Multifaceted approaches to a Berkeley problem: part 2

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Abstract

We once presented a few ways to solve a Berkeley problem without paying attention to initial values, which we try taking into account in this sequel.

1 What if initial value (IV) was given in this problem?: rather meticulous version

In [1], IV of a system of linear differential equations was neglected, since it was unspecified in **Problem 3.4.9 (Sp91)** . Moreover, we considered only one eigenvalue and its corresponding eigenvector . Putting aside such an economy of thoughts, we wish to be rather meticulous throughout this preprint. Specifically, we will consider three eigenvalues and their corresponding eigenvectors and try taking IV's into account [1] .

At the outset, we reproduce the system we considered in [1] for reference:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}. \tag{1}$$

We then compute the characteristic polynomial of this 3×3 matrix, or A, to get

$$-\lambda^3 + 13\lambda^2 - 170\lambda + 173\ ^{2}, \tag{2}$$

which is denoted $A(\lambda)$. And we will sometimes refer to α, β, and γ, *i.e.,* the roots of the equation

$$-A(\lambda) = \lambda^3 - 13\lambda^2 + 170\lambda - 173 = 0\ ^{3}\ . \tag{3}$$

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[1]See, *e.g.,* [2].

[2]*Cf.* (4) in [1].

[3]Since α, β, γ are the roots of (3), we have $\alpha^3 - 13\alpha^2 + 170\alpha - 173 = \beta^3 - 13\beta^2 + 170\beta - 173 = \gamma^3 - 13\gamma^2 + 170\gamma - 173 = 0$. So we also have $-(\alpha^3 - 13\alpha^2 + 170\alpha - 173) = -(\beta^3 - 13\beta^2 + 170\beta - 173) = -(\gamma^3 - 13\gamma^2 + 170\gamma - 173) = 0$. Thus, α, β, γ satisfy not only the equation $-A(\lambda) = 0$, but also $A(\lambda) = 0$. See **2.1** for their detailed computations.

1.1 Getting general solution (GS)

We notice the equation $A(\lambda) = 0$ has the real root $\alpha \approx 1.1$ [4] and factor its left-hand side (LHS) as $-(\lambda - \alpha)(\lambda^2 + p\lambda + q)$. We expand it and equate the resulting polynomial with (2):

$$-(\lambda-\alpha)(\lambda^2+p\lambda+q) = -(\lambda^3+p\lambda^2+q\lambda-\alpha\lambda^2-\alpha p\lambda-\alpha q)$$
$$= -\lambda^3+(\alpha-p)\lambda^2+(\alpha p-q)\lambda+\alpha q = -\lambda^3+13\lambda^2-170\lambda+173.$$

We note we have $\alpha - p = 13$, that is, $p = \alpha + (-13)$. Both α and -13 are real numbers , so is p [5] . Likewise, both 173 and $\frac{1}{\alpha} \approx \frac{1}{1.1} = 0.9\ldots$ are real numbers , so is $173 \times \frac{1}{\alpha}$, or q [6, 7] . Hence, we are able to let β and γ be the roots of the quadratic equation $\lambda^2 + p\lambda + q = 0$, with $p, q \in \mathbb{R}$, the set of real numbers . Because $A(\lambda)$ is a monotonously decreasing function, which consequently crosses the X-axis just once [8] , we regard $\beta, \gamma \in \mathbb{C}$, the set of complex numbers , as the roots of the quadratic equation $x^2 + px + q = 0$ [9] . And it follows from the quadratic formula (QF) that

$$\lambda^2+p\lambda+q = (\lambda - \tfrac{-p+\sqrt{p^2-4q}}{2})(\lambda - \tfrac{-p-\sqrt{p^2-4q}}{2})$$
$$= (\lambda-\beta)(\lambda-\gamma), \text{ with } p^2-4q<0,\ \bar{\beta} = \gamma \text{ (or } \beta = \bar{\gamma}\text{)}.$$

After all, $A(\lambda)$ is factored into $-(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)$, where $\alpha \in \mathbb{R}$, $\beta, \gamma \in \mathbb{C}$, and $\bar{\beta} = \gamma$ (or $\beta = \bar{\gamma}$), which reminds us that α, β, γ correspond to three eigenvalues of A . What about their corresponding eigenvectors ? First, we set $\mathbf{v_1}$, an eigenvector corresponding to the eigenvalue α, to be $(\alpha+62,\ 11\alpha-15,\ \alpha^2-5\alpha+28)^{\mathrm{T}}$, where 'T' stands for the transpose of a matrix [10] . Then, we let eigenvectors corresponding to the eigenvalues β and γ be $\mathbf{v_2}$ and $\mathbf{v_3}$, respectively [11, 12] . Replac-

[4] See Figs. 1 and 2 in [1].

[5] This is because $\mathbb{R}$ is closed under addition . *Cf.* [3].

[6] This is because $\mathbb{R}$ is closed under multiplication . *Cf.* [3].

[7] Alternatively, we can show that q is a real number as follows: Both α and p are real numbers , so is αp. (See footnote 6.) Both αp and 170 are real numbers , so is $\alpha p + 170$. (See footnote 5.) Because $\alpha p + 170 = q$, q is a real number .

[8] See Figs. 1 and 2 in [1] again.

[9] We restrict our attention to $\mathbb{R}$ or $\mathbb{C}$. That is, we put aside $\mathbb{H}$, the set of quaternions , $\mathbb{O}$, the set of octonions , and so forth.

[10] See the arguments made in **2.3** of [1], where setting $B = 1$ for simplicity, one gets $\mathbf{v_1} = (1\times(\alpha+62), 1\times(11\alpha-15), 1\times(\alpha^2-5\alpha+28))^{\mathrm{T}} = (\alpha+62,\ 11\alpha-15,\ \alpha^2-5\alpha+28)^{\mathrm{T}}$. And $A\mathbf{v_1} = \alpha\mathbf{v_1}$ holds, of course. See also footnote 13.

[11] $\bar{\mathbf{v_2}} = \mathbf{v}_3$ (or $\bar{\mathbf{v_3}} = \mathbf{v}_2$). See **Remark.** in [2].

[12] More explicitly, because $\beta, \gamma \in \mathbb{C}$, we can write β (resp . γ) $= \zeta + \eta i$ (resp . $\zeta - \eta i$), where $\zeta, \eta \in \mathbb{R}_{>0}$, and i is the imaginary unit . By the way, why do we demand ζ and η should be greater than 0? Regarding ζ, we recall that the trace of a matrix is the sum of the (complex) eigenvalues . In our case, trace of A, or $\mathrm{tr}(A)$, equals $\alpha+\beta+$ $+\gamma$. That is, $1+4+8 = 13 = \alpha+\beta+\gamma = \alpha+\zeta+\eta i+\zeta-\eta i = \alpha+2\zeta$. So we have $\zeta = \frac{13-\alpha}{2} \approx \frac{13-1}{2} = \frac{12}{2} = 6 > 0$, α approximating 1 for simplicity. Hence, we are justified in setting $\zeta > 0$. What about η, then? It was set to be > 0 just for convenience. Actually, η can be < 0, if we wish. But in this case, given that we replace such η by something negative, *e.g.*, $-\kappa$, where $\kappa \in \mathbb{R}_{>0}$, we get $\beta = \zeta - \kappa i$ and $\gamma = \zeta + \kappa i$. Then, we interchange β with γ to

ing α's in $\mathbf{v_1}$ by β's (resp . γ's) yields $\mathbf{v_2}$ (resp . $\mathbf{v_3}$) $= (\beta+62,\ 11\beta-15, \beta^2-5\beta+28)^{\mathrm{T}}$ (resp . $(\gamma+62,\ 11\gamma-15,\ \gamma^2-5\gamma+28)^{\mathrm{T}}$) [13] .

Now the GS is $\mathbf{x}(t) = C_1 e^{\alpha t}\mathbf{v}_1 + C_2 e^{\beta t}\mathbf{v}_2 + C_3 e^{\gamma t}\mathbf{v}_3$, where C_1, C_2, C_3 are arbitrary constants (AC's) [14] . Writing out, we obtain

$$\begin{pmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix} = C_1 e^{\alpha t}\begin{pmatrix} \alpha+62\\ 11\alpha-15\\ \alpha^2-5\alpha+28\end{pmatrix} + C_2 e^{\beta t}\begin{pmatrix} \beta+62\\ 11\beta-15\\ \beta^2-5\beta+28\end{pmatrix} + C_3 e^{\gamma t}\begin{pmatrix} \gamma+62\\ 11\gamma-15\\ \gamma^2-5\gamma+28\end{pmatrix}$$

$$= \begin{pmatrix} C_1(\alpha+62)e^{\alpha t} + C_2(\beta+62)e^{\beta t} + C_3(\gamma+62)e^{\gamma t}\\ C_1(11\alpha-15)e^{\alpha t} + C_2(11\beta-15)e^{\beta t} + C_3(11\gamma-15)e^{\gamma t}\\ C_1(\alpha^2-5\alpha+28)e^{\alpha t} + C_2(\beta^2-5\beta+28)e^{\beta t} + C_3(\gamma^2-5\gamma+28)e^{\gamma t}\end{pmatrix}. \qquad (4)$$

But we would like to check if (4) satisfies (1). Replacing $x_i(t)$, $i = 1, 2, 3$, in the LHS of (1) by (4), one gets

$$\frac{d}{dt}\begin{pmatrix} C_1(\alpha+62)e^{\alpha t} + C_2(\beta+62)e^{\beta t} + C_3(\gamma+62)e^{\gamma t}\\ C_1(11\alpha-15)e^{\alpha t} + C_2(11\beta-15)e^{\beta t} + C_3(11\gamma-15)e^{\gamma t}\\ C_1(\alpha^2-5\alpha+28)e^{\alpha t} + C_2(\beta^2-5\beta+28)e^{\beta t} + C_3(\gamma^2-5\gamma+28)e^{\gamma t}\end{pmatrix} \qquad (5)$$

$$= \begin{pmatrix} C_1\alpha(\alpha+62)e^{\alpha t} + C_2\beta(\beta+62)e^{\beta t} + C_3\gamma(\gamma+62)e^{\gamma t}\\ C_1\alpha(11\alpha-15)e^{\alpha t} + C_2\beta(11\beta-15)e^{\beta t} + C_3\gamma(11\gamma-15)e^{\gamma t}\\ C_1\alpha(\alpha^2-5\alpha+28)e^{\alpha t} + C_2\beta(\beta^2-5\beta+28)e^{\beta t} + C_3\gamma(\gamma^2-5\gamma+28)e^{\gamma t}\end{pmatrix}$$

$$= \begin{pmatrix} C_1\alpha(\alpha+62)e^{\alpha t}\\ C_1\alpha(11\alpha-15)e^{\alpha t}\\ C_1\alpha(\alpha^2-5\alpha+28)e^{\alpha t}\end{pmatrix} + \begin{pmatrix} C_2\beta(\beta+62)e^{\beta t}\\ C_2\beta(11\beta-15)e^{\beta t}\\ C_2\beta(\beta^2-5\beta+28)e^{\beta t}\end{pmatrix} + \begin{pmatrix} C_3\gamma(\gamma+62)e^{\gamma t}\\ C_3\gamma(11\gamma-15)e^{\gamma t}\\ C_3\gamma(\gamma^2-5\gamma+28)e^{\gamma t}\end{pmatrix}$$

---

obtain $\beta = \zeta+\kappa i$ and $\gamma = \zeta-\kappa i$. These are essentially the same as $\beta = \zeta+\eta i$ and $\gamma = \zeta-\eta i$, respectively, since κ, $\eta \in \mathbb{R}_{>0}$, and η can be identified with κ. See **2.2** for the relationship between determinant of A, or $\det(A)$ and the product $\alpha\beta\gamma$.

[13] This idea has been obtained by simply calculating $A\mathbf{v_1} - \alpha\mathbf{v_1}$, which is meant to be $\mathbf{0}$, in a componentwise and explicit manner. We mean that in the following computations

- $\{1\times(\alpha+62)+6\times(11\alpha-15)+1\times(\alpha^2-5\alpha+28)\}-\alpha(\alpha+62)$
  $= \alpha+62+66\alpha-90+\alpha^2-5\alpha+28-\alpha(\alpha+62) = \alpha^2+62\alpha-\alpha(\alpha+62) = \alpha(\alpha+62)-\alpha(\alpha+62) = 0$
- $\{-4\times(\alpha+62)+4\times(11\alpha-15)+11\times(\alpha^2-5\alpha+28)\}-\alpha(11\alpha-15)$
  $= -4\alpha-248+44\alpha-60+11\alpha^2-55\alpha+308-\alpha(11\alpha-15) = 11\alpha^2-15\alpha-\alpha(11\alpha-15)$
  $= \alpha(11\alpha-15)-\alpha(11\alpha-15) = 0$
- $\{-3\times(\alpha+62)-9\times(11\alpha-15)+8\times(\alpha^2-5\alpha+28)\}-\alpha(\alpha^2-5\alpha+28)$
  $= -3\alpha-186-99\alpha+135+8\alpha^2-40\alpha+224-\alpha^3+5\alpha^2-28\alpha = -\alpha^3+13\alpha^2-170\alpha+173$
  $= -(\alpha^3-13\alpha^2+170\alpha-173) = 0,$

we note the first two hold for any letters such as α, β, γ, x, and so forth, since they just cancel out. We also note the last one is slightly restricted. That is, it holds for α and other letters which satisfy (3), or β and γ. This leads us to replace α and $\mathbf{v}_1$ by β (resp . γ) and $\mathbf{v}_2$ (resp. $\mathbf{v}_3$), and we thus get $A\mathbf{v}_2 - \beta\mathbf{v}_2 = A\mathbf{v}_3 - \gamma\mathbf{v}_3 = \mathbf{0}$, which means $\mathbf{v}_2$ (resp. $\mathbf{v}_3$) has become an eigenvector of A whose eigenvalue is β (resp. γ) as desired.

