Proton Charge Radius Distortion in Dirac Hydrogen by "Electron Zitterbewegung"

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Abstract

The commutator of the Dirac free-particle's velocity operator with its Hamiltonian operator is nonzero and independent of Planck's constant, which starkly violates the quantum correspondence-principle requirement that commutators of observables must vanish when Planck's constant vanishes, as well as violating the extended Newton's First Law principle that relativistic free particles do not accelerate. The consequent nonphysical particle "zitterbewegung" is of course absent altogether when the natural relativistic free-particle square-root Hamiltonian operator, which transparently follows from the free particle's Lorentz-covariant energy-momentum, replaces the free-particle Dirac Hamiltonian. The energy spectrum of the pathology-free relativistic square-root free-particle Hamiltonian is, however, matched perfectly by the positive-energy sector of the Dirac free-particle Hamiltonian's energy spectrum. But when a hydrogen type of potential energy is added to the free particle Dirac Hamiltonian, Foldy-Wouthuysen unitary transformation of the result reveals a "Darwin term" in its positive-energy sector which stems from nonphysical "zitterbewegung"-smearing of that potential energy. This physically nonexistent smearing of the potential energy can alternatively be viewed as having been produced by physically nonexistent smearing of its proton charge density source, which using the Dirac theory for data analysis erroneously compensates, resulting in a misleadingly contracted impression of the proton's charge radius.

Dirac kinematics: motion pathology, but positive-energy spectrum accuracy

The *Lorentz-covariant* free-particle energy-momentum $(\mathbf{p}^\mu c) = ((m^2c^4 + |\mathbf{p}\,c|^2)^{\frac{1}{2}},\ \mathbf{p}\,c)$ *naturally implies*,

$$\widehat{H} = \left(m^2c^4 + |\widehat{\mathbf{p}}\,c|^2\right)^{\frac{1}{2}}. \tag{1a}$$

The *velocity operator* corresponding to this free-particle relativistic square-root Hamiltonian operator is,

$$(d\widehat{\mathbf{r}}/dt) = (-i/\hbar)[\widehat{\mathbf{r}},\, \widehat{H}] = (-i/\hbar)\big[\widehat{\mathbf{r}},\, \left(m^2c^4 + |\widehat{\mathbf{p}}\,c|^2\right)^{\frac{1}{2}}\big] = \nabla_{\widehat{\mathbf{p}}}\left(m^2c^4 + |\widehat{\mathbf{p}}\,c|^2\right)^{\frac{1}{2}} = \widehat{\mathbf{p}}\,c\left(m^2c^2 + |\widehat{\mathbf{p}}|^2\right)^{-\frac{1}{2}}, \tag{1b}$$

and the *acceleration operator* corresponding to $\widehat{H} = (m^2c^4 + |\widehat{\mathbf{p}}\,c|^2)^{\frac{1}{2}}$ consequently *vanishes*,

$$(d^2\widehat{\mathbf{r}}/dt^2) = (-i/\hbar)\big[(d\widehat{\mathbf{r}}/dt),\, \widehat{H}\big] = (-i/\hbar)\big[\widehat{\mathbf{p}}\,c\big(m^2c^2 + |\widehat{\mathbf{p}}|^2\big)^{-\frac{1}{2}},\, \big(m^2c^4 + |\widehat{\mathbf{p}}\,c|^2\big)^{\frac{1}{2}}\big] = \mathbf{0}, \tag{1c}$$

which is consistent *with the absence of acceleration of relativistic free particles*, i.e., Newton's First Law *remains valid in special relativity*. Two *crucial relativistic characteristics* of the Eq. (1b) free-particle velocity $(d\widehat{\mathbf{r}}/dt)$ are (1) that *its magnitude is less than c*,

$$|d\widehat{\mathbf{r}}/dt| = |\widehat{\mathbf{p}}|c\left(m^2c^2 + |\widehat{\mathbf{p}}|^2\right)^{-\frac{1}{2}} < c, \tag{1d}$$

and (2) that its asymptotic form for $|\widehat{\mathbf{p}}| \ll mc$ *is Newtonian*, i.e.,

$$(d\widehat{\mathbf{r}}/dt) \sim (\widehat{\mathbf{p}}/m) \text{ as } \widehat{\mathbf{p}} \to \mathbf{0}, \tag{1e}$$

which is *echoed* for $|\widehat{\mathbf{p}}| \ll mc$ *by the Newtonian asymptotic form of the kinetic-energy part of* $\widehat{H}$,

$$\big(\widehat{H} - mc^2\big) = mc^2\big(\big(1 + |\widehat{\mathbf{p}}/(mc)|^2\big)^{\frac{1}{2}} - 1\big) \sim \big(|\widehat{\mathbf{p}}|^2/(2m)\big) \text{ as } \widehat{\mathbf{p}} \to \mathbf{0}. \tag{1f}$$

