac{kq_{jm}+1}{kq_{ji}+k-1} \text{ for all } 1 \leq i, m \leq j-1, i < m, \tag{4}$$

$$\frac{kq_{mj}+k-1}{kq_{ij}+1} \geq \frac{s_m - s_j}{s_i - s_j} \geq \frac{kq_{mj}+1}{kq_{ij}+k-1} \text{ for all } j+1 \leq i, m \leq k-1, i < m, \tag{5}$$

$$\frac{kq'_{bj}+k-1}{kq'_{ja}+1} \geq \frac{s_b - s_j}{s_j - s_a} \geq \frac{kq'_{bj}+1}{kq'_{ja}+k-1} \text{ for all } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k \tag{6}$$

Intermediate runner $j = k-1$. There exists $m'_{k-1j}, m'_{kk-1} \in \mathbb{N}$ such that

$$\frac{km'_{k-1j}+k-1}{km'_{k-1i}+1} \geq \frac{s_{k-1} - s_j}{s_{k-1} - s_i} \geq \frac{km'_{k-1i}+1}{km'_{k-1i}+k-1} \tag{7}$$

for $3 \leq j, i \leq k$ and $j > i$ and

$$\frac{km_{kk-1}+k-1}{km_{k-1a}+1} \geq \frac{s_k - s_{k-1}}{s_{k-1} - s_a} \geq \frac{km_{kk-1}+1}{km_{k-1a}+k-1} \tag{8}$$

for $1 \leq a \leq k-2$.

Fastest runner. There exists $r_1, r_2, \ldots r_{k-1} \in \mathbb{N}$ such that

$$\frac{kr_m+k-1}{kr_i+1} \geq \frac{s_k - s_m}{s_k - s_i} \geq \frac{kr_m+1}{kr_i+k-1} \tag{9}$$

for all $1 \leq i, m \leq k-1, m > i$.

Proof. See [3]. □

While these inequalities are very complicated their meaning can be summerized in a relatively straightfoward way. Inequality (1) means there is some time T_1 such that the slowest runner becomes lonely. Inequalities (2) and (3) means that there is some time T_2 such that an intermediate runner $j = 2$ becomes lonely from the runners $1, 3, \dots k-1, k$, and so on. We now prove the following lemma.

Lemma 2. Let $k \geq z \geq 2$ where $z \in \mathbb{R}$ and let $1 \leq x_m \leq z-1$ and $x \in \mathbb{R}$. Then

$$\frac{kn_m + k - 1}{kn_i + 1} \geq \frac{zn_m + x_m}{zn_i + x_m} \geq \frac{kn_m + 1}{kn_i + k - 1} \tag{10}$$

where $n_m \in \mathbb{N}$.

Proof. Since $k \geq z \geq 2$ it follows that

$$(k - z)(n_m + n_i + 1) \geq 0. \tag{11}$$

With algebra it follows that

$$\frac{zn_m + 1}{zn_i + z - 1} \geq \frac{kn_m + 1}{kn_i + k - 1}. \tag{12}$$

Also, with algebra it follows that

$$\frac{kn_m + k - 1}{kn_i + 1} \geq \frac{zn_m + z - 1}{zn_i + 1}. \tag{13}$$

Hence,

$$\frac{kn_m + k - 1}{kn_i + 1} \geq \frac{zn_m + z - 1}{zn_i + 1} > \frac{zn_m + 1}{zn_i + z - 1} \geq \frac{kn_m + 1}{kn_i + k - 1}. \tag{14}$$

Since $1 \leq x_m \leq z - 1$, it follows that

$$\frac{zn_m + z - 1}{zn_i + 1} \geq \frac{zn_m + x_m}{zn_i + x_m} \geq \frac{zn_m + 1}{zn_i + z - 1}. \tag{15}$$

Therefore

$$\frac{kn_m + k - 1}{kn_i + 1} \geq \frac{zn_m + x_m}{zn_i + x_m} \geq \frac{kn_m + 1}{kn_i + k - 1}. \square \tag{16}$$

We now prove (Theorem 1) that if the LRC is true for some speeds s_i it must be true for an affine transformation $as_i + b$ where $a > 0, b \geq 0$. These facts are easy to see using Lemma 1.

