Three arcsin type formulas for Pi

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abstract

We give three arcsin type formulas for Pi

The number Pi is defined by

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415926535... \tag{1}$$

An exact formula for π in terms of the inverse tangents is Machin's formula

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) \tag{2}$$

This formula can also be written

$$\pi = 16\sin^{-1}\left(\frac{1}{\sqrt{26}}\right) - 4\sin^{-1}\left(\frac{1}{13^2\sqrt{2}}\right) \tag{3}$$

Where $\sin^{-1} x$ is the inverse sine function. In this note we give three arcsin type formulas for Pi.

Formulas

$$\begin{aligned}\pi = {} & \frac{3}{2}\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3\left(1+e^{-2}+e^{-4}\right)}{1+e^{-4}+e^{-8}}}\right) + \frac{3}{2}\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3\left(1-e^{-2}+e^{-4}\right)}{1+e^{-4}+e^{-8}}}\right) \\ & - \frac{3}{2}\sin^{-1}\left(\frac{e^{-2}}{2}\sqrt{\frac{3\left(1+e^{-2}+e^{-4}\right)}{1+e^{-4}+e^{-8}}}\right) + \frac{3}{2}\sin^{-1}\left(\frac{e^{-2}}{2}\sqrt{\frac{3\left(1-e^{-2}+e^{-4}\right)}{1+e^{-4}+e^{-8}}}\right)\end{aligned} \tag{4}$$

$$\begin{aligned}\pi = {} & 2\sin^{-1}\left(\sqrt{\frac{1+\sqrt{2}\,e^{-2}+e^{-4}}{2\left(1+e^{-8}\right)}}\right) + 2\sin^{-1}\left(\sqrt{\frac{1-\sqrt{2}\,e^{-2}+e^{-4}}{2\left(1+e^{-8}\right)}}\right) \\ & - 2\sin^{-1}\left(e^{-2}\sqrt{\frac{1+\sqrt{2}\,e^{-2}+e^{-4}}{2\left(1+e^{-8}\right)}}\right) + 2\sin^{-1}\left(e^{-2}\sqrt{\frac{1-\sqrt{2}\,e^{-2}+e^{-4}}{2\left(1+e^{-8}\right)}}\right)\end{aligned} \tag{5}$$

$$\pi = 3\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{1+\sqrt{3}\,e^{-2}+e^{-4}}{1-e^{-4}+e^{-8}}}\right) + 3\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{1-\sqrt{3}\,e^{-2}+e^{-4}}{1-e^{-4}+e^{-8}}}\right)$$
$$-3\sin^{-1}\left(\frac{e^{-2}}{2}\sqrt{\frac{1+\sqrt{3}\,e^{-2}+e^{-4}}{1-e^{-4}+e^{-8}}}\right) + 3\sin^{-1}\left(\frac{e^{-2}}{2}\sqrt{\frac{1-\sqrt{3}\,e^{-2}+e^{-4}}{1-e^{-4}+e^{-8}}}\right) \qquad (6)$$

Remark: $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828182...$

References

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