ABOUT ONE GEOMETRIC VARIATION PROBLEM

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In the Euclidean space R_n of n dimensions, let us consider integral

$$\int f ds,$$

taken along some real curve (M) with arc s; f – given function with curvatures $k_i (i = 1, 2, ..., n-1)$ of curve and their derivatives against s up to order r including. It is required to determine extremal curves for which first variation of integral is equal to zero.

For curves of R_3 following results are known. Radon [1] has found that for $f = f(k_i)$ finite equations of extremals may be found by quadratures. De Castro Brzezicki [2] has shown that at

$$f = f\left(k_1, \frac{dk_1}{ds}\right)$$

finite equations may be get by quadrature, if natural equations of curve are found.

In this work, general expressions for variation of all k_i and their derivatives are found (§ 1); equations of Euler-Lagrange are found (§ 2); some considerations are given on integration these equations, and it is shown that results of de Castro Brzezicki work for arbitrary f in R_3, R_4 and R_5 (§ 3). Finally, some particular cases in R_3 are considered, in addition result of Radon is generalized to functions $f(k_1, k_2)$ that satisfy some differential equations.

In what follows it is assumed that all derivatives, which are encountered in the calculations, exist and are continuous. Derivatives against s are indicated by strokes and indicators in brackets; summations always are explicit. Border conditions are assumed to be satisfied.

1 VARIATIONS OF CURVATURES AND THEIR DERIVATIVES

Let $M(s)$ – radius vector of curve (M) to be considered, $t_i(i = 1, ..., n)$ – unit vectors of orthonormal frame, $t_{ij}(j = 1, ..., n)$ – coordinates t_i in some fixed orthonormal system of coordinates.
Using orthogonal matrix

$$T = \|t_{ij}\|_1^n$$

and antisymmetric matrix of Frenet

$$K = \|k_{ij}\|_1^n,$$

where

$$k_{ij} = k_j\delta_{i,j-1} - k_{i-1}\delta_{i,j+1}, k_0 = k_n = 0,$$

and δ_{ij} – symbol of Kronecker, formulae of Frenet may be written in the form

$$T'' = KT. \tag{1}$$

Supposedly, all $k_i(i = 1, ..., n-1)$ are different from zero.
We take radius vector of curve (N) to be varied in the form

$$N = M(s) + \epsilon\sum_{i=1}^{n} u_i t_i, \tag{2}$$

where ϵ – parameter, u_i - arbitrary functions of s.
In what follows, magnitudes we are interested in, that characterize curve (N), decompose in increasing powers of ϵ, where powers higher that first are dropped. Coefficient at ϵ we name variation of corresponding magnitude, and denote by δ.
By differentiation of (2) with respect to s, we get

$$\frac{dN}{ds} = [1 + \epsilon(u_1' - k_1u_2)]t_1 + \epsilon\sum_{i=2}^{n}(k_{i-1}u_{i-1} + u_i' - k_iu_{i+1})\,t_i. \tag{3}$$

Module of this vector gives derivative of arc σ of curve (N) against s:

$$\frac{d\sigma}{ds} = 1 + \epsilon a, \tag{4}$$

where

$$a = u_1' - k_1u_2, \tag{5}$$

but unit vector p_1 tangent to (N) is equal to

$$\frac{dN}{d\sigma} = t_1 + \epsilon\sum_{i=2}^{n} a_{1i}t_i,$$

where

$$a_{1i} = k_{i-1}u_{i-1} + u_i' - k_i u_{i+1}, i = 2, 3, ..., n. \tag{6}$$

Let p_i be unit vectors of accompanied orthonormal frame of curve (N), and

$$P = \|P_{ij}\|_1^n$$

for matrix of their coordinates. Latter is connected with matrix T by expression

$$P = T + \epsilon AT. \tag{7}$$

Due to orthogonality P and T, matrix $A = \|a_{ij}\|_1^n$ is antisymmetric. Expressions (6) give elements of first row (and first column) of matrix A. Remaining elements of A, as well as variations κ_i of curvatures k_i, may be received in the following way: for curve (N), Frenet matrix is equal to

$$K + \epsilon L,$$

where antisymmetric matrix $L = \|\kappa_{ij}\|_1^n$ is composed from magnitudes κ_i in the same way as K is composed from k_i:

$$\kappa_{ij} = \kappa_i \delta_{i,j-1} - \kappa_{i-1}\delta_{i,j+1}, \kappa_0 = \kappa_n = 0.$$