Theorem 1. Let $k \geq 5$ and let $s_i < s_j$ where $i < j$ and $s_i > 0$. Suppose these speeds s_i satisfy the LRC. Then the LRC is true for speeds $as_i + b$.

Proof. Note that for any $s_1, s_2, s_3, s_4 \in \mathbb{R}$ and any $a > 0, b \geq 0$

$$\frac{s_4 - s_3}{s_2 - s_1} = \frac{(as_4 + b) - (as_3 + b)}{(as_2 + b) - (as_1 + b)}. \tag{17}$$

Hence if the LRC is true for speeds s_i it must also be true for speeds $as_i + b$ by Lemma 1. □

We prove a more interesting result next.

Theorem 2. Let $k \geq 5$ and $n_m \in \mathbb{N}$. Let $s_m = kn_m + m$ for $1 \leq m \leq k$ where $n_m \geq n_i$ for $m \geq i$. Then the LRC is true for these speeds.

Proof. *Slowest runner.* Since $s_m = kn_m + m$, it follows that

$$\frac{s_m - s_1}{s_i - s_1} = \frac{k(n_m - n_1) + m - 1}{k(n_i - n_1) + i - 1} = \frac{kn'_m + m - 1}{kn'_i + i - 1} \tag{18}$$

for $2 \leq m, i \leq k, m > i$. Hence there exists some $n''_m \in \mathbb{N}$ such that

$$\frac{kn''_{m-1} + k - 1}{kn''_{i-1} + 1} \geq \frac{kn'_m + m - 1}{kn'_i + i - 1} \geq \frac{kn''_{m-1} + 1}{kn''_{i-1} + k - 1} \tag{19}$$

for $2 \leq m, i \leq k, m > i$. Then by Lemma 1, the slowest runner becomes lonely.

Intermediate runner $j = 2$. It follows that

$$\frac{s_j - s_2}{s_i - s_2} = \frac{k(n_j - n_2) + j - 2}{k(n_i - n_2) + i - 2)} = \frac{km_{j2} + j - 2}{km_{i2} + i - 2)} \tag{20}$$

for $3 \leq j, i \leq k$ and $j > i$ and

$$\frac{s_i - s_2}{s_2 - s_1} = \frac{k(n_i - n_2) + i - 2}{k(n_2 - n_1) + 1} = \frac{km_{i2} + i - 2}{km_{21} + 1} \tag{21}$$

for $3 \leq i \leq k$. Hence there exists $m_{j2}, m_{21} \in \mathbb{N}$ such that

$$\frac{km_{j2} + k - 1}{km_{i2} + 1} \geq \frac{s_j - s_2}{s_i - s_2} \geq \frac{km_{j2} + 1}{km_{i2} + k - 1} \tag{22}$$

for $3 \leq j, i \leq k$ and $j > i$ and

$$\frac{km_{i2} + k - 1}{km_{21} + 1} \geq \frac{s_i - s_2}{s_2 - s_1} \geq \frac{km_{i2} + 1}{km_{21} + k - 1} \tag{23}$$

for $3 \leq i \leq k$. Hence by Lemma 1 runner $j = 2$ becomes lonely.

Intermediate runner $3 \leq j \leq k - 2$. We have that

$$\frac{s_j - s_m}{s_j - s_i} = \frac{k(n_j - n_m) + j - m}{k(n_j - n_i) + j - i} = \frac{kq_{jm} + j - m}{kq_{ji} + j - i} \tag{24}$$

for all $1 \leq i, m \leq j - 1, i < m$ and

$$\frac{s_m - s_j}{s_i - s_j} = \frac{k(n_m - n_j) + m - j}{k(n_i - n_j) + i - j} = \frac{kq_{mj} + m - j}{kq_{ij} + i - j} \tag{25}$$

for all $j + 1 \leq i, m \leq k - 1, i < m$. Also,

$$\frac{s_b - s_j}{s_j - s_a} = \frac{k(n_b - n_j) + b - j}{k(n_j - n_a) + j - a} = \frac{kq'_{bj} + b - j}{kq'_{ja} + j - a} \tag{26}$$

