Covering Problems That Imply the Lonely Runner Conjecture

DVR

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Abstract

We explore a covering problem that implies the lonely runner conjecture. We then prove that this covering has infinite measure.

1 Introduction

The lonely runner conjecture (LRC) is a very deep problem in the theory of Diophantine approximations that asks the following question: Given a circular track and k runners starting from a given starting line with speeds $s_1 < s_2 < \ldots s_k$, is it true that each runner becomes lonely at some time, i.e. seperated from the other runners by a distance of at least $\frac{1}{k}$? While easy to state, this conjecture is remarkably difficult and has eluded proof since it was proposed by T.W. Cusick in 1967. In this paper we will consider some covering problems that imply the LRC. We'll then prove that these coverings have infinite measure.

2 Covering Problems That Imply the LRC

To begin with, we'll define the sequence d_k and the sets I_k, C_{d_k} and E_{d_k}.

Definition 1. Let $k \geq 3$. Then define d_k as

$$d_k = \frac{(k-1)(k-2)}{2}. \tag{1}$$

Let $a, b \ldots c, d \in \mathbb{N}$. Define $I_k(a,b)$ as

$$I_k(a,b) = \left[\frac{ka+1}{kb+k-1}, \frac{ka+k-1}{kb+1}\right] \tag{2}$$

and C_{d_k} as

$$C_{d_k}(a,b,\ldots c) = I_k(a,b) \times \cdots \times I_k(c,d). \tag{3}$$

Define E_{d_k} as

$$E_{d_k}(a,b,\ldots c) = I_3(a,b) \times \cdots \times I_3(c,d). \tag{4}$$

Example. Let $k = 4$ and let $n_1, n_2, n_3 \in \mathbb{N}$. Then

$$C_{d_4}(n_2, n_1, n_3, n_2, n_3, n_1) = I_4(n_2, n_1) \times I_4(n_3, n_2) \times I_4(n_3, n_1) \tag{5}$$
$$= \left[\frac{4n_2+1}{4n_1+3}, \frac{4n_2+3}{4n_1+1}\right] \times \left[\frac{4n_3+1}{4n_2+3}, \frac{4n_3+3}{4n_2+1}\right] \times \left[\frac{4n_3+1}{4n_1+3}, \frac{4n_3+3}{4n_1+1}\right].$$

and

$$E_{d_4}(n_2, n_1, n_3, n_2, n_3, n_1) = I_3(n_2, n_1) \times I_3(n_3, n_2) \times I_3(n_3, n_1) \tag{6}$$
$$= \left[\frac{3n_2+1}{3n_1+2}, \frac{3n_2+2}{3n_1+1}\right] \times \left[\frac{3n_3+1}{3n_2+2}, \frac{3n_3+2}{3n_2+1}\right] \times \left[\frac{3n_3+1}{3n_1+2}, \frac{3n_3+2}{3n_1+1}\right].$$

Theorem 1. Let $k \geq 3$. Then the LRC is true if

$$\bigcup_{n_1, n_2, \ldots n_{k-1}=0}^{\infty} C_{d_k}(n_2, n_1, n_3, n_2 \ldots n_{k-1}, n_{k-2}) = \mathbb{R}_+{}^{d_k}. \tag{7}$$

Proof. Slowest runner. Let $\mathbf{x} \in \cup C_{d_k}$. Then $\exists\ n_2, n_3 \ldots n_k \in \mathbb{N}$ such that

$$\frac{kn_{m-1}+k-1}{kn_{i-1}+1} \geq x_j \geq \frac{kn_{m-1}+1}{kn_{i-1}+k-1} \tag{8}$$

where $2 \leq m, i \leq k, m > i$. Hence for $s_k > \cdots > s_3 > s_2 > s_1\ \exists$ $n_2, n_3 \ldots n_k \in \mathbb{N}$ such that

$$\frac{kn_{m-1}+k-1}{kn_{i-1}+1} \geq \frac{s_m - s_1}{s_i - s_1} \geq \frac{kn_{m-1}+1}{kn_{i-1}+k-1} \tag{9}$$

where $2 \leq m, i \leq k, m > i$. By [1] this implies the slowest runner becomes lonely.