Consequently, for P relationship holds

$$\frac{dP}{d\sigma} = (K + \epsilon L)P.$$

Substituting for P expression (7), and taking into account (1) and by equating coefficients at ϵ, we get

$$L + aK = A' + AK - KA. \tag{8}$$

Due to the fact that in both parts of equation (8) are present antisymmetric matrices, it is equivalent to system of $\frac{n(n-1)}{n}$ scalar independent equations, to be received by equating elements above the main diagonal:

$$\kappa_i + ak_i = a_{i,i+1}' + k_{i-1}a_{i-1,i+1} - k_{i+1}a_{i,i+2}, \tag{9}$$

where $i = 1, 2, ..., n-1$, and

$$k_i a_{i+1,j} = a_{ij}' + k_{i-1,}a_{i-1,j} + k_{j-1}a_{i,j-1} - k_j a_{i,j+1}, \tag{10}$$

where

$$i = 1, 2, ..., n-2; j = i+2, i+3, ..., n.$$

Equation (10) allows to determine all elements of $i+1$-st row of matrix A by use of elements of previous rows and curvatures. Due to fact that (6) determines all a_{1j}, it is possible to express all a_{ij} – and by their use all κ_i too – as functions of curvature k_i, variations u_i and their derivatives against s. Explicit expressions,

received in this way, in what follows are but not necessary.
If variations

$$\delta(ds) = ads,$$

$$\delta(k_i) = \kappa_i,$$

are known, then, using expression (φ – arbitrary function of s)

$$\delta(\varphi') = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\frac{d(\varphi+\epsilon\delta\varphi)}{(1+\epsilon a)ds} - \frac{d\varphi}{ds}\right),$$

i.e.

$$\delta(\varphi') = (d\varphi)' - a\varphi', \tag{11}$$

we have

$$\delta(k_i') = \kappa_i' - ak_i',$$

and, repeatedly applying (11), we get

$$\delta\left(k_i^{(h)})\right) = \kappa_i^{(h)} - (ak_i')^{(h-1)} - (ak_i'')^{(h-2)} - ... - ak_i^{(h)}, \tag{12}$$

for derivatives of order h, or, compressing,

$$\delta\left(k_i^{(h)}\right) = \kappa_i^{(h)} - \sum_{j=0}^{h-1}\binom{j+1}{h}a^{(j)}k_i^{(h-j)},$$

where $a^{(0)} = a$, $s^{(j)} = \frac{d^j a}{ds^j}$.

2 EQUATIONS OF EULER-LAGRANGE

For the integral along curve (M) in R_n

$$J = \int f ds,$$

where f – given function of k_i $(i = 1, ..., n-1)$ and their derivatives against s up to order r including, variation is equal to

$$\delta J = \int\left[af + \sum_{i=1}^{n-1}\sum_{h=0}^{r}\frac{\partial f}{\partial k_i^{(h)}}\delta(k_i^{(h)})\right]ds.$$

Substituting expression (11) for variations of derivatives against curvatures and removing by integration in parts from function in integral derivatives against κ_i and a, we receive this function in this form

$$F = ab + \sum_{i=1}^{n-1}(\kappa_i + ak_i)b_{i,i+1}, \tag{13}$$

where

$$b_{i,i+1} = \sum_{h=0}^{r}(-1)^h f_{ih}^h, \tag{14}$$

$$b = f - \sum_{i=1}^{n-1}\left[k_i b_{i,i+1} + \sum_{h=1}^{r} k_i^{(h)} \sum_{g=0}^{r-h}(-1)^g f_{i,h+g}^g\right], \tag{15}$$

and where

$$f_{ih}^g = \frac{d^g}{ds^g}\left[\frac{\partial f}{\partial k_i^{(h)}}\right].$$