[14] *Cf.* [4].

$$
= C_1 e^{\alpha t}\begin{pmatrix}\alpha(\alpha+62)\\ \alpha(11\alpha-15)\\ \alpha(\alpha^2-5\alpha+28)\end{pmatrix} + C_2 e^{\beta t}\begin{pmatrix}\beta(\beta+62)\\ \beta(11\beta-15)\\ \beta(\beta^2-5\beta+28)\end{pmatrix} + C_3 e^{\gamma t}\begin{pmatrix}\gamma(\gamma+62)\\ \gamma(11\gamma-15)\\ \gamma(\gamma^2-5\gamma+28)\end{pmatrix}. \tag{6}
$$

On the other hand, substituting it into the right-hand side (RHS) of (1), we obtain

$$
\begin{pmatrix}1 & 6 & 1\\ -4 & 4 & 11\\ -3 & -9 & 8\end{pmatrix}\begin{pmatrix}C_1(\alpha+62)e^{\alpha t}+C_2(\beta+62)e^{\beta t}+C_3(\gamma+62)e^{\gamma t}\\ C_1(11\alpha-15)e^{\alpha t}+C_2(11\beta-15)e^{\beta t}+C_3(11\gamma-15)e^{\gamma t}\\ C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2(\beta^2-5\beta+28)e^{\beta t}+C_3(\gamma^2-5\gamma+28)e^{\gamma t}\end{pmatrix} \tag{7}
$$

$$
= \begin{pmatrix}
1\times\{C_1(\alpha+62)e^{\alpha t}+C_2(\beta+62)e^{\beta t}+C_3(\gamma+62)e^{\gamma t}\}\\
+6\times\{C_1(11\alpha-15)e^{\alpha t}+C_2(11\beta-15)e^{\beta t}+C_3(11\gamma-15)e^{\gamma t}\}\\
+1\times\{C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2(\beta^2-5\beta+28)e^{\beta t}+C_3(\gamma^2-5\gamma+28)e^{\gamma t}\}\\
-4\times\{C_1(\alpha+62)e^{\alpha t}+C_2(\beta+62)e^{\beta t}+C_3(\gamma+62)e^{\gamma t}\}\\
+4\times\{C_1(11\alpha-15)e^{\alpha t}+C_2(11\beta-15)e^{\beta t}+C_3(11\gamma-15)e^{\gamma t}\}\\
+11\times\{C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2(\beta^2-5\beta+28)e^{\beta t}+C_3(\gamma^2-5\gamma+28)e^{\gamma t}\}\\
-3\times\{C_1(\alpha+62)e^{\alpha t}+C_2(\beta+62)e^{\beta t}+C_3(\gamma+62)e^{\gamma t}\}\\
-9\times\{C_1(11\alpha-15)e^{\alpha t}+C_2(11\beta-15)e^{\beta t}+C_3(11\gamma-15)e^{\gamma t}\}\\
+8\times\{C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2(\beta^2-5\beta+28)e^{\beta t}+C_3(\gamma^2-5\gamma+28)e^{\gamma t}\}
\end{pmatrix}
$$

$$
= \begin{pmatrix}C_1(\alpha^2+62\alpha)e^{\alpha t}+C_2(\beta^2+62\beta)e^{\beta t}+C_3(\gamma^2+62\gamma)e^{\gamma t}\\ C_1(11\alpha^2-15\alpha)e^{\alpha t}+C_2(11\beta^2-15\beta)e^{\beta t}+C_3(11\gamma^2-15\gamma)e^{\gamma t}\\ C_1(8\alpha^2-142\alpha+173)e^{\alpha t}+C_2(8\beta^2-142\beta+173)e^{\beta t}+C_3(8\gamma^2-142\gamma+173)e^{\gamma t}\end{pmatrix}
$$

$$
= \begin{pmatrix}C_1(\alpha^2+62\alpha)e^{\alpha t}\\ C_1(11\alpha^2-15\alpha)e^{\alpha t}\\ C_1(8\alpha^2-142\alpha+173)e^{\alpha t}\end{pmatrix} + \begin{pmatrix}C_2(\beta^2+62\beta)e^{\beta t}\\ C_2(11\beta^2-15\beta)e^{\beta t}\\ C_2(8\beta^2-142\beta+173)e^{\beta t}\end{pmatrix} + \begin{pmatrix}C_3(\gamma^2+62\gamma)e^{\gamma t}\\ C_3(11\gamma^2-15\gamma)e^{\gamma t}\\ C_3(8\gamma^2-142\gamma+173)e^{\gamma t}\end{pmatrix}
$$

$$
= C_1 e^{\alpha t}\begin{pmatrix}\alpha^2+62\alpha\\ 11\alpha^2-15\alpha\\ 8\alpha^2-142\alpha+173\end{pmatrix} + C_2 e^{\beta t}\begin{pmatrix}\beta^2+62\beta\\ 11\beta^2-15\beta\\ 8\beta^2-142\beta+173\end{pmatrix} + C_3 e^{\gamma t}\begin{pmatrix}\gamma^2+62\gamma\\ 11\gamma^2-15\gamma\\ 8\gamma^2-142\gamma+173\end{pmatrix}. \tag{8}
$$

Therefore,

$$
(6)-(8) = C_1 e^{\alpha t}\begin{pmatrix}\alpha(\alpha+62)\\ \alpha(11\alpha-15)\\ \alpha(\alpha^2-5\alpha+28)\end{pmatrix} + C_2 e^{\beta t}\begin{pmatrix}\beta(\beta+62)\\ \beta(11\beta-15)\\ \beta(\beta^2-5\beta+28)\end{pmatrix} + C_3 e^{\gamma t}\begin{pmatrix}\gamma(\gamma+62)\\ \gamma(11\gamma-15)\\ \gamma(\gamma^2-5\gamma+28)\end{pmatrix}
$$

$$
-\left\{C_1 e^{\alpha t}\begin{pmatrix}\alpha^2+62\alpha\\ 11\alpha^2-15\alpha\\ 8\alpha^2-142\alpha+173\end{pmatrix} + C_2 e^{\beta t}\begin{pmatrix}\beta^2+62\beta\\ 11\beta^2-15\beta\\ 8\beta^2-142\beta+173\end{pmatrix}\right.
$$

$$+C_3 e^{\gamma t}\begin{pmatrix}\gamma^2+62\gamma\\ 11\gamma^2-15\gamma\\ 8\gamma^2-142\gamma+173\end{pmatrix}\Bigg\}$$

$$= C_1 e^{\alpha t}\begin{pmatrix}\alpha(\alpha+62)-(\alpha^2+62\alpha)\\ \alpha(11\alpha-15)-(11\alpha^2-15\alpha)\\ \alpha(\alpha^2-5\alpha+28)-(8\alpha^2-142\alpha+173)\end{pmatrix}$$

$$+C_2 e^{\beta t}\begin{pmatrix}\beta(\beta+62)-(\beta^2+62\beta)\\ \beta(11\beta-15)-(11\beta^2-15\beta)\\ \beta(\beta^2-5\beta+28)-(8\beta^2-142\beta+173)\end{pmatrix}$$

$$+C_3 e^{\gamma t}\begin{pmatrix}\gamma(\gamma+62)-(\gamma^2+62\gamma)\\ \gamma(11\gamma-15)-(11\gamma^2-15\gamma)\\ \gamma(\gamma^2-5\gamma+28)-(8\gamma^2-142\gamma+173)\end{pmatrix}$$

$$= C_1 e^{\alpha t}\begin{pmatrix}0\\ 0\\ \alpha^3-13\alpha^2+170\alpha-173\end{pmatrix}+C_2 e^{\beta t}\begin{pmatrix}0\\ 0\\ \beta^3-13\beta^2+170\beta-173\end{pmatrix}$$

$$+C_3 e^{\gamma t}\begin{pmatrix}0\\ 0\\ \gamma^3-13\gamma^2+170\gamma-173\end{pmatrix}$$

$$= \begin{pmatrix}0\\ 0\\ C_1(\alpha^3-13\alpha^2+170\alpha-173)e^{\alpha t}\\ +C_2(\beta^3-13\beta^2+170\beta-173)e^{\beta t}\\ +C_3(\gamma^3-13\gamma^2+170\gamma-173)e^{\gamma t}\end{pmatrix} \tag{9}$$

$$=^{15} \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

Hence, we have (6) = (8), which means that we also have (5) = (7). So (4) satisfies (1). In what follows, we verify our computations using OpenAxiom and wxMaxima with Zsh unless otherwise noted [16], [17], [18]:

[15] This is because $\alpha^3-13\alpha^2+170\alpha-173=\beta^3-13\beta^2+170\beta-173=\gamma^3-13\gamma^2+170\gamma-173=0$. See footnote 3.

[16] Throughout this preprint, we perform our computations on quad-core Intel processors of an ArcoLinux 19.02.4 machine.

[17] See footnote 4 in [1] for how we verify our computations.

[18] As in footnote 5 of [1], verbatim outputs of software are not always kept intact. For instance, the output (%o6) on p11 isn't always the same as its 'raw' output by wxMaxima, which is not shown for simplicity. Usually, such 'raw' outputs are edited so as to resemble the preceding OpenAxiom outputs.

```
% zsh --version
zsh 5.7.1 (x86_64-pc-linux-gnu)

% open-axiom
            OpenAxiom: The Open Scientific Computation Platform
                    Version: OpenAxiom 1.5.0-2013-06-21
              Built on Sunday December 15, 2013 at 18:59:05

(1) -> )read bp349_part2_1_1_1.input

exp(1)
--This is confirmation.

   (1)  %e
                                                  Type: Expression Integer

exp(1.0)
--Ditto.

   (2)  2.7182818284 590452354

                                                               Type: Float

B1:=matrix[[D(C1*(alpha+62)*exp(alpha*t)+C2*(beta+62)*exp(beta*t)
              +C3*(gamma+62)*exp(gamma*t),t)],
           [D(C1*(11*alpha-15)*exp(alpha*t)+C2*(11*beta-15)*exp(beta*t)
              +C3*(11*gamma-15)*exp(gamma*t),t)],
           [D(C1*(alpha^2-5*alpha+28)*exp(alpha*t)
              +C2*(beta^2-5*beta+28)*exp(beta*t)
              +C3*(gamma^2-5*gamma+28)*exp(gamma*t),t)]]
--This computes (5).

   (3)
   [  [
                 2                 gamma t
        (C3 gamma  + 62C3 gamma)%e
       +
                2                beta t
        (C2 beta  + 62C2 beta)%e
       +
                 2                 alpha t
        (C1 alpha  + 62C1 alpha)%e
       ]
     ,
```

```
      [
                       2                 gamma t
        (11C3 gamma  - 15C3 gamma)%e
       +
                      2                beta t
        (11C2 beta  - 15C2 beta)%e
       +
                      2                 alpha t
        (11C1 alpha  - 15C1 alpha)%e
       ]
      ,
      [
                    3            2                 gamma t
         (C3 gamma  - 5C3 gamma  + 28C3 gamma)%e
       +
                   3           2                beta t
         (C2 beta  - 5C2 beta  + 28C2 beta)%e
       +
                    3            2                 alpha t
         (C1 alpha  - 5C1 alpha  + 28C1 alpha)%e
       ]
      ]

                                       Type: Matrix Expression Integer
```

```
A1:=matrix[[1,6,1],[-4,4,11],[-3,-9,8]]
--This denotes matrix A.
```

```
        + 1    6   1 +
        |            |
   (4)  |- 4   4   11|
        |            |
        +- 3  - 9  8 +

                                                  Type: Matrix Integer
```

```
E1:=matrix[[C1*(alpha+62)*exp(alpha*t)+C2*(beta+62)*exp(beta*t)
            +C3*(gamma+62)*exp(gamma*t)],
           [C1*(11*alpha-15)*exp(alpha*t)+C2*(11*beta-15)*exp(beta*t)
            +C3*(11*gamma-15)*exp(gamma*t)],
           [C1*(alpha^2-5*alpha+28)*exp(alpha*t)
            +C2*(beta^2-5*beta+28)*exp(beta*t)
            +C3*(gamma^2-5*gamma+28)*exp(gamma*t)]]
--This denotes (4).
```

```
(5)
[
  [
                        gamma t
    (C3 gamma + 62C3)%e
   +
                       beta t
   (C2 beta + 62C2)%e
   +
                        alpha t
    (C1 alpha + 62C1)%e
   ]

  ,

  [
                          gamma t
    (11C3 gamma - 15C3)%e
   +
                         beta t
    (11C2 beta - 15C2)%e
   +
                          alpha t
    (11C1 alpha - 15C1)%e
   ]

  ,


  [
             2                       gamma t
    (C3 gamma  - 5C3 gamma + 28C3)%e
   +
            2                     beta t
    (C2 beta  - 5C2 beta + 28C2)%e
   +
             2                       alpha t
    (C1 alpha  - 5C1 alpha + 28C1)%e
   ]

  ]
                                      Type: Matrix Expression Integer
```

```
B1-A1*E1
--This computes (5)-(7) to give (6)-(8).

                                                    Type: Matrix Expression Integer

   (6)
   [[0],

    [0],

     [
                  3              2                              gamma t
        (C3 gamma  - 13C3 gamma  + 170C3 gamma - 173C3)%e
      +
                 3             2                              beta t
        (C2 beta  - 13C2 beta  + 170C2 beta - 173C2)%e
      +
                  3              2                              alpha t
        (C1 alpha  - 13C1 alpha  + 170C1 alpha - 173C1)%e
      ]

     ]
                                                    Type: Matrix Expression Integer.
```

Rewriting this last output, we have

$$(6)-(8)=(5)-(7)=\begin{pmatrix} 0 \\ 0 \\ C_1(\alpha^3-13\alpha^2+170\alpha-173)e^{\alpha t} \\ +C_2(\beta^3-13\beta^2+170\beta-173)e^{\beta t} \\ +C_3(\gamma^3-13\gamma^2+170\gamma-173)e^{\gamma t} \end{pmatrix} \overset{[19]}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again, we have (6) = (8) and (5) = (7). So (4) has been shown to satisfy (1) twice.

