A CONCISE PROOF FOR BEAL'S CONJECTURE

Abstract. In this paper, we show how $a^x - b^y$ can be expressed as a binomial expansion (to an indeterminate power, z), and use it as the basis for a proof for the Beal Conjecture.

Introduction

The Beal Conjecture states that for the equation $a^x - b^y = c^z$, where $gcd(a, b, c) = 1$, integer solutions only exist for the values of x or y or $z = 1, 2$, but not for values of $x, y, z > 2$.[1]

We restate the equation as $a^x - b^y = c^z$ without loss of integrity, and demonstrate how $a^x - b^y$ can be reconfigured as a binomial expansion, containing not only the standard factors for a single power but also an additional non-standard factor. We then give a simple proof for Beal's Conjecture.

Theorem 0.1. *To prove that, for the equation $a^x - b^y = c^z$, where $gcd(a, b, c) = 1$, integer solutions only exist for the values of x or y or $z = 1, 2$, but not for values of $x, y, z > 2$.*

We first observe the following identity for $a^x - b^y$ as a binomial expansion (where the upper index n is an indeterminate integer):

$$a^x - b^y = \sum_{k=0}^{n} \binom{n}{k} (a + b)^{n-k} (-ab)^k (a^{x-n-k} - b^{y-n-k}). \tag{0.1}$$

Regardless of the value of n, the right hand side always equals $a^x - b^y$. So we fix n as z, such that:

$$a^x - b^y = \sum_{k=0}^{z} \binom{z}{k} (a + b)^{z-k} (-ab)^k (a^{x-z-k} - b^{y-z-k}). \tag{0.2}$$

Proof. We now assume that a solution exists for the equation $a^x - b^y = c^z$ for values of $x, y, z > 2$. From (0.2) if $a^x - b^y = c^z$, it follows that:

$$\sum_{k=0}^{z} \binom{z}{k} (a + b)^{z-k} (-ab)^k (a^{x-z-k} - b^{y-z-k}) = [(a + b)|s| - ab|t|]^z. \tag{0.3}$$

for all $s, t \in \mathbb{Q}$, where $gcd(s, ab) = 1$ (since a shared factor would mean that c would no longer be coprime with ab), such that $[(a + b)|s| - ab|t|] = c$. This expands to:

(0.4)

$$\sum_{k=0}^{z} \binom{z}{k} (a + b)^{z-k} (-ab)^k (a^{x-z-k} - b^{y-z-k}) = \sum_{k=0}^{z} \binom{z}{k} (a + b)^{z-k} (-ab)^k |s|^{z-k} |t|^k.$$

Date: June 2019.

2010 *Mathematics Subject Classification.* Primary 11D41.

Key words and phrases. Number theory, Beal's Conjecture, Binomial Theorem.

[1]See www.bealconjecture.com. Last accessed 14.12.17.

Comment: We know that the right hand side is a perfect power; we are assuming the left hand side is but it may not be. In our proof we are hoping that it isn't, and that there can be no equality for $z > 2$.

Since the factors $(a+b)^{z-k}$ and $(-ab)^k$ are constant on both sides, and standard in exponential form for a perfect power, we need only test whether the associated factor on the left, i.e. $(a^{x-z-k} - b^{y-z-k})$, is the same in exponential form as the associated factors on the right, i.e. $|s|^{z-k}|t|^k$. For if it cannot even be represented in the correct standard *form* for a binomial expansion of a perfect power, then there can be no equality in value either (while remaining a perfect power).

It is pointless to test for inequality of *value* when the associated factors, $|s|^{z-k}$ and $|t|^k$, are variable (whose values depend on each other), for even when $s \neq v$ and $t \neq w$, the following equation holds true:

$$[(a+b)|s| - ab|t|]^z = [(a+b)|v| - ab|w|]^z$$

i.e. when $a = 8$, $b = 5$, $s = 120$, $t = 26$, $v = 160$ and $w = 39$. In the binomial expansion of this equation, the exponential *forms* of each counterpart term may be the same even while their *values* differ. So trying to prove inequality of counterpart terms (in terms of value) will pose problems.