However the *Dirac* relativistic free-particle quantum Hamiltonian operator $\widehat{H}_D$, which is given by,

$$\widehat{H}_D = (\beta mc + \vec{\alpha}\cdot\widehat{\mathbf{p}})c, \tag{2a}$$

flouts the Newtonian asymptotic form of its kinetic-energy part because $(\widehat{H}_D - mc^2) = ((\beta - 1)mc + \vec{\alpha}\cdot\widehat{\mathbf{p}})c$. The Eq. (2a) Dirac $\widehat{H}_D$ *as well flouts* the Eq. (1e) *velocity's* Newtonian asymptotic form because,

$$(d\widehat{\mathbf{r}}/dt) = (-i/\hbar)\big[\widehat{\mathbf{r}},\, \widehat{H}_D\big] = (-i/\hbar)\big[\widehat{\mathbf{r}},\, (\beta mc + \vec{\alpha}\cdot\widehat{\mathbf{p}})c\big] = \big(\nabla_{\widehat{\mathbf{p}}}(\beta mc + \vec{\alpha}\cdot\widehat{\mathbf{p}})\big)c = \vec{\alpha}c, \tag{2b}$$

which is *independent* of $\widehat{\mathbf{p}}$. Eq. (2b) *also violates* Eq. (1d) *in a physically wholly unacceptable way because*,

$$|d\widehat{\mathbf{r}}/dt| = |\vec{\alpha}|c = \big((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2\big)^{\frac{1}{2}}c = (1+1+1)^{\frac{1}{2}}c = 3^{\frac{1}{2}}c = 1.732c > c. \tag{2c}$$

That the Eq. (2a) Dirac $\widehat{H}_D$ *is physically wholly unacceptable* is in addition *confirmed by the fact that*,

$$\big[(d\widehat{\mathbf{r}}/dt),\, \widehat{H}_D\big] = c\big(\vec{\alpha}\widehat{H}_D - \widehat{H}_D\vec{\alpha}\big) = c^2\big(2\vec{\alpha}\beta mc + \vec{\alpha}(\vec{\alpha}\cdot\widehat{\mathbf{p}}) - (\vec{\alpha}\cdot\widehat{\mathbf{p}})\vec{\alpha}\big) = c^2\big(2\vec{\alpha}\beta mc - (\vec{\alpha}\times\vec{\alpha})\times\widehat{\mathbf{p}}\big), \tag{2d}$$

is nonzero, yet is independent of $\hbar$, which *flatly violates* the quantum correspondence-principle requirement *that commutators of observables such as* $(d\widehat{\mathbf{r}}/dt)$ *and* $\widehat{H}_D$ *must vanish when* $\hbar \to 0$. The *related fact* that,

$$(d^2\widehat{\mathbf{r}}/dt^2) = (-i/\hbar)\big[(d\widehat{\mathbf{r}}/dt),\, \widehat{H}_D\big] = (-i/\hbar c^2)\big(2\vec{\alpha}\beta mc - (\vec{\alpha}\times\vec{\alpha})\times\widehat{\mathbf{p}}\big) \neq \mathbf{0}, \tag{2e}$$

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violates the Eq. (1c) *extension of Newton's First Law to relativistic free particles*. The *magnitude* $|d^2\widehat{\mathbf{r}}/dt^2|$ of *the physically nonexistent* Eq. (2e) *"zitterbewegung" acceleration* $(d^2\widehat{\mathbf{r}}/dt^2)$ of a Dirac free particle can be worked out by combining $(d^2\widehat{\mathbf{r}}/dt^2) = (-ic/\hbar)(\vec{\alpha}\widehat{H}_D - \widehat{H}_D\vec{\alpha})$ with the two Dirac identities,

$$\vec{\alpha}\widehat{H}_D + \widehat{H}_D\vec{\alpha} = 2\widehat{\mathbf{p}}\,c \quad \text{and} \quad (\vec{\alpha}\cdot\vec{\alpha}) = 3, \tag{2f}$$

and Dirac's signature equality,

$$(\widehat{H}_D)^2 = \widehat{H}^2 = \big(m^2c^4 + |\widehat{\mathbf{p}}\,c|^2\big), \tag{2g}$$

to obtain *the nonphysical* time-independent result,

$$\begin{gathered}|d^2\widehat{\mathbf{r}}/dt^2| = \big((-ic/\hbar)^2(\vec{\alpha}\widehat{H}_D - \widehat{H}_D\vec{\alpha})\cdot(\vec{\alpha}\widehat{H}_D - \widehat{H}_D\vec{\alpha})\big)^{\frac{1}{2}} = \\ \big(-(c/\hbar)^2(-12(\widehat{H})^2 + 2(\widehat{\mathbf{p}}\,c)\cdot(\vec{\alpha}\widehat{H}_D + \widehat{H}_D\vec{\alpha}))\big)^{\frac{1}{2}} = \big(2c/\hbar\big)\big(3m^2c^4 + 2|\widehat{\mathbf{p}}\,c|^2\big)^{\frac{1}{2}} > 0.\end{gathered} \tag{2h}$$