Further transformation of F by integration in parts turns into selection from F full derivations against s of suitable functions in order to remove all derivatives of initial variations u_i. As a result we should receive linear homogeneous form of all u_i. By equating all coefficients of this form to zero we should receive desired equations of Euler-Lagrange.
In order to perform specified transformation, let us consider matrix $B = \|b_{ij}\|_1^n$, determined by the following features:
B – antisymmetric; elements of first over-diagonal have values (14); the following relationship holds

$$B' + BK - KB = -G, \tag{16}$$

where $G = \|g_{ij}\|_1^n$ – matrix with elements distinct from zero only these

$$g_{1j} = g_j; g_{j1} = -g_j (j = 2, 3, ..., n).$$

These features determine matrices B and G unambiguously. Truly, we have $\frac{n(n-1)}{2}$ scalar equations, expressing equality of elements of antisymmetric matrices of both parts of (16):

$$b'_{ij} + k_{i-1}b_{i-1,j} - k_i b_{i+1,j} + k_{j-1}b_{i,j-1} - k_j b_{i,j+1} = -\delta_{1j}g_i,$$

where

$$i = 1, 2, ..., n-1; j = i+1, i+2, ..., n.$$

Taking into account

$$b_{ii} = 0, k_n = 0,$$

we get following order of determination of desired values, based on known elements (14) of first over-diagonal of matrix B.

$$\left\{\begin{array}{c} k_{n-2}b_{n-2,n} = -b'_{n-1,n}, \\ k_{i-1}b_{i-1,i+1} = -b'_{i,i+1} + k_{i+1}b_{i,i+2} \\ (i = n-2, n-3, ..., 2), \\ k_i b_{ij} = -b'_{i+1,j} + k_{i+1}b_{i+2,j} - k_{j-1}b_{i+1,j-1} + k_j b_{i+1,j+1} \\ (i = n-3, n-4, ..., 1; j = i+3, i+4, , .., n). \end{array}\right. \tag{17}$$

At last

$$\left\{\begin{array}{c} g_2 = -b'_{12} + k_2 b_{13}, \\ g_i = -b'_{1j} + k_1 b_{2j} - k_{j-1}b_{1,j-1} + k_j b_{1,j+1} \\ (j = 3, 4, ..., n). \end{array}\right. \tag{18}$$

At $j=n$ last members of right part drop, containing multiple $k_n=0$.
Let us determine product of two matrices, e.g., A and B by use of expression

$$A\times B=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}b_{ij}.$$

Because each of multiples is linear and homogeneous with respect to its elements, this product is additive with respect to each multiple. Due to this, we get by differentiation

$$(A\times B)'=A'\times B+A\times B'. \tag{19}$$

Using explicit expression for scalar product, it is easy to check that because of antisymmetry of K

$$\begin{aligned}(AK)\times B+A\times(BK)=0,\\ (KA)\times B+A\times(KB)=0,\end{aligned} \tag{20}$$

where brackets contain ordinary product of matrices. By substitution in the right part of (19) values of derivatives A' and B' from expressions (8) and (16), and taking into account distributivity of scalar product and expressions (20), we get

$$(L+aK)\times B=(A\times B)'+A\times G,$$

or

$$\sum_{i=1}^{n-1}(\kappa_i+ak_i)b_{i,i+1}=(A\times B)'+\sum_{i=2}^{n}a_{1i}g_i.$$

Then (13) gives

$$F=(A\times B)'+ab+\sum_{i=2}^{n}a_{1i}g_i,$$

or, taking into account values (5) and (6) and assuming

$$g_1=b,$$

$$F=(A\times B)'+\sum_{i=1}^{n}(u_ig_i)'-\sum_{i=1}^{n}u_i(g_i'+k_{i-1}g_{i-1}-k_ig_{i+1}).$$

