Relative Phase and Time States

Masataka Ohta
Email: mohta@necom830.hpcl.titech.ac.jp
Department of Computer Science, School of Information Science and Engineering
Tokyo Institute of Technology
2-12-1-W8-54, O-okayama, Meguro, Tokyo 1528552, JAPAN
Sept. 27, 2019

Abstract: It is pointed out that, while Suskind and Glogower claim "The phase operator for **an oscillator** is shown not to exist.", they actually showed absolute phase operator, which is physically meaningless, does not exist. It is also pointed out that, they showed countably infinitely many trigonometric relative (relative between **two oscillators**) phase states, which form complete basis and are, in a sense, best possible quantum "phase states", exist. In addition, continuously infinitely many phase states are constructed at classical, that is, infinitely many quanta, limit. Similarly, relative time states and operator are constructed with **two oscillators** with different angular velocities.

1. Introduction

There are various attempts to construct phase state with single oscillator, for example, by Dirac [1], Suskind and Glogower [2] and Pegg and Barnett [3]. Though they have difficulties (e.g. [2] and [3] criticize difficulties of [1] and [1, 2], respectively), it is not necessary to discuss them in detail, because state constructed is that of absolute phase, which is not an observable and is physically meaningless.

Usually, it is not a problem to represent quantum state including absolute phase by ket vectors of a single oscillator, it is because value represented is invariant under time translation. For example, even if a number state $|n\rangle$, by time translation, becomes $e^{in\theta}|n\rangle$, both states represent the same number value of n. However, as is stated "at time t the state becomes $|\theta - \omega t\rangle$" in [3] as if it were a desirable property, phase value represented by state constructed with a single oscillator changes by time translation from θ to $\theta - \omega t$, which means the value is that of absolute phase, which is not an observable and is physically meaningless.

Though it is stated "phase (defined as the time since the wave was in a standard phase)" [1], we can't evaluate or control quality of implicit "standard phase" and, in [1, 2], it is implicitly assumed that the "standard phase" should have infinite accuracy with no quantum fluctuations constructed only as a classical state involving infinitely many quanta. It makes states related to absolute phase classical states consisting from infinitely many quanta. For example, consider Hermitian trigonometric absolute phase operator of $\cos\hat{\phi}$ introduced in [4]. In [2], its eigenvectors are computed to be (Eq. (18) of [2]):

$$\sum_n \sin(n+1)\theta|n\rangle$$

which are a zero vector (if θ is multiple of π) or vectors with infinite norm and infinite average number (otherwise), none of which are within Hilbert space representing quantum states and non-zero eigenvectors are located at the classical limit. Though, in section III of [2], it is stated "wave packets in cos space spread with time", classical packets do not spread, "they are states for which the uncertainties in the non-commuting operators $\sin\hat{\phi}$ and $\cos\hat{\phi}$ are zero!", it is of course that uncertainties of classical states are zero, and "any quantum oscillator can be used as an arbitrarily accurate clock", it is because the oscillator considered is not quantum but classical.

Approach of [3] is to have less implicit "standard phase" by finite average number of quanta with reasonable (w.r.t. the number) quality and increase the number (and the quality) to the infinity, which may result in accurate results as long as relative phase between an observed and the "standard phase" is considered. However, as what we can actually observe is relative phase between two oscillators both with finite average number of quanta, the approach is not applicable to the reality except as approximation for extreme cases. Though in section VIII "PHASE DIFFERENCES" of [3], it is argued "Our phase difference operator is simply $\hat{\phi}_{\theta_1} - \hat{\phi}_{\theta_1}$, where again the subscripts 1 and 2 refer to the individual modes.", it is obvious that such phase difference involving the "standard phase" is noisier than that considered in section V "Phase Difference of Two Oscillators" of [2]. Actually, "A phase difference measurement can lead to a countably infinite number of results regardless of total excitation number." [3] means a finite number of results for finitely excited states of [2] is broadened by noise.

Relative phase introduced by [2] and Ban [5] does not have such difficulties, though its importance has not been properly recognized, perhaps because difficulties of absolute phase have not been properly understood.

In sections 2, implication that countably infinitely many eigenvectors of trigonometric relative phase operator introduced in section V of [2] form complete basis is discussed. In section 3, continuous phase states are constructed at classical limit. In section 4, relative time states and operator, similar to those of [5], are constructed and is argued that infiniteness is not a problem for them. Section 5 concludes the letter.