The time-independent angular frequency ω and isotropic root-mean-square spatial deviation $\langle|\delta\widehat{\mathbf{r}}|^2\rangle^{\frac{1}{2}}$ of *the physically nonexistent* Dirac free-particle "zitterbewegung" can be obtained from *the nonphysical* time-independent Eq. (2h) result for $|d^2\widehat{\mathbf{r}}/dt^2|$ together with *the corresponding nonphysical result for* $|d^3\widehat{\mathbf{r}}/dt^3|$. By combining $(d^2\widehat{\mathbf{r}}/dt^2) = (-ic/\hbar)(\vec{\alpha}\widehat{H}_D - \widehat{H}_D\vec{\alpha})$ with Eqs. (2g) and (2f) we obtain that,

$$\begin{gathered}d^3\widehat{\mathbf{r}}/dt^3 = (-i/\hbar)\big[d^2\widehat{\mathbf{r}}/dt^2,\, \widehat{H}_D\big] = -\big(c/\hbar^2\big)\big[(\vec{\alpha}\widehat{H}_D - \widehat{H}_D\vec{\alpha}),\, \widehat{H}_D\big] = \\ \big(2c/\hbar^2\big)\big(\widehat{H}_D\vec{\alpha}\widehat{H}_D - \vec{\alpha}\widehat{H}^2\big) = \big(4c/\hbar^2\big)\big((\widehat{\mathbf{p}}\,c)\widehat{H}_D - \vec{\alpha}\widehat{H}^2\big) \neq \mathbf{0}.\end{gathered} \tag{2i}$$

This together with Eqs. (2f) and (2g) yields *the nonphysical* time-independent result,

$$\begin{gathered}|d^3\widehat{\mathbf{r}}/dt^3| = \big(4c/\hbar^2\big)\big(((\widehat{\mathbf{p}}\,c)\widehat{H}_D - \vec{\alpha}\widehat{H}^2)\cdot((\widehat{\mathbf{p}}\,c)\widehat{H}_D - \vec{\alpha}\widehat{H}^2)\big)^{\frac{1}{2}} = \\ \big(4c\widehat{H}/\hbar^2\big)\big(|\widehat{\mathbf{p}}\,c|^2 + 3\widehat{H}^2 - (\widehat{\mathbf{p}}\,c)\cdot(\widehat{H}_D\vec{\alpha} + \vec{\alpha}\widehat{H}_D)\big)^{\frac{1}{2}} = \big(4c\widehat{H}/\hbar^2\big)\big(3m^2c^4 + 2|\widehat{\mathbf{p}}\,c|^2\big)^{\frac{1}{2}} > 0,\end{gathered} \tag{2j}$$

which together with the $|d^2\widehat{\mathbf{r}}/dt^2|$ of Eq. (2h) determines the time-independent angular frequency ω of *the physically nonexistent* Dirac free-particle "zitterbewegung",

$$\omega = \big(|d^3\widehat{\mathbf{r}}/dt^3|/|d^2\widehat{\mathbf{r}}/dt^2|\big) = \big(2\widehat{H}/\hbar\big) = \big(2mc^2/\hbar\big)\big[(1 + |\widehat{\mathbf{p}}/(mc)|^2)^{\frac{1}{2}}\big], \tag{2k}$$

as well as *physically-nonexistent* "zitterbewegung's" isotropic root-mean-square spatial deviation $\langle|\delta\widehat{\mathbf{r}}|^2\rangle^{\frac{1}{2}}$,

$$\begin{gathered}\big\langle|\delta\widehat{\mathbf{r}}|^2\big\rangle^{\frac{1}{2}} = \big(|d^2\widehat{\mathbf{r}}/dt^2|^3/|d^3\widehat{\mathbf{r}}/dt^3|^2\big) = \big(|d^2\widehat{\mathbf{r}}/dt^2|/\omega^2\big) = \big(|d^3\widehat{\mathbf{r}}/dt^3|/\omega^3\big) = \\ \hbar c\big(3m^2c^4 + 2|\widehat{\mathbf{p}}\,c|^2\big)^{\frac{1}{2}}/\big(2\widehat{H}^2\big) = \big(3^{\frac{1}{2}}/2\big)\big(\hbar/(mc)\big)\big[(1 + (2/3)|\widehat{\mathbf{p}}/(mc)|^2)^{\frac{1}{2}}/(1 + |\widehat{\mathbf{p}}/(mc)|^2)\big].\end{gathered} \tag{2l}$$