As it was notified higher, equations of Euler-Lagrange are received by equating to zero coefficients of individual u_i in the last sum of right side, that gives

$$\begin{cases} g_1'=k_1g_2,\\ g_i'=-k_{i-1}g_{i-1}+k_ig_{i+1}\\ (i=2,3,...,n-1),\\ g_n'=-k_{n-1}g_{n-1}.\end{cases} \tag{21}$$

Due to the fact that u_1 characterize movement along curve (M), that can't cause variation of integral J, first of the equations (21) should hold identically, what is the case. Really, by differentiating values (15) at $g_1=b$, we get

$$g_1'=-\sum_{i=1}^{n-1}k_ib_{i,i+1}'.$$

On the other hand, multiplying from equations (17) and (18) these that contain $b_{i,i+1}$ $(i = 1, 2, ..., n-1)$ with corresponding k_i and summing, after simplification we get

$$k_1 g_2 = -\sum_{i=1}^{n-1} k_i b'_{i,i+1} = g'_1.$$

Remaining $n-1$ equations (21) comprise system of Euler-Lagrange, determining $n-1$ curvatures of extremals that was sought for.

3 ABOUT INTEGRATION OF EQUATIONS OF THE PROBLEM

Let us assume that there exists solution of the system (21); in this case set of values k_i, $b_{ij}$and g_i, satisfying set of equations (17), (18) and (21), exists. Along with consideration of finding of finite equations of curve (M), we will find first integrals of equations (21) too. These expressions must contain k_i, b_{ij} and g_i, but their values will not depend from s.

Integration of natural equations of curve in R_n with curvatures $k_1, k_2, ..., k_{n-1}$ reduces to finding of n independent systems of solutions $x_1, x_2, ..., x_n$ of equations of Frenet

$$\begin{gathered} x'_i = -k_{i-1}x_{i-1} + k_i x_{i+1} \\ (i = 1, 2, ..., n; k_0 = k_n = 0) \end{gathered} \tag{22}$$

If vector using such system of values x_i is built

$$x = \sum_{i=1}^{n} x_i t_i, \tag{23}$$

then this vector must be fixed, i.e., its derivative must be equal to zero. Backwards, if representation (23) exists for a fixed nonzero vector x, then x_i – system of solutions of equations of Frenet (22). $x_1, x_2, ..., x_n$ may be called relative coordinates of vector x, while its length – first integral of system (22). Such fixed vector is g.

Taking n fixed vectors corresponding to n linear independent solutions of the system (22), we may build orthonormal system of fixed vectors

$$t_1 = \sum_{i=1}^{n} \xi_{i1} Z_i,$$

and by integration we get radius vector of curve (M) in the system Z_i.

If functions x_i are known at which vector (23) is not fixed, but are in some fixed plane (π), then by one quadrature we get two linear independent fixed vectors in this plane.

Vector x may be normalized. In plane (π) we find unit vector y orthogonal to x. Then expressions hold

$$x' = \alpha y,$$

$$y' = -\alpha x,$$

where $\alpha = x'y$ is known. We compute function

$$\varphi = \varphi_0 - \int_{s_0}^{s} \alpha ds,$$

where $\varphi_0 = \text{const}$. Then vectors

$$z_1 = xcos\varphi + ysin\varphi,$$

$$z_2 = -xsin\varphi + ycos\varphi$$

are fixed and orthonormal.
In order to find vectors that are fixed or are in the fixed plane, "fixed" tensors may be used, i.e., tensors with constant coordinates in fixed system of coordinates. Vectors and planes, uniquely determined by such tensors, will be fixed too. But, if relative coordinates of tensor are known, i.e., its coordinates with respect to orthonormal frame of the desired curve, by use of them relative coordinates of desired vector may be found, consequently, one or two systems of solutions (22) may be found.
Considering tenson of second order B to be obtained by general product of two fixed vectors, we may get condition of "immobility" or Frenet's formulae for tensors. In matrix notation, these formulae have view of (16), where $G = 0$, but in coordinate denotation – view of last equation of (17), taken for all i, j.
Obviously, matrix or tensor B, considered in previous paragraph, is not fixed, because G must differ from zero. But, combining tensor B with any fixed vector x in way of maltiplication and alternation, as a result we will get fixed tensor of third order D, if the used operation applied to tensors D' and D is the same as in case of combination of fixed tensor with vector x, consequently, corresponding formulae of Frenet will be satisfied.
As x we take$g$, and alternate it with respect to all indices. Dropping denominator 3, we get expression for elements of D