for all $1 \leq a \leq j-1$ and $j+1 \leq b \leq k$. Hence, there exists $q_{jm}, q_{ji} \in \mathbb{N}$ such that

$$\frac{kq_{jm} + k - 1}{kq_{ji} + 1} \geq \frac{s_j - s_m}{s_j - s_i} \geq \frac{kq_{jm} + 1}{kq_{ji} + k - 1} \tag{27}$$

for all $1 \leq i, m \leq j - 1, i < m$ and there exists $q_{mj}, q_{ij} \in \mathbb{N}$ such that

$$\frac{kq_{mj} + k - 1}{kq_{ij} + 1} \geq \frac{s_m - s_j}{s_i - s_j} \geq \frac{kq_{mj} + 1}{kq_{ij} + k - 1} \tag{28}$$

for all $j + 1 \leq i, m \leq k, i < m$. Finally, there exists $q'_{bj}, q'_{ja} \in \mathbb{N}$ such that

$$\frac{kq'_{bj} + k - 1}{kq'_{ja} + 1} \geq \frac{s_b - s_j}{s_j - s_a} \geq \frac{kq'_{bj} + 1}{kq'_{ja} + k - 1} \tag{29}$$

for all $1 \leq a \leq j-1$ and $j+1 \leq b \leq k$. Hence, by Lemma 1, the intermediate runner becomes lonely.

Intermediate runner $j = k - 1$. Similar to the case for $j = 2$.

Fastest runner. Similar to the case for the slowest runner. □

Example 2. Let $k = 10$ and $n_i =$ i-th prime number p_i. Then for speeds $s_i = 10p_i + i$ the LRC is true.

We can now combine Theorems 1 and 2 together to prove the following corollary.

Corollary 1. Let $k \geq 5$ and $n_m \in \mathbb{N}$. Let $s_m = kn_m + m$ for $1 \leq m \leq k$. Then if $a > 0$ and $b \geq 0$ the LRC is true for speeds $as_m + b$.

Proof. Follows from Theorems 1 and 2. □

Example 3. Let $k = 30$ and $n_i = i^2$. Then if $a = \pi$ and $b = e$, for speeds $s_i = 30\pi i^2 + \pi i + e$ the LRC is true.

We now prove the main result of the paper: Given any set of speeds and any number of runners with $k \geq 5$, the LRC is true for a set of speeds that comes arbitrarily close to these speeds.

Theorem 3. Consider any set of speeds $s_1 < s_2 < \cdots < s_k$ for $k \geq 5$ and let $s'_m = c(kn_m + m)$ where $c > 0$ and $n_m \geq n_i$ for $m \geq i$. Then for any $\epsilon > 0$ there exists $c > 0$ and $n_m \in \mathbb{N}$ such that

$$|s_m - s'_m| < \epsilon. \tag{30}$$

That is, for any sequence of speeds $s_1 < s_2 < \cdots < s_k$ there exists a sequence of speeds $s'_1 < s'_2 < \cdots < s'_k$ that is arbitrarily close to the original set of speeds and for which the LRC is true.

Proof. Take $\epsilon > 0$ and choose

$$\left\{ \min_{1 \leq i \leq k} \frac{s_i + \epsilon}{i}, \frac{2\epsilon}{k}, \min_{1 \leq i \leq k-1} s_{i+1} - s_i \right\} \geq c. \tag{31}$$

Then

$$\left\{ \frac{s_i + \epsilon}{i}, \frac{2\epsilon}{k} \right\} \geq c \tag{32}$$

for $1 \leq i \leq k$. Since $\frac{2\epsilon}{k} \geq c$ it follows that

$$\frac{s_i + \epsilon - ci}{ck} - \frac{s_i - \epsilon - ci}{ck} \geq 1. \tag{33}$$