Because of Eq. (2g), there is a *simple* complete set of orthogonal eigenprojectors $P_D^{\pm}$ for the Dirac $\widehat{H}_D$,

$$P_D^{\pm} \stackrel{\text{def}}{=} \tfrac{1}{2}\big(1 \pm \widehat{H}_D\widehat{H}^{-1}\big),\ \big(P_D^{\pm}\big)^2 = P_D^{\pm},\ P_D^{\pm}P_D^{\mp} = 0,\ \big(P_D^{+} + P_D^{-}\big) = 1 \text{ and } \widehat{H}_D\big(P_D^{\pm}\big) = \pm\widehat{H}\big(P_D^{\pm}\big). \tag{2m}$$

The last two Eq. (2m) properties of the eigenprojectors $P_D^{\pm}$ yield *the spectral decomposition of* $\widehat{H}_D$,

$$\widehat{H}_D = \widehat{H}_D\left(P_D^{+} + P_D^{-}\right) = \widehat{H}\left(P_D^{+}\right) - \widehat{H}\left(P_D^{-}\right), \tag{2n}$$

which reveals that *although the spectrum of* $\widehat{H}_D$ *differs starkly from the spectrum of* $\widehat{H}$ in that it has *a nonphysical negative-energy sector* entirely alien to the spectrum of $\widehat{H}$, *the positive-energy sector of the spectrum of* $\widehat{H}_D$ *exactly matches the full spectrum of* $\widehat{H}$. Thus *the physically unacceptable repercussions of the Dirac free-particle Hamiltonian* $\widehat{H}_D$ *revealed by* Eqs. (2b)–(2e), (2h)–(2j) *and* (2n) *can be finessed by eschewing investigation of any consequences of* $\widehat{H}_D$ *other than the positive-energy sector of its spectrum*!

We next wish to ascertain *the extent to which a hydrogen-type Dirac Hamiltonian*, namely $(\widehat{H}_D + eA^0)$, can be expected *to likewise yield positive-energy sector spectrum results that are physically correct*. Just as we *checked the positive-energy sector of the spectrum of the physically wholly unacceptable free-particle Dirac* $\widehat{H}_D$ against *the spectrum of the physically sensible free-particle* $\widehat{H}$, we shall *check the positive-energy sector* *of the unitary Foldy-Wouthuysen transformation of the Dirac* $(\widehat{H}_D + eA^0)$ against *the relativistic upgrade, in the presence of* $A^\mu = (A^0, \mathbf{0})$*, of the nonrelativistic Pauli Hamiltonian*. That relativistic upgrade will be carried out in several steps: The Lagrangian and action functional which correspond to the nonrelativistic Pauli Hamiltonian are worked out, and then that action is *specialized* to the frame *where the particle is instantaneously at rest*, because *in that frame the nonrelativistic description of the particle's dynamics is instantaneously identical to its relativistic description*. The *Lorentz transformation of* $A^\mu = (A^0, \mathbf{0})$ *to the particle's instantaneous rest frame* is inserted into the specialized Pauli action for that frame. This action is then Lorentz transformed to the frame where $A^\mu = (A^0, \mathbf{0})$, *which is greatly facilitated by the fact that the relativistic action is Lorentz-invariant*, although its arguments *aren't* Lorentz-invariant. The *integrand* of the result *is the relativistic Pauli Lagrangian in the presence of* $A^\mu = (A^0, \mathbf{0})$, from which *the corresponding relativistic Pauli Hamiltonian* is worked out in the usual way. The *conversion* of a Lagrangian to a Hamiltonian, however, *always entails the solution of an algebraic equation*, and *if we had been dealing with a case where* $\mathbf{A} \neq \mathbf{0}$*, it wouldn't have been possible to solve that algebraic equation in closed form*, although a satisfactory *successive approximation scheme* apparently *exists*. The *mathematical details* of relativistically upgrading the nonrelativistic Pauli Hamiltonian in the way outlined here *are presented in the next section*, together with mention of the "Thomas precession" subtlety of the Lorentz transformation of a particle *with spin* which is continuously subject *to centripetal acceleration*, as well as "the Thomas half" rule of thumb which applies to that situation.