Therefore, we will seek to prove only inequality in form, and simplify the problem as follows:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k = \sum_{k=0}^{z} (a^{x-z-k} - b^{y-z-k}) \tag{0.5}$$

For this equation to have solutions the associated factor, $(a^{x-z-k} - b^{y-z-k})$, must exactly correspond in form with $|s|^{z-k}|t|^k$ in each counterpart (k^{th}) term, for any given value of z. If it does, then the whole of the left hand side of (0.4) will be a power to z (as we know the right hand side is), and the Beal equation will have solutions. But if just one term of the corresponding binomials exists where $(a^{x-z-k} - b^{y-z-k})$ does not equal $|s|^{z-k}|t|^k$ in form, then not only will the integrity of that particular k^{th} term be compromised as a valid binomial term, but also the whole expression as an expansion of a power to z.

To do this we can assume that all the counterpart (k^{th}) terms in (0.5) are equal and in particular the first and last terms. [Then we will show that when these are equal there is inequality in the second and penultimate terms. This circumvents the need to demonstrate inequality in any further terms however large z becomes.]

So, we can deduce the second term from the equation in (0.5). If the first term is $|s|^z = \pm(a^{x-z} - b^{y-z})$, and the last term is $|t|^z = \pm(a^{x-2z} - b^{y-2z})$, we can raise the powers accordingly and multiply together to get:

$$|s|^{z-1}|t| = \pm(a^{x-z} - b^{y-z})^{(z-1)/z}(a^{x-2z} - b^{y-2z})^{1/z}. \tag{0.6}$$

We can also calculate the second term directly from the right hand side of (0.5). So when $k = 1$, the second term is:

$$\pm(a^{x-z-1} - b^{y-z-1}). \tag{0.7}$$

Putting (0.6) and (0.7) together, we get:

$$\pm(a^{x-z} - b^{y-z})^{(z-1)/z}(a^{x-2z} - b^{y-2z})^{1/z} = \pm(a^{x-z-1} - b^{y-z-1}). \tag{0.8}$$

Now we raise both sides by z and divide both sides by $(a^{x-z} - b^{y-z})^{(z-2)}$ and rearrange to get:

$$\pm(a^{x-z} - b^{y-z})(a^{x-2z} - b^{y-2z}) = \pm\frac{(a^{x-z-1} - b^{y-z-1})^z}{(a^{x-z} - b^{y-z})^{(z-2)}}. \tag{0.9}$$

The procedure for the penultimate term is exactly the same. So, again using $|s|^{z-k}$ and $|t|^k$ as our point of reference, we raise the powers accordingly and multiplying together to get the penultimate term:

$$|s||t|^{z-1} = \pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{(z-1)/z}. \tag{0.10}$$

And *directly* from the binomial formula, when $k = z - 1$, the penultimate term is:

$$\pm(a^{x-2z+1} - b^{y-2z+1}) \tag{0.11}$$

Putting (0.10) and (0.11) together, we get:

$$\pm(a^{x-z} - b^{y-z})^{1/z}(a^{x-2z} - b^{y-2z})^{(z-1)/z} = \pm(a^{x-2z+1} - b^{y-2z+1}). \tag{0.12}$$

This time, we raise both sides by z and divide both sides of by $(a^{x-2z} - b^{y-2z})^{(z-2)}$ and rearrange to get:

$$\pm(a^{x-z} - b^{y-z})(a^{x-2z} - b^{y-2z}) = \pm\frac{(a^{x-2z+1} - b^{y-2z+1})^z}{(a^{x-2z} - b^{y-2z})^{(z-2)}}. \tag{0.13}$$

At this point we can ignore the $\pm$ sign. This was introduced by the absolute values of s and t, which are now no longer necessary. It is self-evident that inequality exists when there is opposite polarity. The harder task is to prove inequality when polarity is the same. So by ignoring the signs, we are not making the proof easier. So we will remove the $\pm$ sign and focus on circumstances where polarity is the same.