Also, since $\frac{s_i + \epsilon}{i} \geq c$ this implies that

$$\frac{s_i + \epsilon - ci}{ck} \geq 0 \tag{34}$$

for $1 \leq i \leq k$. Hence

$$\frac{s_i + \epsilon - ci}{ck} - \frac{s_i - \epsilon - ci}{ck} \geq 1, \tag{35}$$

$$\frac{s_i + \epsilon - ci}{ck} \geq 0. \tag{36}$$

$$\tag{37}$$

From the two inequalities above it follows that there exists $n_i \in \mathbb{N}$ such that

$$\frac{s_i + \epsilon - ci}{ck} > n_i > \frac{s_i - \epsilon - ci}{ck} \tag{38}$$

for $1 \leq i \leq k$. Since

$$\min_{1\leq i\leq k-1} s_{i+1} - s_i \geq c \tag{39}$$

it follows that

$$\frac{s_{i+1} + \epsilon - c(i+1)}{ck} \geq \frac{s_i + \epsilon - ci}{ck}, \tag{40}$$

$$\frac{s_{i+1} - \epsilon - c(i+1)}{ck} \geq \frac{s_i - \epsilon - ci}{ck}. \tag{41}$$

$$\tag{42}$$

These conditions ensure that $n_j \geq n_i$ for $j \geq i$. We have that

$$s_i + \epsilon > ckn_i + ci > s_i - \epsilon \tag{43}$$

for $1 \leq i \leq k$. Therefore

$$|s_i - c(kn_i + i)| < \epsilon. \tag{44}$$

Since $s'_i = c(kn_i + i)$ the statement of the theorem follows. □

Example 4. Let $k = 10$ and take $s_m = \pi^m$ and $\epsilon = 0.01$. Then

$$\min_{1\leq i\leq 10} \frac{s_i + \epsilon}{i} = \frac{100\pi + 1}{100}, \tag{45}$$

$$\frac{2\epsilon}{k} = \frac{1}{500}, \tag{46}$$

$$\min_{1\leq i\leq 9} s_{i+1} - s_i = \pi(\pi - 1). \tag{47}$$

Now, choose $c = \frac{1}{1000}$. Then there exists $n_m \in \mathbb{N}$ such that

$$100\pi^m + 1 - \frac{m}{10} > n_m > 100\pi^m - 1 - \frac{m}{10} \tag{48}$$

and $n_j \geq n_i$ for $j \geq i$ since $\pi(\pi - 1) \geq c$. Using a computer these integers are found to be

$$
\begin{aligned}
n_1 &= 314,\\
n_2 &= 986,\\
n_3 &= 3100,\\
n_4 &= 9740,\\
n_5 &= 30601,\\
n_6 &= 96138,\\
n_7 &= 302028,\\
n_8 &= 948852,\\
n_9 &= 2980909,\\
n_{10} &= 9364803.
\end{aligned}
\tag{49}
$$

Hence

$$|s_m - s'_m| = |\pi^m - \frac{(10n_m + m)}{1000}| < \epsilon. \tag{50}$$

Note also that the LRC is true for speeds $s'_m = c(kn_m + m) = \frac{(10n_m+m)}{1000}$.

We will now prove some statements similar to Theorem 3.

Theorem 4. Consider any set of speeds $s_1 < s_2 < \cdots < s_k$ for $k \geq 5$ and let $s'_m = c(kn_m + m)$ where $c > 0$ and $n_m \geq n_i$ for $m \geq i$. Then for any $\epsilon > 0$ there exists $c > 0$ and $n_m \in \mathbb{N}$ such that

$$0 < s_m - s'_m < \epsilon. \tag{51}$$

That is, there is a set of speeds $s'_m < s_m$ that comes arbitarily close to s_m for which the LRC is true.

Proof. Take $\epsilon > 0$ and choose

$$\left\{ \min_{1\leq i\leq k} \frac{s_i}{i}, \frac{\epsilon}{k}, \min_{1\leq i\leq k-1} s_{i+1} - s_i \right\} \geq c. \tag{52}$$

Hence

$$\frac{s_i}{i} \geq c \tag{53}$$

for $1 \leq i \leq k$ and thus

$$\frac{s_i - ci}{ck} \geq 0 \tag{54}$$

for $1 \leq i \leq k$. Since

$$\frac{\epsilon}{k} \geq c \tag{55}$$

it follows that

$$\frac{s_i - ci}{ck} - \frac{s_i - \epsilon - ci}{ck} \geq 1. \tag{56}$$

Hence there exists $n_i \in \mathbb{N}$ such that

$$\frac{s_i - ci}{ck} > n_i > \frac{s_i - \epsilon - ci}{ck}. \tag{57}$$

Since

$$\min_{1 \leq i \leq k-1} s_{i+1} - s_i \geq c \tag{58}$$

it follows that

$$\frac{s_{i+1} - c(i+1)}{ck} \geq \frac{s_i - ci}{ck}, \tag{59}$$

$$\frac{s_{i+1} - \epsilon - c(i+1)}{ck} \geq \frac{s_i - \epsilon - ci}{ck}. \tag{60}$$