$$d_{ijk} = g_i b_{jk} + g_j b_{ki} + g_k b_{ij}, \tag{24}$$

and tensor G by this turns into zero tenson.
The obtained antisymmetric tensor D in general case is not zero tensor. We will show that in R_3, R_4 and R_5 with the solution of system (21) known determination of curve (M) is completed by quadratures.

3.1 Curve in R_3.

First integrals (21) – length of vector g and pseudoscalar – the only significant coordinate of D. After normalization of g, as vectors x and y in the fixed plane perpendicular to g may be taken

$$x = \left(g_2^2 + g_3^2\right)^{-\frac{1}{2}} \left(g_3 t_2 - g_2 t_3\right),$$

$$y = \left(g_2^2 + g_3^2\right)^{-\frac{1}{2}} \left[-\left(g_2^2 + g_3^2\right) t_1 + g_1 g_2 t_2 + g_1 g_3 t_3\right].$$

Then

$$\alpha = \frac{k_1 g_3}{g_2^2 + g_3^2}$$

and by use of one quadrature we get orthonormal system of fixed vectors g, z_1, z_2.

3.2 Curve in R_4.

Tensor V, additional to D [comp. 3] – of first order, consequently, equivalent to pseudovector. So, fixed will be vector v with coordinates

$$v_h = g_i b_{jk} + g_j b_{ki} + g_k b_{ij},$$

where h, i, j, k – even permutation of indices $1, 2, 3, 4$. In general, v isn't zero vector; its length - second first integral.

Obviously, v and g are orthogonal. Any vector x, perpendicular both to v and g, is parallel to fixed plane and, in accordance with argued higher, allow to determine two fixed orthogonal vectors z_1 and z_2. The last two, together with normalized g and v, are giving fixed frame.

As in the case of R_3, determination of t_1 requires one quadrature.

3.3 Curve in R_5.

We have fixed antisymmetric tensor V of second order, additional at D, with coordinates

$$v_{hm} = g_i b_{jk} + g_j b_{ki} + gk b_{ij},$$

where h, m, i, j, k – even permutation of indices $1, 2, 3, 4, 5$. Vector g – characteristic vector of this tensor, corresponding to zero characteristic number. Remaining characteristic numbers pure imaginar and pairwise conjugate, in general distinct and differing from zero. Each pair of conjugate characteristic numbers has some two dimensional plane in correspondence, and besides, these planes are perpendicular both between themselves and to vector g [comp. 4]. By determining in each of these planes pair of fixed orthonormal vectors, we by use of two quadratures get representation of t_1 in fixed frame.

Product of tensor v_{hm} to itself, alternate against three indices, gives antisymmetric tensor of forth order, consequently, pseudovector too. However, the last is collinear to g, so that in place of new fixed vector we get only first integral – ratio of coordinates of both vectors. We receive that by use of expression

$$\nu = \frac{v_{12}v_{34} - v_{13}v_{24} + v_{14}v_{23}}{g_5},$$

or expressions, received by any even permutation of indices in the right side. Besides length of g, we have also two first integrals, e.g., coefficients of characteristic equation. Coefficient at λ^3 equal to $\sum_{j>i} v_{ij}^2$; coefficient at λ equal to