Intermediate runner. Let $\mathbf{x} \in \cup C_{d_k}$. Then there exists exists $q_1, q_2, \ldots q_{k-1} \in \mathbb{N}$ such that

$$\frac{kq_m+k-1}{kq_i+1} \geq \frac{s_j - s_m}{s_j - s_i} \geq \frac{kq_m+1}{kq_i+k-1} \text{ for } 1 \leq i, m \leq j-1, i < m, \tag{10}$$
$$\frac{kq_m+k-1}{kq_i+1} \geq \frac{s_m - s_j}{s_i - s_j} \geq \frac{kq_m+1}{kq_i+k-1} \text{ for } j+1 \leq i, m \leq k-1, i < m, \tag{11}$$
$$\frac{kq_b+k-1}{kq_a+1} \geq \frac{s_b - s_j}{s_j - s_a} \geq \frac{kq_b+1}{kq_a+k-1} \text{ for } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k-1 \tag{12}$$

Hence by [1] the intermediate runner becomes lonely.

Fastest runner. Let $\mathbf{x} \in \cup C_{d_k}$. Then $\exists\ r_1, r_2 \ldots r_{k-1} \in \mathbb{N}$ such that

$$\frac{kr_m + k - 1}{kr_i + 1} \geq x_j \geq \frac{kr_m + 1}{kr_i + k - 1} \tag{13}$$

where $1 \leq m, i \leq k-1, m > i$. Therefore for $s_k > \cdots > s_3 > s_2 > s_1$ $\exists\ r_1, r_2 \ldots r_{k-1} \in \mathbb{N}$ such that

$$\frac{kr_m + k - 1}{kr_i + 1} \geq \frac{s_k - s_m}{s_k - s_i} \geq \frac{kr_m + 1}{kr_i + k - 1}. \tag{14}$$

where $1 \leq m, i \leq k-1, m > i$. By [1] the fastest runner becomes lonely. $\square$

Lemma 1. Let $k \geq 3$. Then

$$\bigcup_{n_1, n_2, \ldots n_{k-1}=1}^{\infty} E_{d_k}(n_2, n_1, n_3, n_2 \ldots n_{k-1}, n_{k-2}) \subset \tag{15}$$
$$\bigcup_{n_1, n_2, \ldots n_{k-1}=0}^{\infty} C_{d_k}(n_2, n_1, n_3, n_2 \ldots n_{k-1}, n_{k-2}).$$

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots x_{d_k})$ and let $\mathbf{x} \in \cup E_{d_k}$. Then $\exists\, n_1, n_2 \ldots n_{k-1} \in \mathbb{N}$ such that

$$\frac{3n_m + 2}{3n_i + 1} \geq x_j \geq \frac{3n_m + 1}{3n_i + 2} \tag{16}$$

where $1 \leq m, i \leq k-1, m > i$. Since $3(n_m + n_i + 1) \leq k(n_m + n_i + 1)$ for $k \geq 3$ it follows with algebra that

$$\frac{3n_m + 1}{3n_i + 2} \geq \frac{kn_m + 1}{kn_i + k - 1}. \tag{17}$$

Likewise, it follows that

$$\frac{kn_m + k - 1}{kn_i + 1} \geq \frac{3n_m + 2}{3n_i + 1}. \tag{18}$$

Hence

$$\frac{kn_m + k - 1}{kn_i + 1} \geq \frac{3n_m + 2}{3n_i + 1} > x_j > \frac{3n_m + 1}{3n_i + 2} \geq \frac{kn_m + 1}{kn_i + k - 1}. \tag{19}$$