(61)

Hence we can choose $n_{i+i} \geq n_i$ and hence $n_j \geq n_i$ for $j \geq i$. We have that

$$0 < s_i - (ckn_i + ci) < \epsilon \tag{62}$$

and the statement of the theorem follows. □

Theorem 5. Consider any set of speeds $s_1 < s_2 < \cdots < s_k$ for $k \geq 5$ and let $s'_m = c(kn_m + m)$ where $c > 0$ and $n_m \geq n_i$ for $m \geq i$. Then for any $\epsilon > 0$ there exists $c > 0$ and $n_m \in \mathbb{N}$ such that

$$0 < s'_m - s_m < \epsilon. \tag{63}$$

That is, there is a set of speeds $s_m < s'_m$ that comes arbitarily close to s_m for which the LRC is true.

Proof. Identical to the proof of Theorem 4. □

We will now prove some theorems concerning the fastest and slowest runners.

Theorem 6. Let $k \geq z \geq 2$ where $z \in \mathbb{R}$ and let $1 \leq x_m \leq z - 1$ and $x_m \in \mathbb{R}$. Suppose that $s_m = zn_m + x_m + s_1$ where s_1 is the speed of the slowest runner and $n_m \in \mathbb{N}$. Then the slowest runner becomes lonely.

Proof. It follows that

$$\frac{s_m - s_1}{s_i - s_1} = \frac{zn_m + x_m}{zn_i + x_i} \tag{64}$$

for $2 \le m, i \le k, m > i$. By Lemma 2

$$\frac{kn_m + k - 1}{kn_i + 1} \ge \frac{s_m - s_1}{s_i - s_1} \ge \frac{kn_m + 1}{kn_i + k - 1} \tag{65}$$

for $2 \le m, i \le k, m > i$. Set $n'_{m-1} = n_m$. Then

$$\frac{kn'_{m-1} + k - 1}{kn'_{i-1} + 1} \ge \frac{s_m - s_1}{s_i - s_1} \ge \frac{kn'_{m-1} + 1}{kn'_{i-1} + k - 1}. \tag{66}$$

Hence the slowest runner becomes lonely by Lemma 1. □

Corollary 2. Suppose speeds $s_m = zn_m + x_m + s_1$ satisfy the conditions of Theorem 3. Let $a > 0$ and $b \ge 0$. Then for speeds $as_m + b$ the slowest runner becomes lonely.

Proof. Follows from Theorems 1 and 4. □

Example 5. Let $k = 4$, $z = 3$, and $n_m = m$. Let $(x_2, x_3, x_4) = (1, 1.5, 2)$ and let s_1 be the speed of the slowest runner. Then for speeds $(s_1, s_2, s_3, s_4) = (s_1, 7 + s_1, 10.5 + s_1, 14 + s_1)$ the slowest runner becomes lonely.

Theorem 7. Let $k \ge z \ge 2$ where $z \in \mathbb{R}$ and let $1 \le x_m \le z - 1$ and $x_m \in \mathbb{R}$. Suppose that $s_m = s_k - (zn_m + x_m)$ where s_k is the speed of the fastest runner and $n_m \in \mathbb{N}$. Then the fastest runner becomes lonely.

Proof. Identical to proof of Theorem 5. □

Example 6. Let $k = 10, z = 10, n_m = m^3$ and let $x_m = \frac{m+1}{m}$. Then for $s_m = s_k - (10m^3 + \frac{m+1}{m})$ the fastest runner s_k becomes lonely.

Corollary 3. Let $s_m = s_k - (zn_m + x_m)$ as in Theorem 6. Then if $a > 0$ and $b \ge 0$ the fastest runner becomes lonely for speeds $as_m + b$.

Proof. Follows from Theorem 1 and 6. □

The following theorem uses a result proved by Dubickas in [1].