Both the nonrelativistic Pauli Hamiltonian and its relativistic upgrade in the presence of $A^\mu = (A^0, \mathbf{0})$ are 2×2 objects which involve only scalars and the three 2×2 Pauli spin matrices $\vec{\sigma}$. On the other hand, the hydrogen-type Dirac Hamiltonian $(\widehat{H}_D + eA^0)$ involves the scalar (eA^0) and the four 4×4 matrices $\vec{\alpha}$ and β. Its *unitary* Foldy-Wouthuysen transformation, whose eigenvalue spectrum *is identical to that of* $(\widehat{H}_D + eA^0)$ *itself*, is specifically designed *to eliminate all of the dependence of* $(\widehat{H}_D + eA^0)$ *on* $\vec{\alpha}$ in favor of dependence on $(-i/2)(\vec{\alpha} \times \vec{\alpha})$, *whose three components satisfy exactly the same algebraic relations as are satisfied by the three components of* $\vec{\sigma}$, as well as dependence on the matrix β, *which commutes with* $(-i/2)(\vec{\alpha} \times \vec{\alpha})$ (β *anticommutes* with $\vec{\alpha}$), and of course dependence on scalars [1]. Selecting the $+1$ *eigenvalue of* β in this unitary Foldy-Wouthuysen transformation of $(\widehat{H}_D + eA^0)$ *selects its positive-energy sector*, which *we can then directly compare with our relativistically upgraded Pauli Hamiltonian in the presence of* $A^\mu = (A^0, \mathbf{0})$.

Before we proceed to the details of the relativistic upgrade of the nonrelativistic Pauli Hamiltonian in the presence of $A^\mu = (A^0, \mathbf{0})$ in the *next* section, we conclude *this* section with the instructive *construction* of the Foldy-Wouthuysen transformation *of the free-particle Dirac Hamiltonian* $\widehat{H}_D = (\beta mc + \vec{\alpha} \cdot \widehat{\mathbf{p}})c$. That transformation is *generated* by *a suitably normalized product* of the two anticommuting terms βmc^2 and $(\vec{\alpha} \cdot \widehat{\mathbf{p}})c$ which comprise $\widehat{H}_D$, namely by,

$$\xi \stackrel{\text{def}}{=} (\beta \vec{\alpha} \cdot \widehat{\mathbf{p}} / |\widehat{\mathbf{p}}|)\,, \tag{3a}$$

which has the key properties of anticommuting with $\widehat{H}_D$ *and being anti-Hermitian*—an additional *convenient* property of ξ as it is *normalized* in Eq. (3a) is that its square is equal to -1. Being anti-Hermitian, ξ *generates a family of unitary transformations of* $\widehat{H}_D$, which are parameterized by the angle θ, as follows,

$$\exp(\xi\theta/2)\widehat{H}_D \exp(-\xi\theta/2) = \exp(\xi\theta/2)\exp(\xi\theta/2)\widehat{H}_D = \exp(\xi\theta)\widehat{H}_D = (\cos\theta + \xi\sin\theta)\widehat{H}_D, \tag{3b}$$

where the first equality reflects the fact that $\widehat{H}_D$ *anticommutes with* ξ, and the third equality reflects the fact that the square of ξ is equal to -1. Inserting *the definitions of* ξ *and* $\widehat{H}_D$ into $(\cos\theta + \xi\sin\theta)\widehat{H}_D$ yields,

$$\begin{aligned}(\cos\theta + \xi\sin\theta)\widehat{H}_D &= (\cos\theta + (\beta\vec{\alpha} \cdot \widehat{\mathbf{p}}/|\widehat{\mathbf{p}}|)\sin\theta)\,(\beta mc + \vec{\alpha} \cdot \widehat{\mathbf{p}})\,c = \\ &c\,(|\widehat{\mathbf{p}}|\cos\theta - mc\sin\theta)\,(\vec{\alpha} \cdot \widehat{\mathbf{p}}/|\widehat{\mathbf{p}}|) + c\beta\,(mc\cos\theta + |\widehat{\mathbf{p}}|\sin\theta)\,.\end{aligned} \tag{3c}$$

For the last expression of Eq. (3c) *to be the Foldy-Wouthuysen transformation of* $\widehat{H}_D$, the angle parameter θ must of course *be chosen such that the coefficient of* $(\vec{\alpha} \cdot \widehat{\mathbf{p}}/|\widehat{\mathbf{p}}|)$ *vanishes*. That is case for,

$$\theta = \arctan(|\widehat{\mathbf{p}}/(mc)|) \;\Rightarrow\; \cos\theta = \left(1 + |\widehat{\mathbf{p}}/(mc)|^2\right)^{-\frac{1}{2}} \text{ and } \sin\theta = |\widehat{\mathbf{p}}/(mc)|\left(1 + |\widehat{\mathbf{p}}/(mc)|^2\right)^{-\frac{1}{2}}, \tag{3d}$$

which when inserted into Eq. (3c) reveals the Foldy-Wouthuysen transformation of $\widehat{H}_D$ to be,

$$\beta\big(mc^2 + \left(|\widehat{\mathbf{p}}|^2/m\right)\big)\left(1 + |\widehat{\mathbf{p}}/(mc)|^2\right)^{-\frac{1}{2}} = \beta\left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}. \tag{3e}$$

The Eq. (3e) Foldy-Wouthuysen transformation of $\widehat{H}_D$ has the simple eigenprojector *spectral decomposition*,

$$\beta\left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}} = \left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}((1+\beta)/2) - \left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}((1-\beta)/2), \tag{3f}$$

whose *positive-energy sector* is *selected* by *setting* β *to its* $+1$ *eigenvalue*; it is, of course, $(m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}} = \widehat{H}$.