Hence $\mathbf{x} \in \cup C_{d_k}$. $\square$

Corollary 1. The LRC is true if

$$\bigcup_{n_1, n_2, \ldots n_{k-1}=1}^{\infty} E_{d_k}(n_2, n_1, n_3, n_2 \ldots n_{k-1}, n_{k-2}) = {\mathbb{R}_+}^{d_k}. \tag{20}$$

Proof. Let ${\mathbb{R}_+}^{d_k} \subset \cup E_{d_k}$. Then by Lemma 1, ${\mathbb{R}_+}^{d_k} \subset \cup C_{d_k}$. Since $\cup C_{d_k} \subset {\mathbb{R}_+}^{d_k}$, it follows that $\cup C_{d_k} = {\mathbb{R}_+}^{d_k}$ which implies the LRC by Theorem 1. □

Projections. We now consider projections of $\cup E_{d_k}$ onto each of its subspaces. It is intuitively clear that if $\cup E_{d_k}$ covers ${\mathbb{R}_+}^{d_k}$ then the projections of $\cup E_{d_k}$ cover each of its subspaces. We define a projection as follows.

Definition 2. Let $S \subset {\mathbb{R}_+}^{d_k}$. Define a projection P_j of S as

$$P_j(S) = \{\mathbf{x} \in S \mid \mathbf{x} = (x_1, \dots x_{j-1}, 0, x_{j+1}, \dots x_{d_k})\}. \tag{21}$$

Corollary 2. Let $k > 3$. Then if $\cup E_{d_k} = {\mathbb{R}_+}^{d_k}$, the LRC must be true for j runners for any $3 \le j < k$.

Proof. Since $\cup E_{d_k} = {\mathbb{R}_+}^{d_k}$, it follows that for $1 \le m \le d_k$

$$P_m P_{m+1} \dots P_{d_k}(\cup E_{d_k}) = \{\mathbf{x} \in \cup E_{d_k} \mid \mathbf{x} = (x_1, \dots x_{m-1}, 0, 0, \dots 0\} = {\mathbb{R}_+}^{m-1}. \tag{22}$$

Note that

$$P_m P_{m+1} \dots P_{d_k}(\cup E_{d_k}) = \cup P_m P_{m+1} \dots P_{d_k}(E_{d_k}) = {\mathbb{R}_+}^{m-1}. \tag{23}$$

Hence for $1 \le d_j + 1 \le d_k$

$$P_{d_j+1} P_{d_j+2} \dots P_{d_k}(\cup E_{d_k}) = \cup P_{d_j+1} P_{d_j+2} \dots P_{d_k}(E_{d_k}) = {\mathbb{R}_+}^{d_j}. \tag{24}$$

We have that

$$P_{d_j+1} P_{d_j+2} \dots P_{d_k}(E_{d_k}) = \{\mathbf{x} \in E_{d_k} \mid \mathbf{x} = (x_1, \dots x_{d_j}, 0, 0, \dots 0\}. \tag{25}$$

Hence $\exists\ n_1, n_2, \dots n_j$ such that

$$\frac{3n_m + 2}{3n_i + 1} \le x_q \le \frac{3n_m + 1}{3n_i + 2} \tag{26}$$

where $1 \le m, i \le j - 1, m > i$. Therefore, $\exists\ n_1, n_2, \dots n_j$ such that $\mathbf{x} \in E_{d_j}$. Hence

$$P_{d_j+1} P_{d_j+2} \dots P_{d_k}(E_{d_k}) = E_{d_j}. \tag{27}$$

Hence

$$\cup P_{d_j+1} P_{d_j+2} \dots P_{d_k}(E_{d_k}) = \cup E_{d_j} = {\mathbb{R}_+}^{d_j}. \tag{28}$$

Therefore the LRC is true for $3 \le j < k$. □

Example. Let $k = 4$. Then suppose that

$$\bigcup_{n_1,n_2,n_3=0}^{\infty} E_3(n_2,n_1,n_3,n_2,n_3,n_1) = {\mathbb{R}_+}^3. \tag{29}$$