2. Eigenvectors of Trigonometric Relative Phase Operator

In [2], using direct product of number state of two oscillators: $|m\rangle|n\rangle$, where $|m\rangle$ and $|n\rangle$ are number state of oscillators 1 and 2, respectively, exponential relative phase operators are defined as $\widehat{\mathrm{e}}^{i(\phi_1-\phi_2)} = \sum_{n\geq 0, m\geq 1} |m-1\rangle\,|n+1\rangle\langle n|\langle m|$ and $\widehat{\mathrm{e}}^{-i(\phi_1-\phi_2)} = \sum_{n\geq 1, m\geq 0} |m+1\rangle\,|n-1\rangle\langle n|\langle m|$. Then, Hermitian trigonometric relative phase operators are defined as

$$\cos(\hat{\phi}_1 - \hat{\phi}_2) = \frac{\hat{e}^{i(\phi_1-\phi_2)} + \hat{e}^{-i(\phi_1-\phi_2)}}{2}$$

$$\sin(\hat{\phi}_1 - \hat{\phi}_2) = \frac{\hat{e}^{i(\phi_1-\phi_2)} - \hat{e}^{-i(\phi_1-\phi_2)}}{2\mathrm{i}}$$

Obviously, all the operators above preserve the total number $R = m + n$ because $\mathrm{m} + \mathrm{n} = (\mathrm{m} - 1) + (\mathrm{n} + 1) = (\mathrm{m} + 1) + (\mathrm{n} - 1)$. As is pointed out in [2], trigonometric relative phase operators commute with total number operator "$\hat{n}_1 + \hat{n}_2$", which means they share common (finite) eigenvectors, which means trigonometric relative phase is, unlike absolute one in [2], a (quantum) observable.

Then, as all the eigenstates of $\cos(\hat{\phi}_1 - \hat{\phi}_2)$, which have zero trigonometric relative phase and zero total number uncertainty, form a complete basis for the Hilbert space representing states of two oscillators, the eigenstates are best possible quantum "phase states". Any state can be represented by linear combination of the eigenvectors and no additional information can be extracted from the state. Moreover, "state with phase difference θ" [2]:

$$|R,\theta\rangle = \sum_{m=0}^{R} \mathrm{e}^{\mathrm{i}m\theta} |m\rangle\, |R-m\rangle \tag{1}$$

which should have little relative phase uncertainty, is shown to be represented by the eigenstates well as "the uncertainty of $\cos\hat{\phi}$ is" [2]:

$$\langle\cos^2\rangle - \langle\cos\rangle^2 = \frac{1-2R}{(R+1)^2}\cos^2\theta$$

the eigenstates are good enough.

A problem of cos operator eigenstates as phase states is "Each $|\cos\theta\rangle$ state can be thought of as a superposition of $|+\theta\rangle$ and $|-\theta\rangle$ states." [2]. That is, we need 1 bit more information to know the phase. In theoretical analysis, we can apply cos and sin operators to the same state to get the information. In practice, we must divide observed state by two, dilute by vacuum (for light, by a half mirror) and apply sin and cos operators to the divided states. As reduction of signal to noise ratio by half by the division means 1 bit of information loss, it is, in a sense, best possible. Ideally sensitive operators rarely commute.

3. Continuous Relative Phase States

Ban [5] defines the continuous relative phase states with two oscillators and infinite number of quanta for the first time, though the states are a little complicated.

In this letter, instead, by making R of Eq. (1) infinitely large, continuous relative phase states simpler than that of [5] is constructed as follows.

By restricting R of Eq. (1) to be even and introducing N and n as $R=2N$ and $m=N+n$, we obtain:

$$|R,\theta\rangle = |2N,\theta\rangle = \mathrm{e}^{iN\theta}\sum_{n=-N}^{N}\mathrm{e}^{in\theta}|N+n\rangle\,|N-n\rangle$$

where absolute phase of $\mathrm{e}^{iN\theta}$ may be ignored. Then:

$$\lim_{N\to\infty}\langle\theta',2N|2N,\theta\rangle = \lim_{N\to\infty}\sum_{n'=-N}^{N}\sum_{n=-N}^{N}\langle N-n'|\langle N+n'|\mathrm{e}^{i(n\theta-n'\theta')}|N+n\rangle|N-n\rangle$$

$$= \lim_{N\to\infty}\sum_{n=-N}^{N}\mathrm{e}^{in(\theta-\theta')} = \sum_{n=-\infty}^{\infty}\delta(\theta-\theta'+2\pi n)$$

Thus, a (classical) state:

$$|\theta\rangle = \lim_{N\to\infty}\sum_{n=-N}^{N}\mathrm{e}^{\boldsymbol{in\theta}}|N+n\rangle\,|N-n\rangle \tag{2}$$

is a relative phase state. Note that $|\theta\rangle = |\theta+2n\pi\rangle$.

From (2), relative phase operator to observe relative phase value in $[0,2\pi)$ should be:

$$\hat{P} = \lim_{\varepsilon\to+0}\int_{0-\varepsilon}^{2\pi-e}|\theta\rangle\theta\langle\theta|d\theta$$

then, as expected (omitting $\varepsilon\to+0$)

$$\hat{P}|\theta\rangle = \int_0^{2\pi}\left|\theta'\right\rangle\theta'\left\langle\theta'\right|d\theta'\,|\theta\rangle = \int_0^{2\pi}|\,\theta'\rangle\,\theta'\sum_{n=-\infty}^{\infty}\delta\left(\theta-\theta'+2\pi n\right)d\theta'$$

$$= \left(\theta-2\pi\left\lfloor\frac{\theta}{2\pi}\right\rfloor\right)|\theta\rangle$$

however, as the operator involves $\langle\theta|$ and inner product between $\langle\theta|$ and usual quantum states becomes meaningless as $N\to\infty$, the operator is not useful for usual quantum states.