Next,

```
% wxmaxima -v
wxMaxima 19.04.1

% wxmaxima
```

[19]See footnote 3 and (9).

```
(%i6) %e^(1);
/*This is confirmation.*/
bfloat(%e^(1));
/*Ditto.*/
E2:matrix([C1*(alpha+62)*%e^(alpha*t)+C2*(beta+62)*%e^(beta*t)
        +C3*(gamma+62)*%e^(gamma*t)],
       [C1*(11*alpha-15)*%e^(alpha*t)+C2*(11*beta-15)*%e^(beta*t)
        +C3*(11*gamma-15)*%e^(gamma*t)],
       [C1*(alpha^2-5*alpha+28)*%e^(alpha*t)+C2*(beta^2-5*beta+28)
        *%e^(beta*t)+C3*(gamma^2-5*gamma+28)*%e^(gamma*t)]);
/*This denotes (4).*/
B2:diff(E2,t);
/*This computes (5).*/
A2:matrix([1,6,1],[-4,4,11],[-3,-9,8]);
/*This denotes matrix A.*/
ratsimp(expand(B2-A2.E2));
/*This computes (5)-(7) to give (6)-(8).*/
```

(%o1) %e

(%o2) 2.718281828459045b0

(E2)
$$\begin{pmatrix} C3\,(\gamma+62)\,\%e^{t\,\gamma}+C2\,(\beta+62)\,\%e^{\beta\,t}+C1\,(\alpha+62)\,\%e^{\alpha\,t} \\ C3\,(11\,\gamma-15)\,\%e^{t\,\gamma}+C2\,(11\,\beta-15)\,\%e^{\beta\,t}+C1\,(11\,\alpha-15)\,\%e^{\alpha\,t} \\ C3\,(\gamma^2-5\,\gamma+28)\,\%e^{t\,\gamma}+C2\,(\beta^2-5\,\beta+28)\,\%e^{\beta\,t}+C1\,(\alpha^2-5\,\alpha+28)\,\%e^{\alpha\,t} \end{pmatrix}$$

(B2)
$$\begin{pmatrix} C3\,\gamma\,(\gamma+62)\,\%e^{t\,\gamma}+C2\,\beta\,(\beta+62)\,\%e^{\beta\,t}+C1\,\alpha\,(\alpha+62)\,\%e^{\alpha\,t} \\ C3\,\gamma\,(11\,\gamma-15)\,\%e^{t\,\gamma}+C2\,\beta\,(11\,\beta-15)\,\%e^{\beta\,t}+C1\,\alpha\,(11\,\alpha-15)\,\%e^{\alpha\,t} \\ C3\,\gamma\,(\gamma^2-5\,\gamma+28)\,\%e^{t\,\gamma}+C2\,\beta\,(\beta^2-5\,\beta+28)\,\%e^{\beta\,t}+C1\,\alpha\,(\alpha^2-5\,\alpha+28)\,\%e^{\alpha\,t} \end{pmatrix}$$

(A2)
$$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}$$

(%o6) $$\begin{pmatrix} 0 \\ 0 \\ \left(C3\gamma^3-13C3\gamma^2+170C3\gamma-173C3\right)\%e^{\gamma t} \\ +\left(C2\beta^3-13C2\beta^2+170C2\beta-173C2\right)\%e^{\beta t} \\ +\left(C1\alpha^3-13C1\alpha^2+170C1\alpha-173C1\right)\%e^{\alpha t} \end{pmatrix}.$$

Likewise, rewriting (%o6) yields

$$(6)-(8)=(5)-(7)=\begin{pmatrix} 0 \\ 0 \\ C_1(\alpha^3-13\alpha^2+170\alpha-173)e^{\alpha t} \\ +C_2(\beta^3-13\beta^2+170\beta-173)e^{\beta t} \\ +C_3(\gamma^3-13\gamma^2+170\gamma-173)e^{\gamma t} \end{pmatrix} =^{20} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again and again, we have (6) = (8) and (5) = (7). So (4) has been shown to satisfy (1) repeatedly. This leads us to hold that we have got a complex-valued solution of (1), or (4) [5].

We now wish to proceed to think a bit more generally. We assume that

$$\frac{d}{dt}\begin{pmatrix} f_1(t)+ig_1(t) \\ f_2(t)+ig_2(t) \\ f_3(t)+ig_3(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} f_1(t)+ig_1(t) \\ f_2(t)+ig_2(t) \\ f_3(t)+ig_3(t) \end{pmatrix},$$

where $f_n(t)$ and $g_n(t), n=1, 2, 3$, are real functions, and $a_{ij} \in \mathbb{R}$, for $1 \le i, j \le 3$. Then, computing the LHS and RHS of the above yields

$$\begin{pmatrix} \{f_1(t)+ig_1(t)\}' \\ \{f_2(t)+ig_2(t)\}' \\ \{f_3(t)+ig_3(t)\}' \end{pmatrix} = \begin{pmatrix} f_1'(t)+ig_1'(t) \\ f_2'(t)+ig_2'(t) \\ f_3'(t)+ig_3'(t) \end{pmatrix} = \begin{pmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{pmatrix} + i\begin{pmatrix} g_1'(t) \\ g_2'(t) \\ g_3'(t) \end{pmatrix}$$

and

$$\begin{pmatrix} a_{11}\{f_1(t)+ig_1(t)\}+a_{12}\{f_2(t)+ig_2(t)\}+a_{13}\{f_3(t)+ig_3(t)\} \\ a_{21}\{f_1(t)+ig_1(t)\}+a_{22}\{f_2(t)+ig_2(t)\}+a_{23}\{f_3(t)+ig_3(t)\} \\ a_{31}\{f_1(t)+ig_1(t)\}+a_{32}\{f_2(t)+ig_2(t)\}+a_{33}\{f_3(t)+ig_3(t)\} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}f_1(t)+ia_{11}g_1(t)+a_{12}f_2(t)+ia_{12}g_2(t)+a_{13}f_3(t)+ia_{13}g_3(t) \\ a_{21}f_1(t)+ia_{21}g_1(t)+a_{22}f_2(t)+ia_{22}g_2(t)+a_{23}f_3(t)+ia_{23}g_3(t) \\ a_{31}f_1(t)+ia_{31}g_1(t)+a_{32}f_2(t)+ia_{32}g_2(t)+a_{33}f_3(t)+ia_{33}g_3(t) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}f_1(t)+a_{12}f_2(t)+a_{13}f_3(t) \\ a_{21}f_1(t)+a_{22}f_2(t)+a_{23}f_3(t) \\ a_{31}f_1(t)+a_{32}f_2(t)+a_{33}f_3(t) \end{pmatrix} + \begin{pmatrix} ia_{11}g_1(t)+ia_{12}g_2(t)+ia_{13}g_3(t) \\ ia_{21}g_1(t)+ia_{22}g_2(t)+ia_{23}g_3(t) \\ ia_{31}g_1(t)+ia_{32}g_2(t)+ia_{33}g_3(t) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}f_1(t)+a_{12}f_2(t)+a_{13}f_3(t) \\ a_{21}f_1(t)+a_{22}f_2(t)+a_{23}f_3(t) \\ a_{31}f_1(t)+a_{32}f_2(t)+a_{33}f_3(t) \end{pmatrix} + i\begin{pmatrix} a_{11}g_1(t)+a_{12}g_2(t)+a_{13}g_3(t) \\ a_{21}g_1(t)+a_{22}g_2(t)+a_{23}g_3(t) \\ a_{31}g_1(t)+a_{32}g_2(t)+a_{33}g_3(t) \end{pmatrix}$$

[20] Ditto.

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} + i \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix},$$

respectively [21] . So we have

$$\begin{pmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{pmatrix} + i \begin{pmatrix} g_1'(t) \\ g_2'(t) \\ g_3'(t) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} + i \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix}.$$

Equating real part s ($\mathfrak{R}$'s) and imaginary part s ($\mathfrak{I}$'s), we end up with

$$\begin{cases} \begin{pmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}, & (10) \\ \begin{pmatrix} g_1'(t) \\ g_2'(t) \\ g_3'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix}. & (11) \end{cases}$$

We notice both (10) and (11) are composed solely of reals , which leads us to focus on real-valued solutions [22] . Now that we have obtained somewhat general stuff, we wish to turn to the deductive rather than the inductive. First, we try to extract $\mathfrak{R}$'s from (4), the GS of (1), which has been inspired by and is deduced from (10). Prior to doing so, we make some concrete preliminaries for later use, though [23] .

Preparation 1.1.1. $\zeta \pm \eta i$ being the roots of (3) [24] , we have

$$\begin{cases} \quad (\zeta+\eta i)^3 - 13(\zeta+\eta i)^2 + 170(\zeta+\eta i) - 173 \\ =^{25} \ \zeta^3 + 3\zeta^2\eta i - 3\zeta\eta^2 - \eta^3 i - 13\zeta^2 - 26\zeta\eta i + 13\eta^2 + 170\zeta + 170\eta i - 173 = 0, \\ \quad (\zeta-\eta i)^3 - 13(\zeta-\eta i)^2 + 170(\zeta-\eta i) - 173 \\ =^{26} \ \zeta^3 - 3\zeta^2\eta i - 3\zeta\eta^2 + \eta^3 i - 13\zeta^2 + 26\zeta\eta i + 13\eta^2 + 170\zeta - 170\eta i - 173 = 0. \end{cases}$$

[21] $'$ stands for differentiation wrt t.

[22] *Cf.* **Lemma 1.** in [5].

[23] We are going to get relations (12) – (15), which play some role in our later verification. See, *e.g.*, footnotes 53 and 54.

[24] See footnotes 3 and 12.

[25] Here we have performed the so-called expansion .

[26] Ditto.

Equating $\Re$'s and $\Im$'s of these relations, we get the following:

$$
\begin{cases}
\quad \Re\{(\zeta+\eta i)^3 - 13(\zeta+\eta i)^2 + 170(\zeta+\eta i) - 173\} \\
\overset{27}{=} \; \Re(\zeta^3 + 3\zeta^2\eta i - 3\zeta\eta^2 - \eta^3 i - 13\zeta^2 - 26\zeta\eta i + 13\eta^2 + 170\zeta + 170\eta i - 173) \\
= \Re(0), \\
\quad \Im\{(\zeta+\eta i)^3 - 13(\zeta+\eta i)^2 + 170(\zeta+\eta i) - 173\} \\
\overset{28}{=} \; \Im(\zeta^3 + 3\zeta^2\eta i - 3\zeta\eta^2 - \eta^3 i - 13\zeta^2 - 26\zeta\eta i + 13\eta^2 + 170\zeta + 170\eta i - 173) \\
= \Im(0), \\
\quad \Re\{(\zeta-\eta i)^3 - 13(\zeta-\eta i)^2 + 170(\zeta-\eta i) - 173\} \\
\overset{29}{=} \; \Re(\zeta^3 - 3\zeta^2\eta i - 3\zeta\eta^2 + \eta^3 i - 13\zeta^2 + 26\zeta\eta i + 13\eta^2 + 170\zeta - 170\eta i - 173) \\
= \Re(0), \\
\quad \Im\{(\zeta-\eta i)^3 - 13(\zeta-\eta i)^2 + 170(\zeta-\eta i) - 173\} \\
\overset{30}{=} \; \Im(\zeta^3 - 3\zeta^2\eta i - 3\zeta\eta^2 + \eta^3 i - 13\zeta^2 + 26\zeta\eta i + 13\eta^2 + 170\zeta - 170\eta i - 173) \\
= \Im(0).
\end{cases}
$$

After some computations and rearrangements, one gets

$$
\begin{cases}
\zeta^3 - 13\zeta^2 + 170\zeta - 3\eta^2\zeta + 13\eta^2 - 173 = 0, & (12) \\
\eta(3\zeta^2 - 26\zeta - \eta^2 + 170) = 0, & (13) \\
-\eta(3\zeta^2 - 26\zeta - \eta^2 + 170) = 0. & (14)
\end{cases}
$$

(13) being essentially the same as (14), we reduce them to one equation

$$3\zeta^2 - 26\zeta - \eta^2 + 170 = 0 \tag{15}$$

and check (12) and (15) [31], [32], [33].

[27] Ditto.

[28] Ditto.

[29] Ditto.

[30] Ditto.

[31] Since $\eta \neq 0$, we can divide both sides of (13) and (14) by η and $-\eta$, respectively. See, *e.g.,* **2.1**. And doing so yields (15). But what if we assume η to be 0? See **2.3.1** and **2.3.2**.

[32] In what follows, we let (15) be representative of (13), (14), and (15), which are essentially the same, unless otherwise noted. See, *e.g.*, footnote 37.

[33] Solving the system of polynomial equations including (12) and (15) will be of some interest and discussed elsewhere.

```
% open-axiom

(1) -> )read bp349_part2_1_1_2.input

%i^2
--This is confirmation.

   (1)  - 1

                                                      Type: Complex Integer

real((zeta+eta*%i)^3-13*(zeta+eta*%i)^2+170*(zeta+eta*%i)-173)

            3         2          2                  2
   (2)  zeta  - 13zeta  + (- 3eta  + 170)zeta + 13eta  - 173

                                                   Type: Expression Integer

imag((zeta+eta*%i)^3-13*(zeta+eta*%i)^2+170*(zeta+eta*%i)-173)

                 2                   3
   (3)  3eta zeta  - 26eta zeta - eta  + 170eta

                                                   Type: Expression Integer

real((zeta-eta*%i)^3-13*(zeta-eta*%i)^2+170*(zeta-eta*%i)-173)

            3         2          2                  2
   (4)  zeta  - 13zeta  + (- 3eta  + 170)zeta + 13eta  - 173

                                                   Type: Expression Integer

imag((zeta-eta*%i)^3-13*(zeta-eta*%i)^2+170*(zeta-eta*%i)-173)

                   2                   3
   (5)  - 3eta zeta  + 26eta zeta + eta  - 170eta

                                                  Type: Expression Integer.
```

We have checked preliminaries. Furthermore,

```
% wxmaxima
(%i5) %i^2;
/*This is confirmation.*/
expand(realpart((zeta+eta*%i)^3-13*(zeta+eta*%i)^2
+170*(zeta+eta*%i)-173));
factor(imagpart((zeta+eta*%i)^3-13*(zeta+eta*%i)^2
+170*(zeta+eta*%i)-173));
expand(realpart((zeta-eta*%i)^3-13*(zeta-eta*%i)^2
+170*(zeta-eta*%i)-173));
factor(imagpart((zeta-eta*%i)^3-13*(zeta-eta*%i)^2
+170*(zeta-eta*%i)-173));
```

(%o1) -1

(%o2) $\zeta^3-13\zeta^2-3\eta^2\zeta+170\zeta+13\eta^2-173$

(%o3) $\eta(3\zeta^2-26\zeta-\eta^2+170)$

(%o4) $\zeta^3-13\zeta^2-3\eta^2\zeta+170\zeta+13\eta^2-173$

(%o5) $-\eta(3\zeta^2-26\zeta-\eta^2+170)$.

We have checked preliminaries repeatedly.