Theorem 8. Let $k \ge 16341$ and let $a > 0$ and $b \ge 0$. Suppose that

$$s_m \ge s_1 + (m - 1)\left(1 + \frac{33 \log k}{k}\right) \tag{67}$$

Then the LRC is true for speeds $as_m + b$.

Proof. In [1] Dubickas proves that for $k \geq 16341$ and

$$s_{m+1} - s_m \geq 1 + \frac{33 \log k}{k} \tag{68}$$

the LRC is true. Hence the LRC is true for speeds such that

$$s_m \geq s_1 + (m-1)\left(1 + \frac{33 \log k}{k}\right). \tag{69}$$

From Theorem 1 the result follows. □

Example 7. Let $k = 20000$, a $= \frac{1}{e}$ and $b = \pi$. Let

$$s_m = s_1 + \frac{(m-1)}{e}\left(1 + \frac{33 \log 20000}{20000}\right) + \pi. \tag{70}$$

Then the LRC is true for these speeds.

3 Measure Theorems

Our goal in this section is to prove that the slowest and fastest runners become lonely for a set of speeds with infinite measure. To do this we'll first prove the following lemma.

Lemma 3. Consider a parallelotope P in $\mathbb{R}^k$ defined by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} s_1 \\ s_1 + 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ x_3 - 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ x_4 - x_2 \\ \vdots \\ 0 \end{bmatrix}, \ldots \mathbf{v}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ k - 1 - x_{k-1} \end{bmatrix}$$

where $s_1 > 0$ and $1 \leq x_1 < x_2 < \cdots < x_k < k - 1$. Let Vol$(P)$ denote the volume of P. Then Vol$(P) = s_1(x_3 - 1)(x_4 - x_2)\ldots(k - 1 - x_{k-1})$.

Proof. As proven in [2] the volume Vol(P) of a parallelotope defined by vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ is

$$\text{Vol}(P) = \text{Vol}(\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n) = |\det[\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n]|. \tag{71}$$

Hence for the vectors that define P we have that

$$\mathrm{Vol}(P) = \det \begin{bmatrix} s_1 & 0 & 0 & \dots & 0 \\ s_1+1 & x_3-1 & 0 & \dots & 0 \\ 0 & 0 & x_4-x_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & k-1-x_{k-1}. \end{bmatrix} \quad (72)$$

It is easily seen that this determinant is the product of the diagonal entries, hence $\mathrm{Vol}(P) = s_1(x_3-1)(x_4-x_2)\dots(k-1-x_{k-1})$. □

Theorem 9. Suppose speeds $s_m = zn_m + x_m + s_1$ satisfy the conditions of Theorem 3. Then these speeds have infinite measure, and hence for $k \geq 5$ the slowest runner becomes lonely for a set of speeds with infinite measure.

Proof. Define the set S as

$$S = \{zn_m + x_m + s_1 \mid 1 \leq x_m \leq k-1, 2 \leq z \leq k, z \in \mathbb{R}, n_m \in \mathbb{N}, s_1 \in \mathbb{R}\}. \quad (73)$$

Now, consider the subset $S_0 \subset S$

$$S_0 = \{x_m + s_1 \mid 1 \leq x_2 < x_3 < \dots < x_k < k-1, s_1 \in \mathbb{R}\} \quad (74)$$

where we set all the $n_m = 0$ and $s_m = x_m + s_1$. Note that if we set $n_m = 0$ we must take $1 \leq x_2 < x_3 < \dots < x_k < k-1$ so that $s_1 < s_2 < \dots < s_k$. Hence we are choosing

$$s_1 = s_1, \quad (75)$$
$$s_1 + 1 \leq s_2 < s_1 + x_3, \quad (76)$$
$$s_1 + x_2 < s_3 < s_1 + x_4, \quad (77)$$
$$\vdots$$
$$s_1 + x_{k-1} < s_k < s_1 + k - 1. \quad (78)$$

This describes a parallelotope in $\mathbb{R}^k$ defined by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} s_1 \\ s_1+1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ x_3-1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ x_4-x_2 \\ \vdots \\ 0 \end{bmatrix}, \dots \mathbf{v}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ k-1-x_{k-1} \end{bmatrix}.$$