$\nu^2 g^2$. Consequently, independent are only first three integrals.
As a simple example that show distinction from zero of two magnitudes considered in R_5 and R_4, let us consider determination of extremals for function

$$f = k_2 + c, c = const.$$

In R_5, setting $\eta = k_1^{-1} k_3$,

$$g = ct_1 + k_1(1-\eta^2)t_3 + \eta' t_4 + \eta k_4 t_5,$$

and V has following distinct from zero coordinates with $j > i$:

$$v_{14} = \eta k_4, v_{15} = \eta', v_{23} = -\eta^2 k_4,$$

$$v_{25} = \eta k_1(1-\eta^2), v_{45} = c.$$

Here $\nu = -\eta^2 k_4$; g^2 and $\sum v_{ij}^2$ contain $(\eta')^2$, but their difference equal to

$$k_1^2(1-\eta^2)^3 - \nu^2,$$

consequently, three first integrals really are independent.
In R_4 we have

$$g = ct_1 + k_1(1-\eta^2)t_3 + \eta' t_4,$$

$$v = \eta' t_1 - \eta k_1(1-\eta^2)t_2 - ct_4;$$

these vectors are distinct from zero and perpendicular.

4 ABOUT INTEGRATION OF EQUATIONS OF EULER-LAGRANGE IN R_3

Let us clarify the type of operations that are actually necessary for full solution of problem in case

$$f = f(k_1, k_2).$$

To shorten the record, assuming

$$k_1 = x; k_2 = y, \frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = q,$$

we have

$$g_1 = f - xp - yq,$$

$$g_2 = -p' - \frac{y}{x}q',$$

$$g_3 = -\left(\frac{q'}{x}\right)' + xq - yp,$$

$$b_{12} = p, b_{31} = \frac{q'}{x}, b_{23} = q.$$

Solutions of problem satisfy equations, given by first integrals:

$$\begin{cases} g_1^2 + g_2^2 + g_3^2 = const, \\ g_1 b_{23} + g_2 b_{31} + g_3 b_{12} = c_2 = const. \end{cases} \tag{25}$$

Let x and y to be expressible as functions of p and q and the last not fixed. Then, solving system (25) with respect to g_2 and g_3, we obtain expression of view

$$\begin{cases} p' = F_1(p, q', q), \\ q'' = F_2(p, q', q). \end{cases} \tag{26}$$

Taking q as independent variable and introducing new unknown $z = q'$, system (26) may be transformed in canonical system of second order:

$$\frac{dp}{dq} = \frac{1}{z} F_1(p, z, q),$$

$$\frac{dz}{dq} = \frac{1}{z} F_2(p, z, q).$$

As a result of the solution of this system, s is received as a function of q by use of quadrature

$$s = s_0 + \int_{q_0}^{q} \frac{dq}{z}.$$

Solving corresponding equations, we obtain q, p, x and y as functions of s, and complete determination of solution by use of quadrature, as notified in the previous paragraph.

Considered technique is not suitable in case p and q are not independent, i.e., expression below holds

$$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 = 0. \tag{27}$$

In the latter case surface, to be determined by equation

$$f = f(x, y) \tag{28}$$

in cartesian coordinates x, y, f, is unfolding. If q is not fixed, then at independent variables x and q (28) allow following representation

$$\begin{cases} f = (\varphi - q\dot{\varphi})x + \psi - q\dot{\psi}, \\ y = -x\dot{\varphi} - \dot{\psi}, \\ p = \varphi, \end{cases}$$

where φ and ψ – suitable functions of $\varphi_1(q)$, but point is denoting differentiation with respect to q. Excluding g_3 from system (25), we obtain equation

$$\varphi^2 \left[\psi^2 + \left(\frac{\dot{\psi} q'}{x} \right)^2 - c_1 \right] + \left[\psi q + \dot{\psi} \left(\frac{q'}{x} \right)^2 - c_2 \right]^2 = 0,$$

that gives

$$x = \varphi_1(q) q',$$

with certain/determined function $\varphi_1(q)$. Substituting this value x in any (e.g., second) equation (25), we obtain expression containing only q' and known functions of q. After representing q' of view

$$q' = \psi_1(q),$$

s is received by quadrature, and solution is completed as higher.
If $q = q_0 =$ const., i.e.,