Then

$$\begin{aligned} &P_2P_3\Big(\bigcup_{n_1,n_2,n_3=0}^{\infty} E_3(n_2,n_1,n_3,n_2,n_3,n_1)\Big) \\ &= \bigcup_{n_1,n_2,n_3=0}^{\infty} P_2P_3E_3(n_2,n_1,n_3,n_2,n_3,n_1) \\ &= \bigcup_{n_1,n_2=0}^{\infty} E_1(n_2,n_1) = \mathbb{R}_+. \end{aligned} \tag{30}$$

Hence the LRC is true for $j = 3 < 4$.

3 Measure Theorems

We now prove that $\cup E_{d_k}$ and $\cup C_{d_k}$ have infinite measure. While intuitively this seems obvious, this is somewhat tricky to prove. The trick to doing this is to consider sets of the form

$$I_k(0,0)^{d_{k-1}} \times I_k(0,m_i)^{k-2} \tag{31}$$

and to show that there is an infinite number of disjoint sets of this form all of which have a measure larger than some constant. We first prove the following lemmas.

Lemma 2. Let $k \geq 3$ and consider the set

$$I_k(0,0)^n \times I_k(0,j)^l \tag{32}$$

where n and $j \in \mathbb{N}$. Then if

$$m > \frac{(kj+k-1)(k-1)-1}{k} \tag{33}$$

it follows that

$$I_k(0,0)^n \times I_k(0,j)^l \cap I_k(0,0)^n \times I_k(0,m)^l = \emptyset. \tag{34}$$

Proof. Since

$$m > \frac{(kj+k-1)(k-1)-1}{k} \tag{35}$$

it follows that

$$\frac{km+1}{k-1} > kj+k-1 \tag{36}$$

and hence

$$\left[\frac{kj+1}{k-1}, kj+k-1\right] < \left[\frac{km+1}{k-1}, km+k-1\right]. \tag{37}$$

Hence

$$I_k(0,j) \cap I_k(0,m) = \emptyset. \tag{38}$$

This implies that

$$I_k(0,0)^n \times I_k(0,j)^l \cap I_k(0,0)^n \times I_k(0,m)^l = \emptyset. \quad \square \tag{39}$$

Lemma 3. Let $k \geq 3$, $n \in \mathbb{N}$ and let $\lambda(\mathrm{S})$ be the Lebesgue measure of set S. Then

$$\lambda(I_k(0,0)^n \times I_k(0,j)^l) = \left[\frac{k(k-2)}{k-1}\right]^{n+l} (j+1)^l. \tag{40}$$

Proof. With algebra

$$\lambda(I_k(0,0)^n) = \left[k-1-\frac{1}{k-1}\right]^n = \left[\frac{k(k-2)}{k-1}\right]^n. \tag{41}$$

Also,

$$\lambda(I_k(0,j)^l) = \left[kj+k-1-\frac{kj+1}{k-1}\right]^l = \left[\frac{k(k-2)}{k-1}(j+1)\right]^l. \tag{42}$$

Hence

$$\lambda(I_k(0,0)^n \times I_k(0,j)^l) = \left[\frac{k(k-2)}{k-1}\right]^n \left[\frac{k(k-2)}{k-1}(j+1)\right]^l \tag{43}$$

$$= \left[\frac{k(k-2)}{k-1}\right]^{n+l} (j+1)^l. \quad \square \tag{44}$$

Lemma 4. Let $k \geq 3$. Then

$$\bigcup_{n_{k-1}=0}^{\infty} I_k(0,0)^{d_{k-1}} \times I_k(0,n_{k-1})^{k-2} \subset$$
$$\bigcup_{n_1,n_2,\dots n_{k-1}=0}^{\infty} C_{d_k}(n_2,n_1,n_3,n_2,\dots n_{k-1},n_{k-2}). \tag{45}$$