As phase uncertainties of classical states, in general, are zero, any such classical states may, in a sense, be recognized as phase states, which is why phase state of equation (2) and that of [5] differ. Then, the most experimentally practical continuous phase state should be direct product (thus, unentangled) of two coherent states, at infinite quant limit as $\lim_{\alpha\to\infty}\left|\mathrm{e}^{i\theta}\alpha\right\rangle|\alpha\rangle$ $(\alpha\in\mathbb{R})$.

4. Relative Time State

That relative time is an observable is obvious, because, with a clock, we can measure duration of an interval between two events, which is the observation of relative time.

Relative time state is first defined in [5], though it is a little complicated.

Instead, like the previous section, based on Eq. (1) but with angular velocity difference of $\Delta\omega > 0$, relative time state can be constructed starting from $\lim_{N\to\infty} \sum_{n=-N}^{N} \mathrm{e}^{in\Delta\omega t}|N+n\rangle|N-n\rangle$, though, result of inner product of such states is $\sum_{n=-\infty}^{\infty} \delta(\Delta\omega(t-t')+2\pi n)$. That is, to remove periodicity, we must make $\Delta\omega$ infinitely small. By making $\Delta\omega = \omega_0/N$, where ω_0 is the maximum angular velocity difference between the oscillators, and adjusting coefficient, relative time state should be:

$$|t\rangle = \lim_{\omega_0\to\infty} \lim_{N\to\infty} \sum_{n=-N}^{N} \sqrt{\frac{\omega_0}{N}} \mathrm{e}^{\frac{in\omega_0 t}{N}} |N+n\rangle|N-n\rangle \tag{3}$$

as:

$$\langle t'|t\rangle = \lim_{\omega_0\to\infty} \lim_{N\to\infty} \sum_{n'=-N}^{N} \sum_{n=-N}^{N} \langle N-n'|\langle N+n'| \frac{\omega_0}{N} \mathrm{e}^{\frac{i\omega_0(nt-n't')}{N}} |N+n\rangle|N-n\rangle$$

$$= \lim_{\omega_0\to\infty} \lim_{N\to\infty} \sum_{n=-N}^{N} \mathrm{e}^{in\Delta\omega(t-t')}\Delta\omega = \lim_{\omega_0\to\infty} \int_{-\omega_0}^{\omega_0} \mathrm{e}^{i\omega(t-t')} d\omega$$

$$= \delta(t-t')$$

Similar to continuous phase states, relative time states are obtained at infinite quanta limit. As such, relative time states should not be unique allowing variations like those of [5].

The relative time operator should be:

$$\hat{T} = \int_{-\infty}^{\infty} |t\rangle t\langle t| dt$$

Though, like phase operator, it may be argued that the operator is not useful for usual quantum state with finite average number of quanta, remember that infiniteness is necessary to remove periodicity. That is, time can be defined only if a quantum system is unusual including infinitely many quanta. So, it is not a problem that the relative time operator is applicable only to such states.

5. Conclusions

It is pointed out that attempts to derive phase states using a single oscillator is physically meaningless, because, with a single oscillator, only absolute phase, which is physically meaningless, can be represented. Implicitly assumed standard phase without quantum fluctuation makes various absolute phase related states classical (w.r.t. the number of quanta). It is also pointed out that Suskind and Glogower showed, using two oscillators, countably infinitely many eigenstates of trigonometric relative phase operators are complete [2], which means, in a sense, the eigenstates are best possible quantum “phase states”.

It is also shown that continuous phase states can be constructed at classical (w.r.t. the number of quanta) limit and, similarly, relative time states and operator can be constructed. Unlike continuous phase operators, infiniteness related to time operator is not a problem because any quantum system with time must have infinitely many quanta.

As time states must be relative, when Lorentz transformations are involved, position states must also be relative.

References

[1] P. A. M. Dirac, “The quantum theory of the emission and absorption of radiation”, https://royalsocietypublishing.org/doi/abs/10.1098/rspa.1927.0039, 1927.

[2] L. Suskind, J. Glogower, “QUANTUM MECHANICAL PHASE AND TIME OPERATOR”, https://link.aps.org/pdf/10.1103/PhysicsPhysiqueFizika.1.49, 1964.

[3] D. T. Pegg, S. M. Barnett, “Phase properties of the quantized single-mode electromagnetic field”, Phys. Rev. A39,1665, https://journals.aps.org/pra/abstract/10.1103/PhysRevA.39.1665, 1989.

[4] W. H. Louisell, “AMPLITUDE AND PHASE UNCERTAINTY RELATIONS”, Phys. Letters, V. 7, N. 1, https://doi.org/10.1016/0031-9163(63)90442-6, 1963.

[5] M. Ban, “Relative number state representation and phase operator for physical systems”, Journal of Mathematical Physics 32, 3077, https://doi.org/10.1063/1.529054, 1991.