$$f = q_0 y + \psi(x),$$

we have

$$g_1 = \psi - x\dot{\psi}, g_2 = -\ddot{\psi}x', g_3 = xq_0 - y\dot{\psi},$$

$$b_{12} = \dot{\psi}, b_{31} = 0, b_{23} = q_0;$$

excluding g_3, we obtain equation, giving x' having view as a known function of x. As before, s is obtained by quadrature, but y is determined by any equations of (25). Functions f of that type at $q_0 = 0$ are considered in [1].
It should be noted that as opposed to the usual parametric variation tasks in our case solutions

$$x = const., y = const.$$

are allowed.
In this case, the constancy of the first integrals (25) is not yet a sufficient condition for the solution. Returning to differential equations (17), (18) and (21), we see that there must be a relationship

$$(x^2 - y^2)p + 2xyp - xf = 0. \tag{29}$$

For arbitrary functions f, this condition sets relationship between x and y, i.e., it defines a one-parameter family of natural equations of ordinary screw curves, which are extremals. If $f(x, y)$ looks like

$$f = c_0 \frac{y^2}{(\sqrt{x^2+y^2})^2} \left[\ln \frac{\sqrt{x^2+y^2} - x}{y} - \frac{x\sqrt{x^2+y^2}}{y^2} \right] + \frac{y}{\sqrt{x^2+y^2}} f_1 \left(\frac{x^2+y^2}{y} \right), \tag{30}$$

where c_0 – constant, but f_1 – arbitrary function of its argument, then left part of (29) identically equal to $2c_0$. If $c_0 = 0$, any ordinary screw curve is extremal. But if $c_0 \neq 0$, among extremals there is no such line.

1 VI 1957

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References

[1] W. Blaschke. Vorlesungen über Differentialgeometrie I. §§ 26, 27, Berlin, 1930.

[2] A. de Castro Brzezicki. Curvas extremales del problems variacional $\delta \int f(x, x')ds = 0$, Rev. mat. hisp.-amer. 1955, 15, Nr 3-4, 71-78.

[3] P. K. Rashevski, Geometric theory of equations in partial derivatives, in Russian, § 9, Gostehizdat, 1947.

[4] F. R. Gantmacher, Theory of matrices, in Russian, IX, § 13, Gostehizdat, 1954.

Abstract
PAR KĀDU ĢEOMETRISKU VARIĀCIJU PROBLĒMU
Rakstā aplūkota problēma: n dimensiju Eiklīda telpā R_n noteikt līknes, kam integrāļa

$$J = \int f ds$$

pirmā variācija ir nulle; s ir līknes loka garums, f – dota liekumu $k_1, k_2, ..., k_{n-1}$ un to atvasinājumu pēc s funkcija.
Ņemot variētās līknes radijvektoru veidā (2), sakari (5) un (6) dod loka diferenciāļa un pieskares vienības vektora variācijas: ar (9) un (10) ir nosakāmas liekumu k_i variācijas κ_i. Liekumu atvasinājumu variācijas dod (12).
Noteikumi, lai J pirmā variācija katriem u_i ir nulle, ir izsakāmi sekojošā veidā: ar sakariem (14), (15), (17), (18) aprēķina lielumus g_i, kam jāpilda noteikumi (21) (pirmais no tiem ir identitāte). Ja ir zināms šo vienādojumu atrisinājums, telpās R_3, R_4 un R_5 atbilstošās līknes vienādojumi ir nosakāmi ar kvadratūrām; reizē iegūti daži pirmintegrāļi.
Telpā R_3, ja f ir dota liekuma x un vērpes y funkcija, pilns problēmas atrisinājums ir reducējams uz kanonisku otrās kārtas diferenciālvienādojumu sistēmu, pēc kuras atrisināšanas jāizdara vēl kvadratūras. Ja pastāv noteikums (27), viss atrisinājums ir iegūstams ar kvadratūrām. Ekstremāļu starpā ir parastās skrūves līnijas, kam x un y saista (29). Ja f ir ar veidu (30), gadījumā $c_0 = 0$ katra parastā skrūves līnija ir ekstremāle, bet gadījumā $c_0 \neq 0$ – neviena.