Division by Zero Calculus in Multiply Dimensions and Open Problems (an extension)

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Abstract: In this paper, we will introduce the division by zero calculus in multiply dimensions in order to show some wide and new open problems as we see from one dimensional case.

Key Words: Zero, division by zero, division by zero calculus, $0/0 = 1/0 = z/0 = 0$, multiply dimensions, several complex analysis, Laurent expansion.

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1 Division by zero and division by zero calculus

For the long history of division by zero, see [1, 19]. The division by zero with mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $a/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [5] for example.

Division by zero is trivial and clear also from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [5, 29].

The simple field structure containing division by zero was established by M. Yamada ([8]). For a simple introduction, see H. Okumura [17].

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [11, 12, 13, 14, 15, 16] for example.

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.5}$$

we **define** the identity, (by the division by zero)

$$f(a) = C_0. \tag{1.6}$$

(Note that here, there is no problem on any convergence of the expansion (1.5) at the point $z = a$, because all the terms $(z - a)^n$ are zero at $z = a$ for $n \neq 0$.)

Apart from the motivation, we define the division by zero calculus by (1.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [3] and also the references for recent related results.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x=0)=1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019 : 9: 45-10: 00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [3],

he kindly sent the kind message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Meanwhile, on ZERO, the authors S. K. Sen and R. P. Agarwal [26] published its long history and many important properties of zero. See also R. Kaplan [4] and E. Sondheimer and A. Rogerson [28] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [24] that

The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea as an ideal point of one point compactification
and that is represented geometrically with the horn torus model [3].

S. K. Sen and R. P. Agarwal [26] referred to the paper [5] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

"Thou shalt not divide by zero" remains valid eternally.

However, in [23] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

They stated in the book many meanings of zero over mathematics, deeply.

Of course, we will be interested in the case of multiply dimensions, in general, and for multiply dimensional cases, the situations will be very complicated and involved than one dimensional case. At the first step, we would like to consider its origin, simply.

This is an extended version of the paper [25].

2 Division by zero calculus in multiply dimensions

In order to make clear the problem, we give firstly a prototype example. We have the identity by the division by zero calculus, for

$$f(z) = \frac{1+z}{1-z}, \quad f(1) = -1.$$

From the real part and imaginary part of the function, we have, for $z = x+iy$

$$\frac{1-x^2-y^2}{(1-x)^2+y^2} = -1, \quad \text{at} \quad (1,0)$$

and

$$\frac{y}{(1-x)^2+y^2} = 0, \quad \text{at} \quad (1,0),$$

respectively. Why the differences do happen?

In order to solve this problem, we will give the definition of the division by zero calculus in multidimensional spaces.

Definition of the division by zero calculus for multidimensional spaces:

For an analytic function $g(z)$ on a domain D on $\mathbf{C}^n, n \geq 1$, we set

$$E = \{z \in D; g(z) = 0\}.$$

For an analytic function $f(z)$ on the set $D \setminus E$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} C_n(z) g(z)^n, \tag{2.1}$$

for analytic functions $C_n(z)$ on D, we define the division by zero calculus by the correspondence

$$f \longrightarrow F_{f,g=0}(z) := C_0(z)$$

that shows a natural analytic function of the function f on the domain D derived from $D \setminus E$ with respect to $E = \{z \in D; g(z) = 0\}$.

Of course, this definition is a natural extension of one dimensional case. The expression (2.1) may be guaranteed by the general Laurent expansion that is given by the following proposition (that was introduced by Takeo Ohsawa):

Proposition 13.1 *In the Definition of the division by zero calculus for mutidimensional spaces, if the domain D is a regular domain, for any analytic function g, the expansion* (2.1) *is possible.*

See [30] for the related topics.

Since the uniqueness of the expansion is, in general, not valid, the division by zero calculus is not determined uniquely. However, we are very interested in the expansion (2.1) and the property of the function $C_0(z)$ as in one dimensional case.

From the above arguments, we can see the desired results for the examples as follows:

$$\frac{1 - x^2 - y^2}{(1-x)^2 + y^2}$$

$$= -1 + \frac{2(1-x)}{(1-x)^2 + y^2} = -1, \quad \text{at} \quad (1, 0)$$

and

$$\frac{y}{(1-x)^2 + y^2} = 0, \quad \text{at} \quad (1, 0).$$

3 In parameter representations

For example, we will consider the parameter representation

$$x^2 = \frac{a(u-a)(v-a)}{(b-a)(c-a)}, \tag{3.1}$$

$$y^2 = \frac{b(u-b)(v-b)}{(c-b)(a-b)}, \tag{3.2}$$

$$z^2 = \frac{c(u-c)(v-c)}{(a-c)(b-c)}, \tag{3.3}$$

of the ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \quad a, b, c, > 0 \tag{3.4}$$

([2], 112 page).

For the very natural case $b = a$, how will be the parameter representations (3.1)-(3.3)?

At first, we have, by the division by zero, for $b = a$,

$$\frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{c} = 1 \tag{3.5}$$

and

$$x^2 = 0. \tag{3.6}$$

Therefore

$$\frac{y^2}{a} + \frac{z^2}{c} = 1. \tag{3.7}$$

This will mean that (3.5) is the rotation of (3.7) around the z axis and (3.7) is the cut elliptic function of (3.5) by the plane $x = 0$.

Next, by the division by zero calculus, we have

$$y^2 = -\frac{1}{(c-a)^2}[c(u-a)(v-a) - a(c-a)(u-a) - a(c-a)(v-a)]. \tag{3.8}$$

However, since we are considering the case with $x = 0$, the parameters u and v have to be restricted as $u = a$ or $v = a$. We fix as $v = a$ then we have

$$y^2 = \frac{a(a-u)}{a-c}. \tag{3.9}$$

Finally, of course, we have

$$z^2 = \frac{c(u-c)}{a-c}. \tag{3.10}$$

Meanwhile, we will consider the parametric representations with $c = 0$

$$x^2 = \frac{-(u-a)(v-a)}{b-a}, \tag{3.11}$$

$$y^2 = \frac{-(u-b)(v-b)}{a-b}, \tag{3.12}$$

$$z^2 = 0. \tag{3.13}$$

Then, we note that

$$\frac{x^2}{a} + \frac{y^2}{b} = 1 - \frac{uv}{ab} \tag{3.14}$$

and so, for the case $u = 0$ or $v = 0$, we obtain the parameter representation for the elliptic curve

$$\frac{x^2}{a} + \frac{y^2}{b} = 1. \tag{3.15}$$

4 Open problem

In Section 3, we gave natural interpretations for the parameter representations (3.1)-(3.3) for the case $b = a$ by the division by zero and division by zero calculus. However, we wonder

Open problem: *For the parameter representations (3.1)-(3.3), could we derive some parameter representations of (3.5)?*

When, we use (3.5), (3.3) and (3.8), we have the curious representation:

$$x^2 = \frac{1}{(c-a)^2}[uv(c-a)+ac(u+v)+a^2(u+v+c)-a^3-a(c-a)(v-a)]. \tag{4.1}$$

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