Proof. Note that

$$\bigcup_{n_{k-1}=0}^{\infty} I_k(0,0)^{d_{k-1}} \times I_k(0,n_{k-1})^{k-2} = \bigcup_{n_{k-1}=0}^{\infty} C_{d_k}(0,0,0,0,\dots 0,n_{k-1}). \tag{46}$$

Also

$$\bigcup_{n_{k-1}=0}^{\infty} C_{d_k}(0,0,0,0,\dots 0,n_{k-1}) \subset \tag{47}$$
$$\bigcup_{n_1,n_2,\dots n_{k-1}=0}^{\infty} C_{d_k}(n_2,n_1,n_3,n_2,\dots n_{k-1},n_{k-2}).$$

Hence Lemma 3 follows. $\square$

Theorem 2. Let $k \geq 3$. Then $\lambda(\cup C_{d_k}) = \infty$.

Proof. Consider the sequence $\{m_i\}_{i=0}^{n}$ where $m_i \in \mathbb{N}$, m_0 is any natural number and

$$m_{i+1} > \frac{(km_i + k - 1)(k-1) - 1}{k}. \tag{48}$$

Then by Lemma 2 it follows that

$$I_k(0,0)^n \times I_k(0,m_i)^l \cap I_k(0,0)^n \times I_k(0,m_{i+1})^l = \emptyset. \tag{49}$$

Now, consider the following union

$$\bigcup_{i=0}^{n} I_k(0,0)^{d_{k-1}} \times I_k(0,m_i)^{k-2}. \tag{50}$$

This is a finite union of disjoint sets with each set having measure

$$\left[\frac{k(k-2)}{k-1}\right]^{d_k} (m_i+1)^{k-2} \tag{51}$$

by Lemma 3. Denote $\sum m_i = M$. Then

$$\lambda\bigg(\bigcup_{i=0}^{n} I_k(0,0)^{d_{k-1}} \times I_k(0,m_i)^{k-2}\bigg) \geq \left[\frac{k(k-2)}{k-1}\right]^{d_k} (M+1). \tag{52}$$

Now as $n \to \infty$ it is clear from the way m_i were chosen that

$$\sum_{i=0}^{\infty} m_i = \infty. \tag{53}$$

Hence,

$$\lambda\bigg(\bigcup_{i=0}^{\infty} I_k(0,0)^{d_{k-1}} \times I_k(0,m_i)^{k-2}\bigg) = \infty. \tag{54}$$

Note that

$$\bigcup_{i=0}^{\infty} I_k(0,0)^{d_{k-1}} \times I_k(0,m_i)^{k-2} \subset \bigcup_{n_{k-1}=0}^{\infty} I_k(0,0)^{d_{k-1}} \times I_k(0,n_{k-1})^{k-2}. \tag{55}$$

Also note that

$$\bigcup_{n_{k-1}=0}^{\infty} I_k(0,0)^{d_{k-1}} \times I_k(0,n_{k-1})^{k-2} \subset \bigcup_{n_1,n_2,\dots n_{k-1}=0}^{\infty} C_{d_k}(n_2,n_1,n_3,n_2\dots n_{k-1},n_{k-2}) \tag{56}$$

by Lemma 4. This implies that

$$\lambda\Bigg(\bigcup_{n_1,n_2,\dots n_{k-1}=0}^{\infty} C_{d_k}(n_2,n_1,n_3,n_2\dots n_{k-1},n_{k-2})\Bigg) = \infty. \quad \square \tag{57}$$

Corollary 3. Let $k \geq 3$. Then $\lambda(\cup E_{d_k}) = \infty$.

Proof. Same argument as for $\cup C_{d_k}$ but with $k = 3$. $\square$

References

[1] Rudisill, David v. *Diophantine Approximation Problems Equivalent to the Lonely Runner Conjecture.*