Check of the Dirac positive-energy sector against the Pauli relativistic upgrade

The nonrelativistic Pauli Hamiltonian is,

$$H = mc^2 + \left(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\right) + eA^0 - (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}), \tag{4a}$$

in which, to facilitate its relativistic upgrading, we included the particle's rest-energy mc^2—this doesn't alter its equations of motion. We obtain the nonrelativistic action $S_{\rm nr}$ corresponding to H from its Lagrangian L, in which the dependence of H on particle canonical momentum $\mathbf{P}$ is swapped for dependence on particle velocity $\dot{\mathbf{r}}$, which is related to $\mathbf{P}$ by the Heisenberg (and classical Hamiltonian) equation of motion,

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H] = \nabla_{\mathbf{P}}H = (\mathbf{P} - (e/c)\mathbf{A})/m. \tag{4b}$$

We next *solve* Eq. (4b) *for* $\mathbf{P}$, and thereby obtain,

$$\mathbf{P} = m\dot{\mathbf{r}} + (e/c)\mathbf{A}, \tag{4c}$$

which we *insert* into the well-known relation of the Lagrangian L to the Hamiltonian H,

$$L = \dot{\mathbf{r}}\cdot\mathbf{P} - H\Big|_{\mathbf{P}=m\dot{\mathbf{r}}+(e/c)\mathbf{A}} = -mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(A^0 - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}). \tag{4d}$$

This nonrelativistic Lagrangian L immediately yields the nonrelativistic action $S_{\rm nr}$,

$$S_{\rm nr} = \int L dt = \int[-mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(A^0 - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})]dt, \tag{4e}$$

which we now *specialize* to the particle's *instantaneous rest frame where its velocity* $\dot{\mathbf{r}} = \mathbf{0}$,

$$S = \int[-mc^2 - e(A')^0 + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}')]dt'. \tag{4f}$$

Taking the particle to be an electron, we now *in addition* suppose *a proton* to exist, which in its *own* rest frame *produces the hydrogen four-potential* $A^\mu = (A^0, \mathbf{0})$. If in the proton's rest frame the electron's instantaneous velocity is $\dot{\mathbf{r}}$, then *in the electron's instantaneous rest frame* the hydrogen four-potential $A^\mu = (A^0, \mathbf{0})$ *is Lorentz transformed to*,

$$(A')^\mu = \left((A')^0, \mathbf{A}'\right) = \gamma(|\dot{\mathbf{r}}/c|)\left(A^0, -(\dot{\mathbf{r}}/c)A^0\right), \text{ where } \gamma(|\dot{\mathbf{r}}/c|) \stackrel{\text{def}}{=} \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}, \tag{4g}$$

and consequently, *in the electron's instantaneous rest frame*,

$$(A')^0 = \gamma(|\dot{\mathbf{r}}/c|)A^0 \text{ and } \mathbf{B}' = \nabla_{\mathbf{r}}\times\mathbf{A}' = \nabla_{\mathbf{r}}\times\left[\gamma(|\dot{\mathbf{r}}/c|)\left(-(\dot{\mathbf{r}}/c)A^0\right)\right] = \gamma(|\dot{\mathbf{r}}/c|)\left(\mathbf{E}\times(\dot{\mathbf{r}}/c)\right), \tag{4h}$$

where $\mathbf{E} = -\nabla_{\mathbf{r}}A^0$. The Eq. (4h) effective magnetic field $\mathbf{B}' = \gamma(|\dot{\mathbf{r}}/c|)(\mathbf{E}\times(\dot{\mathbf{r}}/c))$ in the electron's instantaneous rest frame will cause its spin $(\hbar/2)\vec{\sigma}$ to precess in consonance with the presence of the energy term $(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}')$ in the integrand of the Eq. (4f) instantaneous rest frame action functional. However, in case the electron *is as well undergoing acceleration* $\ddot{\mathbf{r}}$ *such that* $(\ddot{\mathbf{r}}\times\dot{\mathbf{r}}) \neq \mathbf{0}$, this analysis of the relativistic physics *is incomplete*: in that case the transformation between the coordinate systems *in addition entails a rotation of their coordinate axes relative to each other*—successive Lorentz boosts *in different directions* don't resolve into *only* a net Lorentz boost; a relative rotation of the coordinate axes of the two systems *always occurs in addition*. The *effect* of such a coordinate axis rotation often tends *to partially cancel out* the spin precession caused by a magnetic field $\mathbf{B}'$ which is induced by a particle's velocity $\dot{\mathbf{r}}$ through a longitudinal electric field $\mathbf{E} = -\nabla_{\mathbf{r}}A^0$, such as the magnetic field $\mathbf{B}' = \gamma(|\dot{\mathbf{r}}/c|)(\mathbf{E}\times(\dot{\mathbf{r}}/c))$ described by Eq. (4h). This phenomenon is especially pronounced for particles with spin *traveling in circles*, in which case *their centripetal acceleration is orthogonal to their velocity*, the situation that is *the most favorable* to relativistic generation of relative coordinate axis rotation via successive Lorentz boosts *in different directions*.