1.2 Extracting $\Re$'s from GS

Having checked preliminaries repeatedly, we replace β and γ in (4) by $\zeta+\eta i$ and $\zeta-\eta i$, respectively [34] , and extract $\Re$'s from $x_n(t)$, $n = 1, 2, 3$ [35] :

$$\begin{aligned}
&\Re\{x_1(t)\}\\
&=\Re\{C_1(\alpha+62)e^{\alpha t}+C_2(\beta+62)e^{\beta t}+C_3(\gamma+62)e^{\gamma t}\}\\
&=\Re\{C_1(\alpha+62)e^{\alpha t}+C_2(\zeta+\eta i+62)e^{(\zeta+\eta i)t}+C_3(\zeta-\eta i+62)e^{(\zeta-\eta i)t}\}\\
&=\Re\{C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}(\zeta+\eta i+62)e^{\eta i t}+C_3e^{\zeta t}(\zeta-\eta i+62)e^{-\eta i t}\}\\
&=\Re\{C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}(\zeta+62+\eta i)e^{\eta i t}+C_3e^{\zeta t}(\zeta+62-\eta i)e^{-\eta i t}\}\\
&=\Re\{C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}(\zeta+62+\eta i)(\cos\eta t+i\sin\eta t)+C_3e^{\zeta t}(\zeta+62-\eta i)(\cos\eta t-i\sin\eta t)\}
\end{aligned}$$

[34] See footnote 12.

[35] In the following, Euler's formula (EF) will be frequently used. See *e.g.*, footnote 93.

$$= \Re[C_1(\alpha+62)e^{\alpha t} + C_2 e^{\zeta t}\{(\zeta+62)(\cos\eta t + i\sin\eta t) + \eta i(\cos\eta t + i\sin\eta t)\}$$
$$+ C_3 e^{\zeta t}\{(\zeta+62)(\cos\eta t - i\sin\eta t) - \eta i(\cos\eta t - i\sin\eta t)\}]$$
$$= \Re[C_1(\alpha+62)e^{\alpha t} + C_2 e^{\zeta t}\{(\zeta+62)\cos\eta t - \eta\sin\eta t + \eta i\cos\eta t + (\zeta+62) i\sin\eta t\}$$
$$+ C_3 e^{\zeta t}\{(\zeta+62)\cos\eta t - \eta\sin\eta t - \eta i\cos\eta t - (\zeta+62) i\sin\eta t\}]$$
$$= C_1(\alpha+62)e^{\alpha t} + C_2 e^{\zeta t}\{(\zeta+62)\cos\eta t - \eta\sin\eta t\} + C_3 e^{\zeta t}\{(\zeta+62)\cos\eta t - \eta\sin\eta t\}$$
$$= C_1(\alpha+62)e^{\alpha t} + (C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t - \eta\sin\eta t\}.$$

$\Re\{x_2(t)\}$
$$= \Re\{C_1(11\alpha-15)e^{\alpha t} + C_2(11\beta-15)e^{\beta t} + C_3(11\gamma-15)e^{\gamma t}\}$$
$$= \Re[C_1(11\alpha-15)e^{\alpha t} + C_2\{11(\zeta+\eta i)-15\}e^{(\zeta+\eta i)t} + C_3\{11(\zeta-\eta i)-15\}e^{(\zeta-\eta i)t}]$$
$$= \Re\{C_1(11\alpha-15)e^{\alpha t} + C_2 e^{\zeta t}(11\zeta+11\eta i-15)e^{\eta i t} + C_3 e^{\zeta t}(11\zeta-11\eta i-15)e^{-\eta i t}\}$$
$$= \Re\{C_1(11\alpha-15)e^{\alpha t} + C_2 e^{\zeta t}(11\zeta-15+11\eta i)e^{\eta i t} + C_3 e^{\zeta t}(11\zeta-15-11\eta i)e^{-\eta i t}\}$$
$$= \Re\{C_1(11\alpha-15)e^{\alpha t} + C_2 e^{\zeta t}(11\zeta-15+11\eta i)(\cos\eta t + i\sin\eta t)$$
$$+ C_3 e^{\zeta t}(11\zeta-15-11\eta i)(\cos\eta t - i\sin\eta t)\}$$
$$= \Re[C_1(11\alpha-15)e^{\alpha t} + C_2 e^{\zeta t}\{(11\zeta-15)(\cos\eta t + i\sin\eta t) + 11\eta i(\cos\eta t + i\sin\eta t)\}$$
$$+ C_3 e^{\zeta t}\{(11\zeta-15)(\cos\eta t - i\sin\eta t) - 11\eta i(\cos\eta t - i\sin\eta t)\}]$$
$$= \Re[C_1(11\alpha-15)e^{\alpha t} + C_2 e^{\zeta t}\{(11\zeta-15)\cos\eta t - 11\eta\sin\eta t + 11\eta i\cos\eta t + (11\zeta-15) i\sin\eta t\}$$
$$+ C_3 e^{\zeta t}\{(11\zeta-15)\cos\eta t - 11\eta\sin\eta t - 11\eta i\cos\eta t - (11\zeta-15) i\sin\eta t\}]$$
$$= C_1(11\alpha-15)e^{\alpha t} + C_2 e^{\zeta t}\{(11\zeta-15)\cos\eta t - 11\eta\sin\eta t\} + C_3 e^{\zeta t}\{(11\zeta-15)\cos\eta t - 11\eta\sin\eta t\}$$
$$= C_1(11\alpha-15)e^{\alpha t} + (C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t - 11\eta\sin\eta t\}.$$

$\Re\{x_3(t)\}$
$$= \Re\{C_1(\alpha^2-5\alpha+28)e^{\alpha t} + C_2(\beta^2-5\beta+28)e^{\beta t} + C_3(\gamma^2-5\gamma+28)e^{\gamma t}\}$$
$$= \Re[C_1(\alpha^2-5\alpha+28)e^{\alpha t} + C_2\{(\zeta+\eta i)^2 - 5(\zeta+\eta i) + 28\}e^{(\zeta+\eta i)t}$$
$$+ C_3\{(\zeta-\eta i)^2 - 5(\zeta-\eta i) + 28\}e^{(\zeta-\eta i)t}]$$
$$= \Re[C_1(\alpha^2-5\alpha+28)e^{\alpha t} + C_2 e^{\zeta t}\{(\zeta+\eta i)^2 - 5(\zeta+\eta i) + 28\}e^{\eta i t}$$
$$+ C_3 e^{\zeta t}\{(\zeta-\eta i)^2 - 5(\zeta-\eta i) + 28\}e^{-\eta i t}]$$
$$= \Re\{C_1(\alpha^2-5\alpha+28)e^{\alpha t} + C_2 e^{\zeta t}(\zeta^2 + 2\zeta\eta i - \eta^2 - 5\zeta - 5\eta i + 28)e^{\eta i t}$$
$$+ C_3 e^{\zeta t}(\zeta^2 - 2\zeta\eta i - \eta^2 - 5\zeta + 5\eta i + 28)e^{-\eta i t}\}$$
$$= \Re[C_1(\alpha^2-5\alpha+28)e^{\alpha t} + C_2 e^{\zeta t}\{\zeta^2 - 5\zeta - \eta^2 + 28 + \eta(2\zeta-5)i\}e^{\eta i t}$$
$$+ C_3 e^{\zeta t}\{\zeta^2 - 5\zeta - \eta^2 + 28 + \eta(5-2\zeta)i\}e^{-\eta i t}]$$
$$= \Re[C_1(\alpha^2-5\,\alpha+28)e^{\alpha t} + C_2 e^{\zeta t}\{\zeta^2 - 5\zeta - \eta^2 + 28 + \eta(2\zeta-5)i\}(\cos\eta t + i\sin\eta t)$$
$$+ C_3 e^{\zeta t}\{\zeta^2 - 5\zeta - \eta^2 + 28 + \eta(5-2\zeta)i\}(\cos\eta t - i\sin\eta t)]$$

$$
\begin{aligned}
&= \Re[C_1(\alpha^2-5\alpha+28)e^{\alpha t} \\
&\qquad +C_2e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t \\
&\qquad\qquad +\eta(2\zeta-5)i\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)i\sin\eta t\} \\
&\qquad +C_3e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t+\eta(5-2\zeta)\sin\eta t \\
&\qquad\qquad -\eta(2\zeta-5)i\cos\eta t-(\zeta^2-5\zeta-\eta^2+28)i\sin\eta t\}] \\
&= C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\} \\
&\quad +C_3e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\} \\
&= C_1(\alpha^2-5\alpha+28)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}.
\end{aligned}
$$

Hence, we have extracted the following real-valued functions from (4):

$$
\begin{cases}
F_1(t)=C_1(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}, \\
F_2(t)=C_1(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}, \\
F_3(t)=C_1(\alpha^2-5\alpha+28)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}.
\end{cases} \quad (16)
$$ [36]

We check if (16) satisfies (1). Replacing $x_i(t)$, $i=1, 2, 3$, in the LHS of (1) by $F_i(t)$, $i=1, 2, 3$, respectively, one gets

$$
\frac{d}{dt}\begin{pmatrix}F_1(t)\\F_2(t)\\F_3(t)\end{pmatrix}=\begin{pmatrix}F_1'(t)\\F_2'(t)\\F_3'(t)\end{pmatrix}
$$

$$
=\begin{pmatrix}
[C_1(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}]' \\
[C_1(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}]' \\
[C_1(\alpha^2-5\alpha+28)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}]'
\end{pmatrix} \quad (17)
$$

$$
=\begin{pmatrix}
C_1\alpha(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta^2+62\zeta-\eta^2)\cos\eta t-2\eta(\zeta+31)\sin\eta t\} \\
C_1\alpha(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta^2-15\zeta-11\eta^2)\cos\eta t-\eta(22\zeta-15)\sin\eta t\} \\
C_1\alpha(\alpha^2-5\alpha+28)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(\zeta^3-5\zeta^2+28\zeta-3\eta^2\zeta+5\eta^2)\cos\eta t-\eta(3\zeta^2-10\zeta-\eta^2+28)\sin\eta t\}
\end{pmatrix}. \quad (18)
$$

On the other hand, applying a similar procedure to its RHS, we get

$$
\begin{pmatrix}1&6&1\\-4&4&11\\-3&-9&8\end{pmatrix}\begin{pmatrix}F_1(t)\\F_2(t)\\F_3(t)\end{pmatrix} \quad (19)
$$

$$
=\begin{pmatrix}1&6&1\\-4&4&11\\-3&-9&8\end{pmatrix}\begin{pmatrix}
C_1(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\} \\
C_1(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\} \\
C_1(\alpha^2-5\alpha+28)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}
\end{pmatrix}
$$

[36] If we let C_2+C_3 equal 0, $(F_1(t), F_2(t), F_3(t))$ becomes $(C_1(\alpha+62)e^{\alpha t}, C_1(11\alpha-15)e^{\alpha t}, C_1(\alpha^2-5\alpha+28)e^{\alpha t})$, which is reminiscent of **2.3** in [1]. But what if $\zeta=\eta=0$? See **2.3.1**.

$$
= \begin{pmatrix}
1\times[C_1(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}] \\
+6\times[C_1(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}] \\
+1\times[C_1(\alpha^2-5\alpha+28)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}] \\
\\
-4\times[C_1(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}] \\
+4\times[C_1(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}] \\
+11\times[C_1(\alpha^2-5\alpha+28)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}] \\
\\
-3\times[C_1(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}] \\
-9\times[C_1(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}] \\
+8\times[C_1(\alpha^2-5\alpha+28)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}]
\end{pmatrix}
$$

$$
= \begin{pmatrix}
C_1(\alpha^2+62\alpha)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(\zeta^2+62\zeta-\eta^2)\cos\eta t-2\eta(\zeta+31)\sin\eta t\} \\
C_1(11\alpha^2-15\alpha)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(11\zeta^2-15\zeta-11\eta^2)\cos\eta t-\eta(22\zeta-15)\sin\eta t\} \\
C_1(8\alpha^2-142\alpha+173)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(8\zeta^2-142\zeta-8\eta^2+173)\cos\eta t-2\eta(8\zeta-71)\sin\eta t\}
\end{pmatrix}. \tag{20}
$$

Therefore,

$$
(18)-(20) = \begin{pmatrix}
0 \\
0 \\
C_1(\alpha^3-13\alpha^2+170\alpha-173)e^{\alpha t} \\
+(C_2+C_3)e^{\zeta t}\{(\zeta^3-13\zeta^2+170\zeta-3\eta^2\zeta+13\eta^2-173)\cos\eta t \\
-\eta(3\zeta^2-26\zeta-\eta^2+170)\sin\eta t\}
\end{pmatrix} \stackrel{37}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{21}
$$

Hence, we have (18) = (20), which means that we also have (17) = (19). So (16) satisfies (1), which we check as follows:

```
% open-axiom

(1) -> )read bp349_part2_1_2_1.input

G1:=matrix[[D(C1*(alpha+62)*exp(alpha*t)+(C2+C3)*exp(zeta*t)
             *((zeta+62)*cos(eta*t)-eta*sin(eta*t)),t)],
          [D(C1*(11*alpha-15)*exp(alpha*t)+(C2+C3)*exp(zeta*t)
             *((11*zeta-15)*cos(eta*t)-11*eta*sin(eta*t)),t)],
          [D(C1*(alpha^2-5*alpha+28)*exp(alpha*t)
             +(C2+C3)*exp(zeta*t)*((zeta^2-5*zeta-eta^2+28)*cos(eta*t)
             -eta*(2*zeta-5)*sin(eta*t)),t)]]
--This computes (17) to yield (18).
```

[37] See footnote 3, (12), and (15).

```
(1)
[
  [
                                                             t zeta
    ((- 2C3 - 2C2)eta zeta + (- 62C3 - 62C2)eta)%e       sin(eta t)
   +
                   2                                          2
     ((C3 + C2)zeta  + (62C3 + 62C2)zeta + (- C3 - C2)eta )cos(eta t)
    *
        t zeta
      %e
   +
              2                 alpha t
    (C1 alpha  + 62C1 alpha)%e
   ]
  ,


  [
                                                             t zeta
    ((- 22C3 - 22C2)eta zeta + (15C3 + 15C2)eta)%e       sin(eta t)
   +
                        2                                            2
      ((11C3 + 11C2)zeta  + (- 15C3 - 15C2)zeta + (- 11C3 - 11C2)eta )
    *
                t zeta
      cos(eta t)%e
   +
                2                 alpha t
    (11C1 alpha  - 15C1 alpha)%e
   ]

  ,


  [
                             2                                      3
        (- 3C3 - 3C2)eta zeta  + (10C3 + 10C2)eta zeta + (C3 + C2)eta
      +
        (- 28C3 - 28C2)eta
    *
        t zeta
      %e      sin(eta t)
```

```
      +
                          3                     2
            (C3 + C2)zeta  + (- 5C3 - 5C2)zeta
          +
                              2                                    2
            ((- 3C3 - 3C2)eta  + 28C3 + 28C2)zeta + (5C3 + 5C2)eta
       *
                       t zeta
          cos(eta t)%e
      +
                 3            2                  alpha t
        (C1 alpha  - 5C1 alpha  + 28C1 alpha)%e
      ]

    ]
```