Now, we note that in (0.9) and (0.13) the left hand sides are exactly the same. This means we can subtract (0.9) from (0.13) and rearrange to get:

$$\left(\frac{a^{x-z-1} - b^{y-z-1}}{a^{x-2z+1} - b^{y-2z+1}}\right)^z = \left(\frac{a^{x-z} - b^{y-z}}{a^{x-2z} - b^{y-2z}}\right)^{(z-2)} \tag{0.14}$$

Solutions will exist to this equation
a) *either* if the large bracketed fractions on each side of have a value of 1 (since the outer exponents are not equal),
b) *or* if the numerators (to their respective powers) on both sides are equal, *and* simultaneously if the denominators (to their respective powers) on both sides are equal.

Taking these two options in turn (still when $x, y, z > 2$):

a) since $(a^{x-z-1} - b^{y-z-1}) \neq (a^{x-2z+1} - b^{y-2z+1})$, and $(a^{x-2z} - b^{y-2z}) \neq (a^{x-z} - b^{y-z})$, neither side in (0.14) has a value of 1, eliminating this option;

b) even without their respective powers, the base value of the left hand numerator $(a^{x-2z+1} - b^{y-2z+1})$ is greater than its right hand counterpart, $(a^{x-2z} - b^{y-2z})$;

but when the power is greater, (i.e. $z > (z-2)$), then the inequality is even greater. So it follows that: $(a^{x-2z+1} - b^{y-2z+1})^z \neq (a^{x-2z} - b^{y-2z})^{(z-2)}$. We do not even need to bother with the denominators.

Having now eliminated both options it follows that, for all values of $x, y, z > 2$:

$$\sum_{k=0}^{z} |s|^{z-k}|t|^k \neq \sum_{k=0}^{z}(a^{x-z-k} - b^{y-z-k}). \tag{0.15}$$

However, this contradicts our equation in (0.4). Under these circumstances there is no equality of form, let alone value. Therefore the left hand side of (0.4) cannot be a perfect power (as we assumed it was). And so our initial assumption that for any value of $x, y, z > 2$ solutions exist for the equation $c^z = a^x - b^y$ is false.

□

What then happens for the case $z = 1, 2$? Well, from (0.14), when $z = 1$ it follows that:

$$\left(\frac{a^{x-2} - b^{y-2}}{a^{x-1} - b^{y-1}}\right)^1 = \left(\frac{a^{x-1} - b^{y-1}}{a^{x-2} - b^{y-2}}\right)^{-1}, \tag{0.16}$$

$$\Rightarrow \left(\frac{a^{x-2} - b^{y-2}}{a^{x-1} - b^{y-1}}\right) = \left(\frac{a^{x-2} - b^{y-2}}{a^{x-1} - b^{y-1}}\right). \tag{0.17}$$

No contradiction.

And again from (0.14), when $z = 2$, it follows that:

$$\left(\frac{a^{x-3} - b^{y-3}}{a^{x-3} - b^{y-3}}\right)^2 = \left(\frac{a^{x-2} - b^{y-2}}{a^{x-4} - b^{y-4}}\right)^0, \tag{0.18}$$

$$\Rightarrow 1 = 1. \tag{0.19}$$

Again, no contradiction.

So in both cases, when $z = 1$ and when $z = 2$, the standard rules of binomial expansion can be applied to our non-standard binomial expression without contradiction such that $(a^{x-z-k} - b^{y-z-k})$ *is* equal to $|s|^{z-k}|t|^k$ in these two cases, and therefore that in these cases solutions to the original equation exist.

References

1. Peter Schorer, *Is There a "Simple" Proof of Fermat's Last Theorem? Part (1) Introduction and Several New Approaches*, 2014, www.occampress.com/fermat.pdf last accessed 14.12.17.
2. Andrew Wiles, *Modular Elliptic Curves and Fermat's Last Theorem*, Ann. of Math (2), Vol. 141, No.3 (May, 1995), 443-551.
3. www.bealconjecture.com. Last accessed 14.12.17.

The Rectory, Village Road, Waverton, Chester Ch3 7QN, UK
E-mail address: julianbeauchamp47@gmail.com