For an electron circling the proton at a speed much less than c in a bound state, "the Thomas half" rule of thumb for this Thomas precession phenomenon is that relativistic relative coordinate axis rotation *halves the spin precession effect produced by the* $\mathbf{B}'$ *of* Eq. (4h) inserted into the spin energy term $(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}')$ of the integrand of Eq. (4f). However for an electron which is *not* in a bound state circling the proton, but is merely *being slightly deflected* (slightly elastically scattered) by the proton's longitudinal electric field $\mathbf{E} = -\nabla_{\mathbf{r}}A^0$, one would expect *negligible deviation* from the spin precession effect given by the $\mathbf{B}'$ of Eq. (4h) inserted into the spin energy term $(e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}')$ of the integrand of Eq. (4f).

Let us now work out the relativistic Lagrangian and Hamiltonian which follow from simply inserting the $(A')^0$ and $\mathbf{B}'$ of Eq. (4h) into the integrand of Eq. (4f), *bearing in mind* that this *ignores* the Thomas-precession consequence of particle acceleration not being parallel to particle velocity, and requires *correction* of its particle spin precession *prediction* which *ranges* from *negligible* for small-angle scattering to *"the Thomas half" rule of thumb* for bound states. The insertion of the $(A')^0$ and $\mathbf{B}'$ of Eq. (4h) into the integrand of Eq. (4f) yields,

$$S = \int[-mc^2 - \gamma(|\dot{\mathbf{r}}/c|)eA^0 + \gamma(|\dot{\mathbf{r}}/c|)(e\hbar/(2mc))(\vec{\sigma}\cdot(\mathbf{E}\times(\dot{\mathbf{r}}/c)))]dt'. \tag{4i}$$

In Eq. (4i) dt' is time in the particle instantaneous rest frame, which in terms of time dt in the proton rest frame is $dt' = (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}}dt = (1/\gamma(|\dot{\mathbf{r}}/c|))dt$. So in the proton rest frame the Lorentz-invariant action is,

$$S_{\rm rel} = \int[-mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - eA^0 + (e\hbar/(2mc))((\vec{\sigma}\times\mathbf{E})\cdot(\dot{\mathbf{r}}/c))]dt, \tag{4j}$$

where we *interchanged* the Eq. (4i) "dot" $\cdot$ and "cross" $\times$. The $S_{\rm rel}$ *integrand* is the relativistic Lagrangian,

$$L_{\rm rel} = -mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - eA^0 + (e\hbar/(2mc))((\vec{\sigma}\times\mathbf{E})\cdot(\dot{\mathbf{r}}/c)). \tag{4k}$$

Passage from $L_{\rm rel}$ to its corresponding relativistic Hamiltonian $H_{\rm rel}$ requires the canonical momentum $\mathbf{P}$,

$$\mathbf{P} = \nabla_{\dot{\mathbf{r}}}\, L_{\rm rel} = m\dot{\mathbf{r}}(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E}). \tag{4l}$$

It is convenient to define kinetic momentum $\mathbf{p}$ in terms of canonical momentum $\mathbf{P}$ as,

$$\mathbf{p} \stackrel{\rm def}{=} \big(\mathbf{P} - (e\hbar/(2mc^2))(\vec{\sigma}\times\mathbf{E})\big), \tag{4m}$$

which permits us *to compactly solve* Eq. (4l) *for* $\dot{\mathbf{r}}$,

$$\dot{\mathbf{r}} = (\mathbf{p}/m)\left(1 + |\mathbf{p}/(mc)|^2\right)^{-\frac{1}{2}}. \tag{4n}$$

With this and the aid of Eqs. (4k) and (4m), we obtain $H_{\rm rel}$ from $L_{\rm rel}$ via their standard relationship,

$$H_{\rm rel} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\rm rel}\Big|_{\dot{\mathbf{r}}=(\mathbf{p}/m)(1+|\mathbf{p}/(mc)|^2)^{-\frac{1}{2}}} = \tag{4o}$$

$$\left(m^2c^4 + |\mathbf{p}\,c|^2\right)^{\frac{1}{2}} + eA^0 = \left(m^2c^4 + |\mathbf{P}\,c - (e\hbar/(2mc))(\vec{\sigma}\times\mathbf{E})|^2\right)^{\frac{1}{2}} + eA^0.$$