Type: Matrix Expression Integer

```
A1:=matrix[[1,6,1],[-4,4,11],[-3,-9,8]]
--This corresponds to matrix A.
```

```
       + 1    6   1 +
       |            |
   (2) |- 4   4   11|
       |            |
       +- 3  - 9  8 +
```

Type: Matrix Integer

```
H1:=matrix[[C1*(alpha+62)*exp(alpha*t)+(C2+C3)*exp(zeta*t)
            *((zeta+62)*cos(eta*t)-eta*sin(eta*t))],
           [C1*(11*alpha-15)*exp(alpha*t)+(C2+C3)*exp(zeta*t)
            *((11*zeta-15)*cos(eta*t)-11*eta*sin(eta*t))],
           [C1*(alpha^2-5*alpha+28)*exp(alpha*t)+(C2+C3)*exp(zeta*t)
            *((zeta^2-5*zeta-eta^2+28)*cos(eta*t)
               -eta*(2*zeta-5)*sin(eta*t))]]
--This corresponds to (16).
```

```
   (3)
   [
     [
                         t zeta
        (- C3 - C2)eta %e      sin(eta t)
```

```
  +
                                                 t zeta
    ((C3 + C2)zeta + 62C3 + 62C2)cos(eta t)%e
  +
                       alpha t
    (C1 alpha + 62C1)%e
  ]

,

[
                         t zeta
    (- 11C3 - 11C2)eta %e      sin(eta t)
  +
                                                     t zeta
    ((11C3 + 11C2)zeta - 15C3 - 15C2)cos(eta t)%e
  +
                         alpha t
    (11C1 alpha - 15C1)%e
  ]

,

[
                                                  t zeta
    ((- 2C3 - 2C2)eta zeta + (5C3 + 5C2)eta)%e      sin(eta t)
  +
                    2
      ((C3 + C2)zeta  + (- 5C3 - 5C2)zeta

     +
                    2
      (- C3 - C2)eta  + 28C3 + 28C2)

   *
                  t zeta
     cos(eta t)%e
  +
             2                        alpha t
    (C1 alpha  - 5C1 alpha + 28C1)%e
  ]

]
                                     Type: Matrix Expression Integer
```

```
G1-A1*H1
--This computes (17)-(19) to give (18)-(20).

   (4)
   [[0], [0],

     [
                                          2                                     3
              (- 3C3 - 3C2)eta zeta  + (26C3 + 26C2)eta zeta + (C3 + C2)eta
            +
              (- 170C3 - 170C2)eta
        *
              t zeta
            %e      sin(eta t)
        +
                              3                       2
              (C3 + C2)zeta  + (- 13C3 - 13C2)zeta
            +
                                      2                                      2
              ((- 3C3 - 3C2)eta  + 170C3 + 170C2)zeta + (13C3 + 13C2)eta
            +
              - 173C3 - 173C2
        *
                        t zeta
            cos(eta t)%e
        +
                    3             2                           alpha t
          (C1 alpha  - 13C1 alpha  + 170C1 alpha - 173C1)%e
        ]

     ]

                                            Type: Matrix Expression Integer.
```

Rewriting this last output, we have

$$(18)-(20) = (17)-(19) = \begin{pmatrix} 0 \\ 0 \\ C_1(\alpha^3 - 13\alpha^2 + 170\alpha - 173)e^{\alpha t} \\ +(C_2+C_3)(\zeta^3 - 13\zeta^2 + 170\zeta - 3\eta^2\zeta + 13\eta^2 - 173)e^{\zeta t}\cos(\eta t) \\ -\eta(C_2+C_3)(3\zeta^2 - 26\zeta - \eta^2 + 170)e^{\zeta t}\sin(\eta t) \end{pmatrix} \stackrel{38}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[38] See footnote 3, (12), (15), and (21).

Again, we have (18) = (20) and (17) = (19). So (16) has been shown to satisfy (1) twice. Next,

```
% wxmaxima
```

```
(%i4)   H2:matrix([C1·(alpha+62)·%e^(alpha·t)
                   +(C2+C3)·%e^(zeta·t)·((zeta+62)·cos(eta·t)−eta·sin(eta·t))],
                  [C1·(11·alpha−15)·%e^(alpha·t)
                   +(C2+C3)·%e^(zeta·t)·((11·zeta−15)·cos(eta·t)−11·eta·sin(eta·t))],
                  [C1·(alpha^2−5·alpha+28)·%e^(alpha·t)
                   +(C2+C3)·%e^(zeta·t)·((zeta^2−5·zeta−eta^2+28)·cos(eta·t)
                   −eta·(2·zeta−5)·sin(eta·t))]);
        /·This corresponds to (16).·/
        G2:diff(H2,t);
        /·This computes (17) to yield (18).·/
        A2:matrix([1,6,1],[−4,4,11],[−3,−9,8]);
        /·This corresponds to matrix A.·/
        ratsimp(expand(G2−A2.H2));
        /·This computes (17)−(19) to give (18)−(20).·/
```

(H2) $$\begin{pmatrix} (C3+C2)(\cos(\eta t)(\zeta+62)-\eta\sin(\eta t))\%e^{\zeta t} + C1(\alpha+62)\%e^{\alpha t} \\ (C3+C2)(\cos(\eta t)(11\zeta-15)-11\eta\sin(\eta t))\%e^{\zeta t} + C1(11\alpha-15)\%e^{\alpha t} \\ (C3+C2)(\cos(\eta t)(\zeta^2-5\zeta-\eta^2+28)-\eta\sin(\eta t)(2\zeta-5))\%e^{\zeta t} + C1(\alpha^2-5\alpha+28)\%e^{\alpha t} \end{pmatrix}$$

(G2) $$\begin{pmatrix} (C3+C2)(-\eta(\zeta+62)\sin(\eta t)-\eta^2\cos(\eta t))\%e^{\zeta t} + (C3+C2)\zeta((\zeta+62)\cos(\eta t)-\eta\sin(\eta t))\%e^{\zeta t} + C1\alpha(\alpha+62)\%e^{\alpha t} \\ (C3+C2)(-(11\zeta-15)\eta\sin(\eta t)-11\eta^2\cos(\eta t))\%e^{\zeta t} + (C3+C2)\zeta((11\zeta-15)\cos(\eta t)-11\eta\sin(\eta t))\%e^{\zeta t} + C1\alpha(11\alpha-15)\%e^{\alpha t} \\ (C3+C2)(-(\zeta^2-5\zeta-\eta^2+28)\eta\sin(\eta t)-\eta^2(2\zeta-5)\cos(\eta t))\%e^{\zeta t} + (C3+C2)\zeta((\zeta^2-5\zeta-\eta^2+28)\cos(\eta t)-\eta(2\zeta-5)\sin(\eta t))\%e^{\zeta t} + C1\alpha(\alpha^2-5\alpha+28)\%e^{\alpha t} \end{pmatrix}$$

(A2) $$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}$$

(%o4)

$$\begin{pmatrix} 0 \\ 0 \\ \Big((C3+C2)\cos(\eta t)\,\zeta^3 \\ +((-3C3-3C2)\eta\sin(\eta t)\ +(-13C3-13C2)\cos(\eta t))\,\zeta^2 \\ +\Big((26C3+26C2)\eta\sin(\eta t) \\ +((-3C3-3C2)\eta^2+170C3+170C2)\cos(\eta t)\Big)\zeta \\ +((C3+C2)\eta^3+(-170C3-170C2)\eta)\sin(\eta t) \\ +((13C3+13C2)\eta^2-173C3-173C2)\cos(\eta t)\Big)\%e^{\zeta t} \\ +(C1\alpha^3-13C1\alpha^2+170C1\alpha-173C1)\%e^{\alpha t} \end{pmatrix}.$$

Likewise, rewriting (%o4), we have

$$(18)-(20) = (17)-(19) = \begin{pmatrix} 0 \\ 0 \\ C_1(\alpha^3-13\alpha^2+170\alpha-173)e^{\alpha t} \\ +(C_2+C_3)(\zeta^3-13\zeta^2+170\zeta-3\eta^2\zeta+13\eta^2-173)e^{\zeta t}\cos(\eta t) \\ -\eta(C_2+C_3)(3\zeta^2-26\zeta-\eta^2+170)e^{\zeta t}\sin(\eta t) \end{pmatrix} =^{39} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again and again, we have (18) = (20) and (17) = (19). So (16) has been shown to satisfy (1) repeatedly. This leads us to hold that (16) is a real-valued solution of (1) devoid of IV and pay attention to trigonometric functions (TF's), which (16) contains [40] . What about the scalar multiplication (SM) of (16)? It follows from (16) that we have

$$\begin{cases} \mu F_1(t) = \mu[C_1(\alpha+62)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}] \\ \qquad\quad = \mu C_1(\alpha+62)e^{\alpha t}+(\mu C_2+\mu C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}, \\ \mu F_2(t) = \mu[C_1(11\alpha-15)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}] \\ \qquad\quad = \mu C_1(11\alpha-15)e^{\alpha t}+(\mu C_2+\mu C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}, \\ \mu F_3(t) = \mu[C_1(\alpha^2-5\alpha+28)e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}] \\ \qquad\quad = \mu C_1(\alpha^2-5\alpha+28)e^{\alpha t}+(\mu C_2+\mu C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}, \end{cases} \quad (22)$$

μ being a scalar . Rewriting (22) yields

$$\begin{cases} I_1(t) = C_4(\alpha+62)e^{\alpha t}+(C_5+C_6)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}, \\ I_2(t) = C_4(11\alpha-15)e^{\alpha t}+(C_5+C_6)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}, \\ I_3(t) = C_4(\alpha^2-5\alpha+28)e^{\alpha t}+(C_5+C_6)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}, \end{cases} \quad (23)$$

where C_i, $i = 4, 5, 6$, are AC's, too. (23) being essentially the same as (16), scalar multiples (sm's) of (16) also satisfy (1) [41] . In other words, (16) satisfies (1) up to an sm.

[39]Ditto.

[40]Though we didn't mention TF's in [1] *explicitly*, the astute reader might notice that if we replace a of e^{at}, which frequently appears therein, by i, we get e^{it}, which amounts to $\cos t + i\sin t$. So we are able to imagine that we have already mentioned them in [1] *implicitly*.

[41]*Cf.* **2.4**.

1.2.1 Taking IV into consideration

We now set $t=0$ in (4) in order to get the IV$(x_1(0),\ x_2(0),\ x_3(0))$, which we once neglected, as follows:

$$
\begin{cases}
x_1(0) = C_1(\alpha+62)e^{\alpha\cdot 0} + C_2(\beta+62)e^{\beta\cdot 0} + C_3(\gamma+62)e^{\gamma\cdot 0} \\
\quad = C_1(\alpha+62)e^{0} + C_2(\beta+62)e^{0} + C_3(\gamma+62)e^{0} \\
\quad = C_1(\alpha+62)\cdot 1 + C_2(\beta+62)\cdot 1 + C_3(\gamma+62)\cdot 1 \\
\quad = C_1(\alpha+62) + C_2(\beta+62) + C_3(\gamma+62) \\
\quad =^{42}\ C_1(\alpha+62) + C_2(\zeta+\eta i+62) + C_3(\zeta-\eta i+62) \\
\quad = C_1(\alpha+62) + C_2(\zeta+62+\eta i) + C_3(\zeta+62-\eta i) \\
\quad = C_1(\alpha+62) + (C_2+C_3)(\zeta+62) + (C_2-C_3)\eta i, \\
x_2(0) = C_1(11\alpha-15)e^{\alpha\cdot 0} + C_2(11\beta-15)e^{\beta\cdot 0} + C_3(11\gamma-15)e^{\gamma\cdot 0} \\
\quad = C_1(11\alpha-15)e^{0} + C_2(11\beta-15)e^{0} + C_3(11\gamma-15)e^{0} \\
\quad = C_1(11\alpha-15)\cdot 1 + C_2(11\beta-15)\cdot 1 + C_3(11\gamma-15)\cdot 1 \\
\quad = C_1(11\alpha-15) + C_2(11\beta-15) + C_3(11\gamma-15) \\
\quad =^{43}\ C_1(11\alpha-15) + C_2\{11(\zeta+\eta i)-15\} + C_3\{11(\zeta-\eta i)-15\} \\
\quad = C_1(11\alpha-15) + C_2(11\zeta+11\eta i-15) + C_3(11\zeta-11\eta i-15) \\
\quad = C_1(11\alpha-15) + C_2(11\zeta-15+11\eta i) + C_3(11\zeta-15-11\eta i) \\
\quad = C_1(11\alpha-15) + (C_2+C_3)(11\zeta-15) + 11(C_2-C_3)\eta i, \\
x_3(0) = C_1(\alpha^2-5\alpha+28)e^{\alpha\cdot 0} + C_2(\beta^2-5\beta+28)e^{\beta\cdot 0} + C_3(\gamma^2-5\gamma+28)e^{\gamma\cdot 0} \\
\quad = C_1(\alpha^2-5\alpha+28)e^{0} + C_2(\beta^2-5\beta+28)e^{0} + C_3(\gamma^2-5\gamma+28)e^{0} \\
\quad = C_1(\alpha^2-5\alpha+28)\cdot 1 + C_2(\beta^2-5\beta+28)\cdot 1 + C_3(\gamma^2-5\gamma+28)\cdot 1 \\
\quad = C_1(\alpha^2-5\alpha+28) + C_2(\beta^2-5\beta+28) + C_3(\gamma^2-5\gamma+28) \\
\quad =^{44}\ C_1(\alpha^2-5\alpha+28) + C_2\{(\zeta+\eta i)^2-5(\zeta+\eta i)+28\} \\
\qquad + C_3\{(\zeta-\eta i)^2-5(\zeta-\eta i)+28\} \\
\quad = C_1(\alpha^2-5\alpha+28) + C_2(\zeta^2+2\zeta\eta i-\eta^2-5\zeta-5\eta i+28) \\
\qquad + C_3(\zeta^2-2\zeta\eta i-\eta^2-5\zeta+5\eta i+28) \\
\quad = C_1(\alpha^2-5\alpha+28) + (C_2+C_3)(\zeta^2-\eta^2-5\zeta+28) + (C_2-C_3)\eta(2\zeta-5)i,
\end{cases}
\tag{24}
$$

from which we extract $\Re$'s as follows [45] :

