On Some Isoperimetric Inequalities for Dirichlet Integrals; Green's Function and Dirichlet Integrals

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Abstract: In this paper, as a direct application of Q. Guan's result on the conjugate analytic Hardy H_2 norm we will derive a new type isoperimetric inequality for Dirichlet integrals of analytic functions.

Key Words: Dirichlet integral, conjugate analytic Hardy H_2 norm, Bergman kernel, Rudin kernel, Green function, isoperimetric inequality.

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1 Introduction

Q. Guan ([1]) derived the surprising identity of the conjugate analytic Hardy norm and the Bergman norm. Let $H^2(D)$ denote the analytic Hardy class on D defined as the set of all analytic functions $f(z)$ on D such that the subharmonic functions $|f(z)|^2$ have harmonic majorants $U(z)$, i.e. $|f(z)|^2 \leq U(z)$ on D.

Here, for simplicity, we assume that the domain D is a regular domain surrounded by a finite number of analytic Jordan curves. All the results in this paper are valid on any compact bordered Riemann surface.

Let $G(z,t)$ be the Green's function on D such that $G(z,t) + \log|z-t|$ is analytic on $D \times D$ with pole at a fixed point t of D.

Let $\partial/\partial\nu$ denote the inner normal derivative on the boundary ∂D and $\partial G(z,t)/\partial\nu$ is positive and real analytic on the boundary ∂D.

Then, Q. Guan ([1]) derived the surprising identity:

Lemma 2.3. *For any fixed $t \in D$ and for any fixed analytic function f on D which is continuous on $D \cup \partial D$, the identity, for $z = x + iy$*

$$\lim_{r \to 1-0} \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)} \geq r\}} |f(z)|^2 dxdy$$

$$= \frac{1}{2} \int_{\partial D} |f(z)|^2 (\partial G(z,t)/\partial\nu)^{-1} |dz|$$

holds.

This surprising result was derived from very difficult problems on some comparison of the magnitudes of the Bergman and the Rudin (Hardy H^2) reproducing kernels, however, here, we would like to consider some interesting and direct application of the identity. For the great and deep background for the problems, see the original paper ([1]). Its source was given by [2] and then the topics was cited in the book [4] in details.

2 Main Result

The following inequalities seem to be interesting on its own sense:

Theorem: *For any given $\epsilon > 0$ and for any fixed analytic function $f(z)$ on $D \cup \partial D$, there exists $r : (0 < r < 1)$ satisfying the inequality*

$$\int\int_D |f'(z)|^2 dxdy - \epsilon \leq \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)} \geq r\}} |f'(z)|^2 dxdy.$$

This inequality may be looked as an isoperimetric inequality, because the Dirichlet integral on a domain is estimated (restricted) by the Dirichlet integral on some small boundary neighborhood of the domain. Here, the neighborhood size and estimation are stated by the level curve of the Green function, precisely.

Even the case of the identity function $f(z) = z$, we can enjoy the senses of the estimation and the result.

3 Proof of the Main Result

The integral of the conjugate analytic Hardy $H^2(D)$ norm seems to be not popular, however, the norm will have a beautiful structure and the norm is conformally invariant. Furthermore, we obtain the inequality:

For any analytic function $f(z)$ on $D \cup \partial D$, we have the inequality

$$\int\int_D |f'(z)|^2 dxdy \leq \frac{1}{2}\int_{\partial D} |f'(z)|^2 (\partial G(z,t)/\partial \nu)^{-1} |dz|. \tag{3.1}$$

This result was derived from some complicated theory of reproducing kernels in ([3]). The equality problem in the inequality was also established; that is, equality holds if and only if the domain is simply-connected and the function $f'(z)$ is expressible in the form $CK(z,\overline{t})$ for the Bergman reproducing kernel $K(z,\overline{u})$ on the domain D and for a constant C.

We can obtain the main result by combining of the Guan identity and this inequality.

4 Remarks

We note the following interesting problems:

Problem 1 : *How will be some generalization of the Guan identity for a general weight for* $(\partial G(z,t)/\partial \nu)^{-1}$*?*

Problem 2 : *We can consider similar inequalities for various function spaces. For example, how will be the case for the harmonic Dirichlet integrals?*

Problem 3 : *The structure and proof of the inequality (3.1) are very complicated (involved) and the Guan identity is very unique. So, we are interested in some direct proof of the theorem.*

Problem 4 : *The theorem seems to be valid for a general domain* D *and for general analytic functions with finite Dirichlet integrals on* D *apart from the proof in this paper. However, in the theorem* r *is depending on the function* f *and therefore the generalization of the theorem is not simple.*

For the inequality (3.1), note the inequality

$$\left(\frac{1}{\pi}\int\int_D |f'(z)|^2 dxdy\right)^2 = \left(\frac{1}{2\pi i}\int_{\partial D} \overline{f(z)} f'(z) dz\right)^2$$

$$\leq \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \frac{\partial G(z,t)}{\partial \nu} |dz| \frac{1}{2\pi} \int_{\partial D} |f'(z)|^2 (\partial G(z,t)/\partial \nu)^{-1} |dz|.$$

Note that the relation of the Bergman norm and the weighted H^2 norm is very delicate. See [2, 3, 4].

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