The $H_{\rm rel}$ of Eq. (4o) is appropriate for small angle scattering, but for bound states it needs to be brought into line with "the Thomas half" rule of thumb for spin precession by its modification to,

$$H_{\rm rel} = \left(m^2c^4 + |\mathbf{P}\,c - (e\hbar/(4mc))(\vec{\sigma}\times\mathbf{E})|^2\right)^{\frac{1}{2}} + eA^0. \tag{4p}$$

We now *read off* from Ref. [1] that the $\beta = +1$ *positive-energy sector* of the Foldy-Wouthuysen transformation of the Dirac Hamiltonian $(\widehat{H}_D + eA^0)$ *agrees with* Eq. (4p), *except that it has an additional "Darwin term",* $-(1/8)e(\hbar/(mc))^2(\nabla_{\mathbf{r}}\cdot\mathbf{E})$. By Coulomb's Law, this "Darwin term" equals $-(\pi/2)e(\hbar/(mc))^2\rho$, where ρ is the proton's charge density, which is around 50,000 times smaller than the hydrogen atom's Bohr radius. A term of such short range *isn't usually discernible in hydrogen atomic physics*, but for experiments *expressly designed to determine the proton's charge radius from the proton charge density's effect on hydrogen atomic physics* [2], *this "Darwin term" cannot be ignored*.

This term *occurs only in physically unacceptable Dirac theory*; it definitely *isn't a feature of the* Eq. (4p) *relativistic upgrade of the Pauli Hamiltonian*. It has been traced *to averaged smearing of the electron potential energy* (eA^0) *by the electron's physically nonexistent free-electron "zitterbewegung"* [1], which has mean vector displacement $\langle\delta\widehat{\mathbf{r}}\rangle = \mathbf{0}$ with, according to Eq. (2l), isotropic mean-square deviation,

$$\big\langle|\delta\widehat{\mathbf{r}}|^2\big\rangle = (3/4)(\hbar/(mc))^2, \tag{5a}$$

when $|\widehat{\mathbf{p}}/(mc)| \ll 1$, *which is the case for the hydrogen atom's nonrelativistic electron*. (The scenario of *averaged* smearing of (eA^0) *is also appropriate for that case*, where according to Eq. (2k) the angular frequency of "zitterbewegung" oscillation of the free electron is $\omega = (2mc^2/\hbar)$, whereas the angular frequency of the ground-state oscillation of the hydrogen atom's electron is $\omega_1 = (-E_1/\hbar) = (1/2)(mc^2/\hbar)(e^2/(\hbar c))^2 = 1.33\times10^{-5}\,\omega$, namely 75,000 times *slower*.) For $\langle\delta\widehat{\mathbf{r}}\rangle = \mathbf{0}$ and $\langle|\delta\widehat{\mathbf{r}}|^2\rangle = (3/4)(\hbar/(mc))^2$, the *physically nonexistent* "zitterbewegung" isotropic smearing of (eA^0) is [1],

$$\big\langle eA^0(\mathbf{r}+\delta\widehat{\mathbf{r}}) - eA^0(\mathbf{r})\big\rangle \approx e\big\langle\delta\widehat{\mathbf{r}}\cdot\left(\nabla_{\mathbf{r}}\,A^0(\mathbf{r})\right) + (1/2)\textstyle\sum_{i,j=1}^{3}(\delta\widehat{\mathbf{r}})^i(\delta\widehat{\mathbf{r}})^j\left(\partial^2A^0(\mathbf{r})/\partial((\mathbf{r})^i)\partial((\mathbf{r})^j)\right)\big\rangle = \tag{5b}$$

$$e(1/6)\big\langle|\delta\widehat{\mathbf{r}}|^2\big\rangle\left(\nabla^2_{\mathbf{r}}\,A^0(\mathbf{r})\right) = -(1/8)e(\hbar/(mc))^2\left(\nabla_{\mathbf{r}}\cdot\mathbf{E}\right),$$

which *indeed* has produced the Dirac theory's "Darwin term". This *physically nonexistent smearing of* (eA^0) can alternatively be viewed *as having been produced by physically nonexistent smearing of its proton charge density source*, which *using the Dirac theory for data analysis erroneously compensates*, resulting in *a misleadingly contracted impression of the proton's charge radius*.

References

[1] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Eqs. (4.5)–(4.7), pp. 51–52.

[2] S. Schmidt et al., "The next generation of laser spectroscopy experiments using light muonic atoms", arXiv:1808.07240 [physics.atom-ph] (2018).