[42] See footnote 12.
[43] Ditto.
[44] Ditto.
[45] Alternatively, setting $t=0$ in (16), we obtain

$$\begin{cases} x_{F1}(0) = \Re\{x_1(0)\} = C_1(\alpha+62) + (C_2+C_3)(\zeta+62)\,, \\ x_{F2}(0) = \Re\{x_2(0)\} = C_1(11\alpha-15) + (C_2+C_3)(11\zeta-15)\,, \\ x_{F3}(0) = \Re\{x_3(0)\} = C_1(\alpha^2-5\alpha+28) + (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)^{46}\,. \end{cases} \tag{25}$$

Eliminating $C_1(\alpha+62)$, $C_1(11\alpha-15)$, and $C_1(\alpha^2-5\alpha+28)$ between (16) and (25), we get

$$\begin{cases} J_1(t) = \{x_{F1}(0) - (C_2+C_3)(\zeta+62)\}e^{\alpha t} + (C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t - \eta\sin\eta t\}, \\ J_2(t) = \{x_{F2}(0) - (C_2+C_3)(11\zeta-15)\}e^{\alpha t} + (C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t - 11\eta\sin\eta t\}, \\ J_3(t) = \{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}e^{\alpha t} \\ \qquad + (C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t - \eta(2\zeta-5)\sin\eta t\}. \end{cases} \tag{26}$$

Setting $t = 0$ in (26) yields

$$\begin{cases} J_1(0) = \{x_{F1}(0) - (C_2+C_3)(\zeta+62)\}e^{\alpha\cdot 0} + (C_2+C_3)e^{\zeta\cdot 0}\{(\zeta+62)\cos\eta\cdot 0 - \eta\sin\eta\cdot 0\} \\ \qquad = \{x_{F1}(0) - (C_2+C_3)(\zeta+62)\}e^{0} + (C_2+C_3)e^{0}\{(\zeta+62)\cos 0 - \eta\sin 0\} \\ \qquad = \{x_{F1}(0) - (C_2+C_3)(\zeta+62)\}\cdot 1 + (C_2+C_3)\cdot 1\cdot\{(\zeta+62)\cdot 1 - \eta\cdot 0\} \\ \qquad = x_{F1}(0) - (C_2+C_3)(\zeta+62) + (C_2+C_3)(\zeta+62) \\ \qquad = x_{F1}(0)\,, \\ J_2(0) = \{x_{F2}(0) - (C_2+C_3)(11\zeta-15)\}e^{\alpha\cdot 0} + (C_2+C_3)e^{\zeta\cdot 0}\{(11\zeta-15)\cos\eta\cdot 0 - 11\eta\sin\eta\cdot 0\} \\ \qquad = \{x_{F2}(0) - (C_2+C_3)(11\zeta-15)\}e^{0} + (C_2+C_3)e^{0}\{(11\zeta-15)\cos 0 - 11\eta\sin 0\} \\ \qquad = \{x_{F2}(0) - (C_2+C_3)(11\zeta-15)\}\cdot 1 + (C_2+C_3)\cdot 1\cdot\{(11\zeta-15)\cdot 1 - 11\eta\cdot 0\} \\ \qquad = x_{F2}(0) - (C_2+C_3)(11\zeta-15) + (C_2+C_3)(11\zeta-15) \\ \qquad = x_{F2}(0)\,, \\ J_3(0) = \{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}e^{\alpha\cdot 0} \\ \qquad\quad + (C_2+C_3)e^{\zeta\cdot 0}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta\cdot 0 - \eta(2\zeta-5)\sin\eta\cdot 0\} \\ \qquad = \{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}e^{0} \\ \qquad\quad + (C_2+C_3)e^{0}\{(\zeta^2-5\zeta-\eta^2+28)\cos 0 - \eta(2\zeta-5)\sin 0\} \\ \qquad = \{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}\cdot 1 \\ \qquad\quad + (C_2+C_3)\cdot 1\cdot\{(\zeta^2-5\zeta-\eta^2+28)\cdot 1 - \eta(2\zeta-5)\cdot 0\} \\ \qquad = x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28) + (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28) \\ \qquad = x_{F3}(0)\,. \end{cases}$$

$$\begin{cases} F_1(0) = x_{F4}(0) = C_1(\alpha+62)e^{\alpha\cdot 0} + (C_2+C_3)e^{\zeta\cdot 0}\{(\zeta+62)\cos\eta\cdot 0 - \eta\sin\eta\cdot 0\} \\ \qquad = C_1(\alpha+62)e^{0} + (C_2+C_3)e^{0}\{(\zeta+62)\cos 0 - \eta\sin 0\} \\ \qquad = C_1(\alpha+62)\cdot 1 + (C_2+C_3)\cdot 1\cdot\{(\zeta+62)\cdot 1 - \eta\cdot 0\} \\ \qquad = C_1(\alpha+62) + (C_2+C_3)(\zeta+62)\,, \\ F_2(0) = x_{F5}(0) = C_1(11\alpha-15)e^{\alpha\cdot 0} + (C_2+C_3)e^{\zeta\cdot 0}\{(11\zeta-15)\cos\eta\cdot 0 - 11\eta\sin\eta\cdot 0\} \\ \qquad = C_1(11\alpha-15)e^{0} + (C_2+C_3)e^{0}\{(11\zeta-15)\cos 0 - 11\eta\sin 0\} \\ \qquad = C_1(11\alpha-15)\cdot 1 + (C_2+C_3)\cdot 1\cdot\{(11\zeta-15)\cdot 1 - 11\eta\cdot 0\} \\ \qquad = C_1(11\alpha-15) + (C_2+C_3)(11\zeta-15)\,, \\ F_3(0) = x_{F6}(0) = C_1(\alpha^2-5\alpha+28)e^{\alpha\cdot 0} + (C_2+C_3)e^{\zeta\cdot 0}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta\cdot 0 - \eta(2\zeta-5)\sin\eta\cdot 0\} \\ \qquad = C_1(\alpha^2-5\alpha+28)e^{0} + (C_2+C_3)e^{0}\{(\zeta^2-5\zeta-\eta^2+28)\cos 0 - \eta(2\zeta-5)\sin 0\} \\ \qquad = C_1(\alpha^2-5\alpha+28)\cdot 1 + (C_2+C_3)\cdot 1\cdot\{(\zeta^2-5\zeta-\eta^2+28)\cdot 1 - \eta(2\zeta-5)\cdot 0\} \\ \qquad = C_1(\alpha^2-5\alpha+28) + (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\,. \end{cases}$$

We see that these coincide with (25).

[46]By the way, are C_1, C_2, C_3 still arbitrary at this stage regardless of the constants α, ζ, and η? See **2.5**.

Having thus regot (and confirmed) the IV $(x_{F1}(0), x_{F2}(0), x_{F3}(0))$ (or $(x_{F4}(0), x_{F5}(0), x_{F6}(0))$) [47], we wish to know whether/how taking IV into account affects the arguments we have made so far. Specifically, we check if (26) satisfies (1) as follows:

1.2.2 Checking solution with IV in a componentwise manner

We substitute (26) into both sides of (1) and check each row, which is divided in three parts, or terms containing $e^{\alpha t}$, $e^{\zeta t}\cos\eta t$, and $e^{\zeta t}\sin\eta t$ [48].

Comparison of both sides of the first row

$e^{\alpha t}$ part

$$
\begin{aligned}
&\text{LHS} = \alpha \times \{x_{F1}(0) - (C_2 + C_3)(\zeta + 62)\}.\\
&\text{RHS} = 1 \times \{x_{F1}(0) - (C_2 + C_3)(\zeta + 62)\} + 6 \times \{x_{F2}(0) - (C_2 + C_3)(11\zeta - 15)\}\\
&\qquad + 1 \times \{x_{F3}(0) - (C_2 + C_3)(\zeta^2 - 5\zeta - \eta^2 + 28)\}.\\
&\text{LHS} - \text{RHS}\\
&= \alpha \times \{x_{F1}(0) - (C_2 + C_3)(\zeta + 62)\}\\
&\quad - [1 \times \{x_{F1}(0) - (C_2 + C_3)(\zeta + 62)\} + 6 \times \{x_{F2}(0) - (C_2 + C_3)(11\zeta - 15)\}\\
&\qquad + 1 \times \{x_{F3}(0) - (C_2 + C_3)(\zeta^2 - 5\zeta - \eta^2 + 28)\}]\\
&=^{49}\ \alpha \times \{C_1(\alpha + 62) + (C_2 + C_3)(\zeta + 62) - (C_2 + C_3)(\zeta + 62)\}\\
&\quad - [1 \times \{C_1(\alpha + 62) + (C_2 + C_3)(\zeta + 62) - (C_2 + C_3)(\zeta + 62)\}\\
&\qquad + 6 \times \{C_1(11\alpha - 15) + (C_2 + C_3)(11\zeta - 15) - (C_2 + C_3)(11\zeta - 15)\}\\
&\qquad + 1 \times \{C_1(\alpha^2 - 5\alpha + 28) + (C_2 + C_3)(\zeta^2 - 5\zeta - \eta^2 + 28)\\
&\qquad\qquad - (C_2 + C_3)(\zeta^2 - 5\zeta - \eta^2 + 28)\}]\\
&= C_1\alpha(\alpha + 62) - \{1 \times C_1(\alpha + 62) + 6 \times C_1(11\alpha - 15) + 1 \times C_1(\alpha^2 - 5\alpha + 28)\}\\
&= C_1\alpha(\alpha + 62) - C_1(\alpha + 62 + 66\alpha - 90 + \alpha^2 - 5\alpha + 28)\\
&= C_1\alpha(\alpha + 62) - C_1(\alpha^2 + 62\alpha)\\
&= 0.
\end{aligned}
$$

$e^{\zeta t}\cos\eta t$ part

$$
\begin{aligned}
&\text{LHS} = (C_2 + C_3)\{\zeta(\zeta + 62) - \eta \cdot \eta\}.\\
&\text{RHS} = 1 \times (C_2 + C_3)(\zeta + 62) + 6 \times (C_2 + C_3)(11\zeta - 15) + 1 \times (C_2 + C_3)(\zeta^2 - 5\zeta - \eta^2 + 28).\\
&\text{LHS} - \text{RHS}\\
&= (C_2 + C_3)\{\zeta(\zeta + 62) - \eta \cdot \eta\}\\
&\quad - \{1 \times (C_2 + C_3)(\zeta + 62) + 6 \times (C_2 + C_3)(11\zeta - 15) + 1 \times (C_2 + C_3)(\zeta^2 - 5\zeta - \eta^2 + 28)\}\\
&= (C_2 + C_3)(\zeta^2 + 62\zeta - \eta^2) - (C_2 + C_3)\{\zeta + 62 + 6(11\zeta - 15) + \zeta^2 - 5\zeta - \eta^2 + 28\}\\
&= (C_2 + C_3)(\zeta^2 + 62\zeta - \eta^2) - (C_2 + C_3)(\zeta^2 + 62\zeta - \eta^2)\\
&= 0.
\end{aligned}
$$

[47] See footnote 45.

[48] In what follows, 'X part' means comparing the coefficient of X in the LHS with that in the RHS and subtracting the latter from the former.

[49] See (25) for the replacement of $x_{Fi}(0)$, where $i = 1, 2, 3$.

$\underline{e^{\zeta t}\sin\eta t \text{ part}}$

$$\begin{aligned}
&\text{LHS} = (C_2+C_3)\{-\eta(\zeta+62)-\zeta\cdot\eta\}.\\
&\text{RHS} = 1\times(C_2+C_3)(-\eta)+6\times(C_2+C_3)(-11\eta)+1\times(C_2+C_3)\{-\eta(2\zeta-5)\}.\\
&\text{LHS}-\text{RHS}\\
&= (C_2+C_3)\{-\eta(\zeta+62)-\zeta\cdot\eta\}\\
&\quad -[1\times(C_2+C_3)(-\eta)+6\times(C_2+C_3)(-11\eta)+1\times(C_2+C_3)\{-\eta(2\zeta-5)\}]\\
&= (C_2+C_3)(-\zeta\eta-62\eta-\zeta\eta)-(C_2+C_3)(-\eta-66\eta-2\zeta\eta+5\eta)\\
&= (C_2+C_3)(-2\zeta\eta-62\eta)-(C_2+C_3)(-62\eta-2\zeta\eta)\\
&= 0.
\end{aligned}$$

So we have

$$\boxed{\text{the LHS of the first row} - \text{the RHS of the first row} = 0\cdot e^{\alpha t}+0\cdot e^{\zeta t}\cos\eta t+0\cdot e^{\zeta t}\sin\eta t = 0},$$

which means that the first row holds.