By Lemma 3 the volume of this parallelotope is

$$\text{Vol}(P) = s_1(x_3 - 1)(x_4 - x_2)\dots(k - 1 - x_{k-1}). \tag{79}$$

Hence as $s_1 \to \infty$ it follows that

$$\lim_{s_1\to\infty} \text{Vol}(P) = \lim_{s_1\to\infty} s_1(x_3 - 1)(x_4 - x_2)\dots(k - 1 - x_{k-1}) = \infty. \tag{80}$$

Hence, as $s_1 \to \infty$ the volume of the parallelotope increases without bound and hence has infinite measure. Therefore $\mu(S_0) = \infty$ where $\mu(S_0)$ is the Lebesgue measure of S_0. Since $S_0 \subset S$ it follows that $\mu(S_0) \leq \mu(S)$ and therefore $\mu(S) = \infty$. □

Corollary 4. Suppose speeds $s_m = zn_m + x_m + s_1$ satisfy the conditions of Theorem 3. Let $a > 0$ and $b \geq 0$. Then the set of speeds $as_m + b$ have infinite measure. That is, under affine transformations the the set of speeds s_m have infnite measure.

Proof. Define

$$S' = \{as_m + b \mid s_m \in S, a > 0, b \geq 0\}. \tag{81}$$

and

$$S'_0 = \{as_m + b \mid s_m \in S_0, a > 0, b \geq 0\}. \tag{82}$$

Note that when we take the transformation $s_m \to s'_m = as_m + b$ we have that

$$s'_1 = s'_1, \tag{83}$$
$$a(s_1 + 1) + b \leq s'_2 < a(s_1 + x_3) + b, \tag{84}$$
$$a(s_1 + x_2) + b < s'_3 < a(s_1 + x_4) + b, \tag{85}$$
$$\vdots$$
$$a(s_1 + x_{k-1}) + b < s'_k < a(s_1 + k - 1) + b. \tag{86}$$

This describes a parallelotope P' defined by vectors

$$\mathbf{v}'_1 = \begin{bmatrix} as_1 + b \\ a(s_1 + 1) + b \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v}'_2 = \begin{bmatrix} 0 \\ a(x_3 - 1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v}'_3 = \begin{bmatrix} 0 \\ 0 \\ a(x_4 - x_2) \\ \vdots \\ 0 \end{bmatrix}, \dots \mathbf{v}'_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a(k - 1 - x_{k-1}) \end{bmatrix}.$$

the vectors $\mathbf{v}_m$ defined in Lemma 3 are transformed to $a\mathbf{v}_m + \mathbf{b}_m$. These transformed vectors then define a transformed parallelotope P'. Hence

$$\begin{aligned} \text{Vol}(P') &= \text{Vol}(a\mathbf{v}_1 + \mathbf{b}_1, a\mathbf{v}_2, \dots a\mathbf{v}_k) \\ &= |\det[a\mathbf{v}_1 + \mathbf{b}_1, a\mathbf{v}_2, \dots a\mathbf{v}_k]|. \end{aligned} \tag{87}$$

It is easy to see that

$$\text{Vol}(P') = (as_1 + b)(a[x_3 - 1])(a[x_4 - x_2]) \dots (a[k - 1 - x_{k-1}]). \tag{88}$$

Hence as $s_1 \to \infty$ it follows that

$$\lim_{s_1 \to \infty} \text{Vol}(P') = \infty. \tag{89}$$

Since $S'_0 \subset S'$ it follows that $\mu(S'_0) \leq \mu(S')$ and therefore $\mu(S') = \infty$. □

Theorem 10. Suppose speeds $s_m = s_k - (zn_m + x_m)$ satisfy the conditions of Theorem 6. Then these speeds have infinite measure and hence for $k \geq 5$ the fastest runner becomes lonely for a set of speeds with infinite measure.

Proof. Similar to the argument in Theorem 9. □

Theorem 11. Suppose speeds s_m satisfy the conditions of Theorem 6. Let $a > 0$ and $b \geq 0$. Then speeds $as_m + b$ have infinite measure. That is, under affine transformations the set of speeds s_m have infinite measure.

Proof. Similar to the argument in Corollary 4. □

References

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