Some infinite sum series for Pi

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Abstract. We give some infinite series for Pi.

1. Introduction

The number Pi is defined by (Gregory and Leibniz): $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592...$, in this note we recall six series for Pi.

2. Series

Entry 1. If $u^2 = \frac{3\left(e^{-2}+e^{-4}+e^{-6}\right)}{1+e^{-4}+e^{-8}}$, then

$$\pi = 3\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}u^{2n+1}\sinh(2n+1)}{2n+1} \tag{1}$$

Entry 2. If $u^2 = \frac{3\left(e^{-2}-e^{-4}+e^{-6}\right)}{1+e^{-4}+e^{-8}}$, then

$$\pi = 3\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}u^{2n+1}\cosh(2n+1)}{2n+1} \tag{2}$$

Entry 3. If $u^2 = \frac{e^{-2}+e^{-4}\sqrt{3}+e^{-6}}{1-e^{-4}+e^{-8}}$, then

$$\pi = 6\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}u^{2n+1}\sinh(2n+1)}{2n+1} \tag{3}$$

Entry 4. If $u^2 = \frac{e^{-2}-e^{-4}\sqrt{3}+e^{-6}}{1-e^{-4}+e^{-8}}$, then

$$\pi = 6\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-4n}u^{2n+1}\cosh(2n+1)}{2n+1} \tag{4}$$

Entry 5. If $u^2 = \frac{e^{-2}+e^{-4}\sqrt{2}+e^{-6}}{2\left(1+e^{-8}\right)}$, then

$$\pi = 8\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}u^{2n+1}\sinh(2n+1)}{2n+1} \tag{5}$$

Entry 6. If $u^2 = \frac{e^{-2} - e^{-4}\sqrt{2} + e^{-6}}{2\left(1+e^{-8}\right)}$, then

$$\pi = 8\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}u^{2n+1}\cosh(2n+1)}{2n+1} \tag{6}$$

Remarks:

- $e = \sum_{n=0}^{\infty}\frac{1}{n!} = 2.718281\ldots$ (Euler number).
- $\sinh x = \frac{e^x - e^{-x}}{2}$ is the hyperbolic-sine.
- $\cosh x = \frac{e^x + e^{-x}}{2}$ is the hyperbolic-cosine.

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