Comparison of both sides of the second row

$\underline{e^{\alpha t} \text{ part}}$

$$\begin{aligned}
&\text{LHS} = \alpha\times\{x_{F2}(0)-(C_2+C_3)(11\zeta-15)\}.\\
&\text{RHS} = -4\times\{x_{F1}(0)-(C_2+C_3)(\zeta+62)\}+4\times\{x_{F2}(0)-(C_2+C_3)(11\zeta-15)\}\\
&\qquad\quad +11\times\{x_{F3}(0)-(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}.\\
&\text{LHS}-\text{RHS}\\
&= \alpha\times\{x_{F2}(0)-(C_2+C_3)(11\zeta-15)\}-[-4\times\{x_{F1}(0)-(C_2+C_3)(\zeta+62)\}\\
&\quad +4\times\{x_{F2}(0)-(C_2+C_3)(11\zeta-15)\}+11\times\{x_{F3}(0)-(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}]\\
&\overset{50}{=} \alpha\times\{C_1(11\alpha-15)+(C_2+C_3)(11\zeta-15)-(C_2+C_3)(11\zeta-15)\}\\
&\quad -[-4\times\{C_1(\alpha+62)+(C_2+C_3)(\zeta+62)-(C_2+C_3)(\zeta+62)\}\\
&\qquad +4\times\{C_1(11\alpha-15)+(C_2+C_3)(11\zeta-15)-(C_2+C_3)(11\zeta-15)\}\\
&\qquad +11\times\{C_1(\alpha^2-5\alpha+28)+(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\\
&\qquad\qquad -(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}]\\
&= C_1(11\alpha^2-15\alpha)-\{-4\times C_1(\alpha+62)+4\times C_1(11\alpha-15)+11\times C_1(\alpha^2-5\alpha+28)\}\\
&= C_1(11\alpha^2-15\alpha)-C_1(-4\alpha-248+44\alpha-60+11\alpha^2-55\alpha+308)\\
&= C_1(11\alpha^2-15\alpha)-C_1(11\alpha^2-15\alpha)\\
&= 0.
\end{aligned}$$

$\underline{e^{\zeta t}\cos\eta t \text{ part}}$

$$\begin{aligned}
&\text{LHS} = (C_2+C_3)\{\zeta(11\zeta-15)-11\eta\cdot\eta\}.\\
&\text{RHS} = -4\times(C_2+C_3)(\zeta+62)+4\times(C_2+C_3)(11\zeta-15)+11\times(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28).
\end{aligned}$$

[50] Ditto.

$$\begin{aligned}
&\text{LHS} - \text{RHS} \\
&= (C_2+C_3)\{\zeta(11\zeta-15) - 11\eta\cdot\eta\} \\
&\quad -\{-4\times(C_2+C_3)(\zeta+62) + 4\times(C_2+C_3)(11\zeta-15) + 11\times(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\} \\
&= (C_2+C_3)(11\zeta^2-15\zeta-11\eta^2) \\
&\quad -(C_2+C_3)(-4\zeta-248+44\zeta-60+11\zeta^2-55\zeta-11\eta^2+308) \\
&= (C_2+C_3)(11\zeta^2-15\zeta-11\eta^2) - (C_2+C_3)(11\zeta^2-15\zeta-11\eta^2) \\
&= 0.
\end{aligned}$$

$e^{\zeta t}\sin\eta t$ part

$$\begin{aligned}
&\text{LHS} = (C_2+C_3)\{-\eta(11\zeta-15) - \zeta\cdot 11\eta\}. \\
&\text{RHS} = -4\times(C_2+C_3)(-\eta) + 4\times(C_2+C_3)(-11\eta) + 11\times(C_2+C_3)\{-\eta(2\zeta-5)\}. \\
&\text{LHS} - \text{RHS} \\
&= (C_2+C_3)\{-\eta(11\zeta-15) - \zeta\cdot 11\eta\} \\
&\quad -[-4\times(C_2+C_3)(-\eta) + 4\times(C_2+C_3)(-11\eta) + 11\times(C_2+C_3)\{-\eta(2\zeta-5)\}] \\
&= (C_2+C_3)(-11\zeta\eta+15\eta-11\zeta\eta) - (C_2+C_3)(4\eta-44\eta-22\zeta\eta+55\eta) \\
&= (C_2+C_3)(-22\zeta\eta+15\eta) - (C_2+C_3)(-22\zeta\eta+15\eta) \\
&= 0.
\end{aligned}$$

So we have

$$\boxed{\text{the LHS of the second row} - \text{the RHS of the second row} = 0\cdot e^{\alpha t} + 0\cdot e^{\zeta t}\cos\eta t + 0\cdot e^{\zeta t}\sin\eta t = 0},$$

which means that the second row holds.

Comparison of both sides of the third row

$e^{\alpha t}$ part

$$\begin{aligned}
&\text{LHS} = \alpha\times\{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}. \\
&\text{RHS} = -3\times\{x_{F1}(0) - (C_2+C_3)(\zeta+62)\} - 9\times\{x_{F2}(0) - (C_2+C_3)(11\zeta-15)\} \\
&\qquad\quad +8\times\{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}. \\
&\text{LHS} - \text{RHS} \\
&= \alpha\times\{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\} \\
&\quad -[-3\times\{x_{F1}(0) - (C_2+C_3)(\zeta+62)\} - 9\times\{x_{F2}(0) - (C_2+C_3)(11\zeta-15)\} \\
&\qquad +8\times\{x_{F3}(0) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}] \\
&\overset{51}{=} \alpha\times\{C_1(\alpha^2-5\alpha+28) + (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\} \\
&\quad -[-3\times\{C_1(\alpha+62) + (C_2+C_3)(\zeta+62) - (C_2+C_3)(\zeta+62)\} \\
&\qquad -9\times\{C_1(11\alpha-15) + (C_2+C_3)(11\zeta-15) - (C_2+C_3)(11\zeta-15)\} \\
&\qquad +8\times\{C_1(\alpha^2-5\alpha+28) + (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28) - (C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}]
\end{aligned}$$

[51] Ditto.

$= C_1\alpha(\alpha^2-5\alpha+28)-\{-3C_1(\alpha+62)-9C_1(11\alpha-15)+8C_1(\alpha^2-5\alpha+28)\}$
$= C_1(\alpha^3-5\alpha^2+28\alpha)-C_1(-3\alpha-186-99\alpha+135+8\alpha^2-40\alpha+224)$
$= C_1(\alpha^3-5\alpha^2+28\alpha)-C_1(8\alpha^2-142\alpha+173)$
$= C_1(\alpha^3-13\alpha^2+170\alpha-173)$
$=$ [52] $0.$

<u>$e^{\zeta t}\cos\eta t$ part</u>

LHS $= (C_2+C_3)\{\zeta(\zeta^2-5\zeta-\eta^2+28)-\eta^2(2\zeta-5)\}.$
RHS $= -3\times(C_2+C_3)(\zeta+62)-9\times(C_2+C_3)(11\zeta-15)+8\times(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28).$
LHS − RHS
$= (C_2+C_3)\{\zeta(\zeta^2-5\zeta-\eta^2+28)-\eta^2(2\zeta-5)\}$
$\quad -\{-3\times(C_2+C_3)(\zeta+62)-9\times(C_2+C_3)(11\zeta-15)+8\times(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}$
$= (C_2+C_3)(\zeta^3-5\zeta^2-3\zeta\eta^2+28\zeta+5\eta^2)$
$\quad -(C_2+C_3)(-3\zeta-186-99\zeta+135+8\zeta^2-40\zeta-8\eta^2+224)$
$= (C_2+C_3)(\zeta^3-5\zeta^2+28\zeta-3\eta^2\zeta+5\eta^2)-(C_2+C_3)(8\zeta^2-142\zeta-8\eta^2+173)$
$= (C_2+C_3)(\zeta^3-13\zeta^2+170\zeta-3\eta^2\zeta+13\eta^2-173)$
$=$ [53] $0.$

<u>$e^{\zeta t}\sin\eta t$ part</u>

LHS $= (C_2+C_3)\{-\eta(\zeta^2-5\zeta-\eta^2+28)-\zeta\eta(2\zeta-5)\}.$
RHS $= -3\times(C_2+C_3)(-\eta)-9\times(C_2+C_3)(-11\eta)+8\times(C_2+C_3)\{-\eta(2\zeta-5)\}.$
LHS − RHS
$= (C_2+C_3)\{-\eta(\zeta^2-5\zeta-\eta^2+28)-\zeta\eta(2\zeta-5)\}$
$\quad -[-3\times(C_2+C_3)(-\eta)-9\times(C_2+C_3)(-11\eta)+8\times(C_2+C_3)\{-\eta(2\zeta-5)\}]$
$= (C_2+C_3)\{-\eta(3\zeta^2-10\zeta-\eta^2+28)\}-(C_2+C_3)(3\eta+99\eta-16\zeta\eta+40\eta)$
$= -\eta(C_2+C_3)(3\zeta^2-10\zeta-\eta^2+28)-\eta(C_2+C_3)(-16\zeta+142)$
$= -\eta(C_2+C_3)(3\zeta^2-26\zeta-\eta^2+170)$
$=$ [54] $0.$

So we have

$$\boxed{\text{the LHS of the third row} - \text{the RHS of the third row} = 0\cdot e^{\alpha t}+0\cdot e^{\zeta t}\cos\eta t+0\cdot e^{\zeta t}\sin\eta t = 0},$$

which means that the third row holds. (26) has thus been shown to satisfy (1). Before we make computational verification as usual, we try to simplify (26) as follows:

At the outset, replacing C_2+C_3 by C_7, an AC, we make it a bit simpler:

[52] See footnote 3.
[53] See (12).
[54] See (15).

$$\begin{cases} K_1(t) = \{x_{F1}(0) - C_7(\zeta+62)\}e^{\alpha t} + C_7 e^{\zeta t}\{(\zeta+62)\cos\eta t - \eta\sin\eta t\}, \\ K_2(t) = \{x_{F2}(0) - C_7(11\zeta-15)\}e^{\alpha t} + C_7 e^{\zeta t}\{(11\zeta-15)\cos\eta t - 11\eta\sin\eta t\}, \\ K_3(t) = \{x_{F3}(0) - C_7(\zeta^2-5\zeta-\eta^2+28)\}e^{\alpha t} \\ \qquad\qquad + C_7 e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t - \eta(2\zeta-5)\sin\eta t\}. \end{cases} \tag{27}$$

Then, we prepare the following for later use:

Table 1. Some notational substitutions [55] , [56]

$x_{F1}(0)$	L	ζ	P
C_1	M	$x_{F2}(0)$	Q
α	N	$x_{F3}(0)$	R
$C_2 + C_3\,(= C_7)$	O	η	S

Making use of this table, we rewrite (25) and (27) (or 26) as

$$\begin{cases} L = M(N+62) + O(P+62), \\ Q = M(11N-15) + O(11P-15), \\ R = M(N^2-5N+28) + O(P^2-5P-S^2+28), \end{cases} \tag{28}$$

and

$$\begin{cases} T_1(t) = \{L - O(P+62)\}e^{Nt} + Oe^{Pt}\{(P+62)\cos St - S\sin St\}, \\ T_2(t) = \{Q - O(11P-15)\}e^{Nt} + Oe^{Pt}\{(11P-15)\cos St - 11S\sin St\}, \\ T_3(t) = \{R - O(P^2-5P-S^2+28)\}e^{Nt} \\ \qquad\qquad + Oe^{Pt}\{(P^2-5P-S^2+28)\cos St - S(2P-5)\sin St\}, \end{cases} \tag{29}$$

respectively. We now verify that (29) satisfies (1) as follows:

```
% open-axiom

(1) -> )read bp349_part2_1_2_2_1.input

U1:=matrix[[D((L-O*(P+62))*exp(N*t)+O*exp(P*t)*((P+62)*cos(S*t)
                -S*sin(S*t)),t)],
           [D((Q-O*(11*P-15))*exp(N*t)+O*exp(P*t)*((11*P-15)*cos(S*t)
                -11*S*sin(S*t)),t)],
           [D((R-O*(P^2-5*P-S^2+28))*exp(N*t)+O*exp(P*t)
                *((P^2-5*P-S^2+28)*cos(S*t)-S*(2*P-5)*sin(S*t)),t)]]
--(29) is plugged into the LHS of (1) and subjected to differentiation.
```

[55] Here we avoid using the notations that have already been employed in case for example, C, F, etc. should be confused with C_1 on p5, $F_1(t)$ on p17, and so forth. *Cf.* footnote 79.

[56] For example, the pair $(x_{F1}(0), L)$ reads '$x_{F1}(0)$ corresponds to L and can thus be replaced by it'. *Cf.* footnote 80.

```
   (1)
[
 [
                      P t                    2      2                      P t
   (- 20 P - 620)S %e    sin(S t) + (- O S   + O P   + 620 P)cos(S t)%e
    +
                                   N t
   (- N O P - 62N O + L N)%e
  ]

    ,

  [
                           P t
       (- 220 P + 150)S %e    sin(S t)
     +
               2         2                   P t
       (- 110 S   + 110 P   - 150 P)cos(S t)%e
     +
                                 N t
       (N Q - 11N O P + 15N O)%e

   ]

    ,


    [
            3          2                   P t
        (O S   + (- 30 P   + 100 P - 280)S)%e    sin(S t)
      +
                       2      3       2                     P t
        ((- 30 P + 50)S   + O P   - 50 P   + 280 P)cos(S t)%e
      +
             2              2                    N t
        (N O S   + N R - N O P   + 5N O P - 28N O)%e
      ]

    ]
                                                   Type: Matrix Expression Integer
```

```
A1:=matrix[[1,6,1],[-4,4,11],[-3,-9,8]]
--This corresponds to matrix A.

        + 1    6   1 +
        |            |
   (2)  |- 4   4   11|
        |            |
        +- 3  - 9  8 +

                                                            Type: Matrix Integer

V1:=matrix[[(L-O*(P+62))*exp(N*t)+O*exp(P*t)*((P+62)*cos(S*t)
             -S*sin(S*t))],
           [(Q-O*(11*P-15))*exp(N*t)+O*exp(P*t)*((11*P-15)*cos(S*t)
             -11*S*sin(S*t))],
           [(R-O*(P^2-5*P-S^2+28))*exp(N*t)
             +O*exp(P*t)*((P^2-5*P-S^2+28)*cos(S*t)
             -S*(2*P-5)*sin(S*t))]]
--This corresponds to (29).

   (3)
             P t                                 P t
   [[- O S %e    sin(S t) + (O P + 62O)cos(S t)%e

       +

                          N t
         (- O P - 62O + L)%e    ],


     [
                   P t                                   P t
         - 11O S %e    sin(S t) + (11O P - 15O)cos(S t)%e

       +

                          N t
         (Q - 11O P + 15O)%e
       ]

     ,
```

```
     [
                        P t
        (- 20 P + 50)S %e    sin(S t)
      +
              2       2                           P t
        (- O S   + O P   - 50 P + 280)cos(S t)%e
      +
            2           2                   N t
        (O S   + R - O P   + 50 P - 280)%e
      ]
     ]
```

Type: Matrix Expression Integer

```
U1-A1*V1
```

```
   (4)
        2                2                                      N t
   [[(- O S   - R - 6Q + O P   + (- N + 62)O P - 62N O + L N - L)%e   ],

            2                          2
    [(- 110 S   - 11R + (N - 4)Q + 110 P

                                     N t
      + (- 11N - 15)O P + 15N O + 4L)%e   ],

     [
            3           2                      P t
        (O S   + (- 30 P   + 260 P - 1700)S)%e    sin(S t)
      +
                          2       3         2                            P t
        ((- 30 P + 130)S   + O P   - 130 P   + 1700 P - 1730)cos(S t)%e
      +
                        2                                 2
            (N - 8)O S   + (N - 8)R + 9Q + (- N + 8)O P   + (5N - 142)O P
          +
            (- 28N + 173)O + 3L
       *
            N t
          %e
      ]

     ]
```

Type: Matrix Expression Integer.

