Central Limit Theorems for Other Types of Means

It is relatively easy to transform the standard version of the Central Limit Theorem for other types of means.

Take the standard (arithmetic) Central Limit Theorem:

$$For\ S_n = X_1 + X_2 + \ldots + X_n$$

$$\Pr\left(\frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} < \beta \right) \to \Phi(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} \exp(-t^2/2)dt$$

For Geometric means, let:

$$P_n = X_1 * X_2 * \ldots * X_n$$

Rewrite:

$$\frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} < \beta \quad as \quad \frac{S_n}{n} < \mu_X + \beta \frac{\sigma_X}{\sqrt{n}}$$

Replace x with ln(x) (x>0), hence:

$$\mu_X\ becomes\ \mu_{\ln(X)} = E(\ln(X))\ and$$

$$\sigma_X\ becomes\ \sigma_{\ln(X)} = \sqrt{E(\ln(X)^2) - \left[E(\ln(X))\right]^2}$$

And thus in the above inequality:

$$\frac{\ln(X_1) + \ln(X_2) + \ldots + \ln(X_n)}{n} < \mu_{\ln(X)} + \beta \frac{\sigma_{\ln(X)}}{\sqrt{n}}$$

Which implies:

$$\exp(\frac{\ln(P_n)}{n}) < \exp(\mu_{\ln(X)} + \beta \frac{\sigma_{\ln(X)}}{\sqrt{n}})$$

That is:

$$P^{1/n} < \exp(\mu_{\ln(X)} + \beta \frac{\sigma_{\ln(X)}}{\sqrt{n}})$$

But:

$$\begin{aligned} \exp(\mu_{\ln(X)}) &= \exp(E(\ln(X)) \\ &= \exp(\int \ln(X) pr(X) dx)) \\ &= \prod_{x\in\mathbb{R}} x^{pr(x)dx} \end{aligned}$$

where the product is a "Product Integral" (a continuous product) – see the Wikipedia article "Product Integral"

So:

$$\Pr\left(P^{1/n} < \exp(\beta \frac{\sigma_{\ln(X)}}{\sqrt{n}}) * \prod_{x\in\mathbb{R}} x^{pr(x)dx} \right) \to \Phi(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} \exp(-t^2/2) dt$$

as $n \to \infty$

Set beta=0 and you see that:

$$median(P^{1/n}) \to \prod_{x\in\mathbb{R}} x^{pr(x)dx}$$

For a Harmonic Mean CLT:

Replace x with 1/x. - hence:

$$\begin{aligned} &\mu_X \quad becomes \quad \mu_{1/X} = E(1/X) \quad and \\ &\sigma_X \quad becomes \quad \sigma_{1/X} = \sqrt{E(1/X^2) - [E(1/X)]^2} \end{aligned}$$

Then

$$\frac{\frac{S_n}{n} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} < \beta \quad becomes \quad \frac{\frac{1/X_1 + 1/X_2 + \ldots + 1/X_n}{n} - \mu_{1/X}}{\left(\frac{\sigma_{1/X}}{\sqrt{n}}\right)} < \beta$$

Or:

$$\frac{n}{1/X_1 + 1/X_2 + \ldots + 1/X_n} > \frac{1}{\mu_{1/X} + \beta\left(\sigma_{1/X}/\sqrt{n}\right)}$$

Thus

$$\Pr\left[\frac{n}{1/X_1+1/X_2+\ldots+1/X_n} > \frac{1}{\mu_{1/X}+\beta\left(\sigma_{1/X}/\sqrt{n}\right)}\right] \to \Phi(\beta)\ as\ n\to\infty$$

Set beta=0, then Pr→1/2, thus:

$$median\left[\frac{n}{1/X_1+1/X_2+\ldots+1/X_n}\right] \to \frac{1}{\mu_{1/X}}\ as\ n\to\infty$$

With the above, standard conditions apply (e.g: no division by zero, finite expectations and variance, etc)

In general, replacing x with g(x) should provide expressions for other more generalised means, provided the usual conditions are satisfied.

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