Rewriting this last output, one gets

$$
\left(\begin{array}{r}
\{-OS^2 - R - 6Q + OP^2 + (62-N)OP - 62NO + LN - L\}e^{Nt} \\
\{-11OS^2 - 11R + (N-4)Q + 11OP^2 - (11N+15)OP + 15NO + 4L\}e^{Nt} \\
\{(N-8)OS^2 + (N-8)R + 9Q + (8-N)OP^2 + (5N-142)OP + (173-28N)O + 3L\}e^{Nt} \\
+\{(13O - 3OP)S^2 + OP^3 - 13OP^2 + 170OP - 173O\}e^{Pt}\cos(St) \\
+\{OS^3 - (3OP^2 - 26OP + 170O)S\}e^{Pt}\sin(St)
\end{array}\right).
$$

Using (28) for replacement, we try to simplify the coefficients of e^{Nt}, $e^{Pt}\cos(St)$, and $e^{Pt}\sin(St)$ in the above in a row-by-row manner:

```
(1) -> )read bp349_part2_1_2_2_2.input

row1:=-O*S^2-R-6*Q+O*P^2+(62-N)*O*P-62*N*O+L*N-L

              2                 2
   (1)  - O S  - R - 6Q + O P  + (- N + 62)O P - 62N O + L N - L

                                                Type: Polynomial Integer

subst(%,L=M*(N+62)+O*(P+62))
--See (28).

              2                 2                      2
   (2)  - O S  - R - 6Q + O P  + 61O P - 62O + M N  + 61M N - 62M

                                                Type: Expression Integer

subst(%,Q=M*(11*N-15)+O*(11*P-15))
--Ditto.

              2            2                    2
   (3)  - O S  - R + O P  - 5O P + 28O + M N  - 5M N + 28M

                                                Type: Expression Integer

subst(%,R=M*(N^2-5*N+28)+O*(P^2-5*P-S^2+28))
--Ditto.

   (4)  0

                                                Type: Expression Integer.
```

Since this last output is 0, the first row becomes $0 \times e^{Nt} = 0$. We proceed to the second row.

```
(1) -> )read bp349_part2_1_2_2_3.input


row2:=-11*O*S^2-11*R+(N-4)*Q+11*O*P^2-(11*N+15)*O*P+15*N*O+4*L

                 2                                2
 (1)   - 11O S   - 11R + (N - 4)Q + 11O P   + (- 11N - 15)O P + 15N O + 4L

                                                          Type: Polynomial Integer


subst(%,L=M*(N+62)+O*(P+62))
--See (28).

   (2)
      2                                  2
- 11O S   - 11R + (N - 4)Q + 11O P   + (- 11N - 11)O P + (15N + 248)O + 4M N
 +
     248M

                                                          Type: Expression Integer


subst(%,Q=M*(11*N-15)+O*(11*P-15))
--Ditto.

                   2                  2                             2
   (3)   - 11O S   - 11R + 11O P   - 55O P + 308O + 11M N   - 55M N + 308M

                                                          Type: Expression Integer


subst(%,R=M*(N^2-5*N+28)+O*(P^2-5*P-S^2+28))
--Ditto.

   (4)  0

                                                          Type: Expression Integer.
```

Likewise, since $0 \times e^{Nt} = 0$, the second row becomes 0, too. We divide the third row in three portions and try to simplify the coefficients in it. Such portions are designated as rows $3-5$.

Portion containing e^{Nt}

```
(1) -> )read bp349_part2_1_2_2_4.input

row3:=(N-8)*O*S^2+(N-8)*R+9*Q+(8-N)*O*P^2+(5*N-142)*O*P+(173-28*N)*O+3*L

   (1)
                 2                                 2
     (N - 8)O S   + (N - 8)R + 9Q + (- N + 8)O P   + (5N - 142)O P
   +
     (- 28N + 173)O + 3L

                                                  Type: Polynomial Integer


subst(%,L=M*(N+62)+O*(P+62))
--See (28).

   (2)
                 2                                 2
     (N - 8)O S   + (N - 8)R + 9Q + (- N + 8)O P   + (5N - 139)O P
   +
     (- 28N + 359)O + 3M N + 186M

                                                  Type: Expression Integer


subst(%,Q=M*(11*N-15)+O*(11*P-15))
--Ditto.

   (3)
                 2                            2
     (N - 8)O S   + (N - 8)R + (- N + 8)O P   + (5N - 40)O P + (- 28N + 224)O
   +
     102M N + 51M

                                                  Type: Expression Integer
```

```
subst(%,R=M*(N^2-5*N+28)+O*(P^2-5*P-S^2+28))
--Ditto. And the below output
```

```
           3        2
   (4)  M N  - 13M N  + 170M N - 173M
```

```
                                                    Type: Expression Integer
```

```
--is rewritten as M*N^3-13*M*N^2+170*M*N-173*M
--and subjected to factorization.
```

```
factor(M*N^3-13*M*N^2+170*M*N-173*M)
```

```
           3      2
   (5)  M(N  - 13N  + 170N - 173)
```

```
                                           Type: Factored Polynomial Integer.
```

This last output $M(N^3 - 13N^2 + 170N - 173)$ amounts to 0, since $N^3 - 13N^2 + 170N - 173 = \alpha^3 - 13\alpha^2 + 170\alpha - 173 = 0$ [57] . So this portion equals $0 \times e^{Nt} = 0$.

Portion containing $e^{Pt}\cos(St)$

```
(1) ->  )read bp349_part2_1_2_2_5.input

row4:=(13*O-3*O*P)*S^2+O*P^3-13*O*P^2+170*O*P-173*O
```

```
                          2      3        2
   (1)  (- 3O P + 13O)S  + O P  - 13O P  + 170O P - 173O
```

```
                                                    Type: Polynomial Integer
```

[57] See **Table 1** and footnote 3.

```
factor(%)

                          2    3      2
   (2)  - O((3P - 13)S  - P  + 13P  - 170P + 173)

                                          Type: Factored Polynomial Integer.
```

We observe this last output also amounts to 0, because (3P − 13) S^2 − P^3 + 13P^2 − 170P + 173 = − (P^3 − 13P^2 + 170P − 3S^2P + 13S^2 − 173) = $-(\zeta^3 - 13\zeta^2 + 170\zeta - 3\eta^2\zeta + 13\eta^2 - 173) = 0$ [58] . So this portion equals $0 \times e^{Pt}\cos(St) = 0$.

Portion containing $e^{Pt}\sin(St)$

```
(1) -> )read bp349_part2_1_2_2_6.input

row5:=O*S^3-(3*O*P^2-26*O*P+170*O)*S

             3          2
   (1)  O S  + (- 30 P  + 260 P - 170O)S

                                                 Type: Polynomial Integer


factor(%)

                 2     2
   (2)  O S(S  - 3P  + 26P - 170)

                                          Type: Factored Polynomial Integer.
```

This last output amounts to 0, too, because S^2 − 3P^2 + 26P − 170 = − (3P^2 − 26P − S^2 + 170) = $-(3\zeta^2 - 26\zeta - \eta^2 + 170) = 0$ [59] . So this portion equals $0 \times e^{Pt}\sin(St) = 0$. Hence, the third row equals row 3 + row 4 + row 5 = 0 + 0 + 0 = 0. Eventually, all the rows become 0, which means U1−A1*V1 amounts to $(0, 0, 0)^T$. (29) has thus been shown to satisfy (1). Since (26) (or (27)) has been simplified to become (29), (26) has been shown to satisfy (1) twice. Next,

```
% wxmaxima
```

[58]See **Table 1** and (12).
[59]See **Table 1** and (15).

```
(%i4) V2:matrix([(L−O·(P+62))·%e^(N·t)+O·%e^(P·t)·((P+62)·cos(S·t)−S·sin(S·t))],
                [(Q−O·(11·P−15))·%e^(N·t)
                 +O·%e^(P·t)·((11·P−15)·cos(S·t)−11·S·sin(S·t))],
                [(R−O·(P^2−5·P−S^2+28))·%e^(N·t)
                 +O·%e^(P·t)·((P^2−5·P−S^2+28)·cos(S·t)−S·(2·P−5)·sin(S·t))]);
      /·This corresponds to (29).·/
      U2:diff(V2,t);
      /·(29) is plugged into the LHS of (1) and subjected to differentiaion.·/
      A2:matrix([1,6,1],[−4,4,11],[−3,−9,8]);
      /·This corresponds to matrix A.·/
      ratsimp(expand(U2−A2.V2));
```

(V2)
$$\begin{pmatrix} O\%e^{Pt}((P+62)\cos(St)-S\sin(St))+(L-O(P+62))\%e^{Nt} \\ O\%e^{Pt}((11P-15)\cos(St)-11S\sin(St)) \\ +(Q-O(11P-15))\%e^{Nt} \\ O\%e^{Pt}((-S^2+P^2-5P+28)\cos(St)-(2P-5)S\sin(St)) \\ +(R-O(-S^2+P^2-5P+28))\%e^{Nt} \end{pmatrix}$$

(U2)
$$\begin{pmatrix} O\%e^{Pt}(-(P+62)S\sin(St)-S^2\cos(St)) \\ +OP\%e^{Pt}((P+62)\cos(St)-S\sin(St))+N(L-O(P+62))\%e^{Nt} \\ O\%e^{Pt}(-(11P-15)S\sin(St)-11S^2\cos(St)) \\ +OP\%e^{Pt}((11P-15)\cos(St)-11S\sin(St))+N(Q-O(11P-15))\%e^{Nt} \\ O\%e^{Pt}(-S(-S^2+P^2-5P+28)\sin(St)-(2P-5)S^2\cos(St)) \\ +OP\%e^{Pt}((-S^2+P^2-5P+28)\cos(St)-(2P-5)S\sin(St)) \\ +N(R-O(-S^2+P^2-5P+28))\%e^{Nt} \end{pmatrix}$$

(A2)
$$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}$$

(%o4)
$$\begin{pmatrix} (-OS^2-R-6Q+OP^2+(62-N)OP-62NO+LN-L)\%e^{Nt} \\ (-11OS^2-11R+(N-4)Q+11OP^2+(-11N-15)OP+15NO+4L)\%e^{Nt} \\ ((N-8)OS^2+(N-8)R+9Q+(8-N)OP^2+(5N-142)OP+(173-28N)O+3L)\%e^{Nt} \\ +((13O-3OP)S^2+OP^3-13OP^2+170OP-173O)\%e^{Pt}\cos(St) \\ +(OS^3+(-3OP^2+26OP-170O)S)\%e^{Pt}\sin(St) \end{pmatrix}.$$

Using (28) for replacement, we try to simplify the coefficients of e^{Nt}, $e^{Pt}\cos(St)$, and $e^{Pt}\sin(St)$ in the above in a row-by-row manner. As mentioned earlier, the third row is divided in three portions, which are designated as rows $3-5$.

```
(%i4)   row1:-O·S^2-R-6·Q+O·P^2+(62-N)·O·P-62·N·O+L·N-L;
        ratsubst(M·(N+62)+O·(P+62),L,row1);
        /·See (28).·/
        ratsubst(M·(11·N-15)+O·(11·P-15),Q,%);
        /·Ditto.·/
        ratsubst(M·(N^2-5·N+28)+O·(P^2-5·P-S^2+28),R,%);
        /·Ditto.·/
```

(row1) $-O\,S^2-R-6\,Q+O\,P^2+(62-N)\,O\,P-62\,N\,O+L\,N-L$

(%o2) $-O\,S^2-R-6\,Q+O\,P^2+61\,O\,P-62\,O+M\,N^2+61\,M\,N-62\,M$

(%o3) $-O\,S^2-R+O\,P^2-5\,O\,P+28\,O+M\,N^2-5\,M\,N+28\,M$

(%o4) 0.

Since this last output is 0, the first row becomes $0\times e^{Nt}=0$. We proceed to the second row.

```
(%i4)   row2:-11·O·S^2-11·R+(N-4)·Q+11·O·P^2+(-11·N-15)·O·P+15·N·O+4·L;
        ratsubst(M·(N+62)+O·(P+62),L,row2);
        /·See (28).·/
        ratsubst(M·(11·N-15)+O·(11·P-15),Q,%);
        /·Ditto.·/
        ratsubst(M·(N^2-5·N+28)+O·(P^2-5·P-S^2+28),R,%);
        /·Ditto.·/
```

(row2) $-11\,O\,S^2-11\,R+(N-4)\,Q+11\,O\,P^2+(-11\,N-15)\,O\,P+15\,N\,O+4\,L$

(%o2) $-11\,O\,S^2-11\,R+(N-4)\,Q+11\,O\,P^2+(-11\,N-11)\,O\,P+(15\,N+248)\,O+4\,M\,N+248\,M$

(%o3) $-11\,O\,S^2-11\,R+11\,O\,P^2-55\,O\,P+308\,O+11\,M\,N^2-55\,M\,N+308\,M$

(%o4) 0.

Likewise, since $0\times e^{Nt}=0$, the second row becomes 0, too. We divide the third row in three portions and try to simplify the coefficients in it:

Portion containing e^{Nt}

```
(%i6)  row3:(N−8)·O·S^2+(N−8)·R+9·Q+(8−N)·O·P^2+(5·N−142)·O·P+(173−28·N)·O+3·L;
       ratsubst(M·(N+62)+O·(P+62),L,row3);
       /·See (28).·/
       ratsubst(M·(11·N−15)+O·(11·P−15),Q,%);
       /·Ditto.·/
       ratsubst(M·(N^2−5·N+28)+O·(P^2−5·P−S^2+28),R,%);
       /·Ditto.·/
       factor(%);
       ratsubst(0,N^3−13·N^2+170·N−173,%);
       /·See Table 1 and footnote 3.·/
```

(row3) $(N-8)\,O\,S^2+(N-8)\,R+9\,Q+(8-N)\,O\,P^2+(5\,N-142)\,O\,P+(173-28\,N)\,O+3\,L$

(%o2) $(N-8)\,O\,S^2+(N-8)\,R+9\,Q+(8-N)\,O\,P^2+(5\,N-139)\,O\,P+(359-28\,N)\,O+3\,M\,N+186\,M$

(%o3) $(N-8)\,O\,S^2+(N-8)\,R+(8-N)\,O\,P^2+(5\,N-40)\,O\,P+(224-28\,N)\,O+102\,M\,N+51\,M$

(%o4) $M\,N^3-13\,M\,N^2+170\,M\,N-173\,M$

(%o5) $M\,(N^3-13\,N^2+170\,N-173)$

(%o6) 0.

Since $0 \times e^{Nt} = 0$, this portion amounts to 0.

Portion containing $e^{Pt}\cos(St)$

```
(%i3)  row4:(13·O−3·O·P)·S^2+O·P^3−13·O·P^2+170·O·P−173·O;
       factor(%);
       ratsubst(0,P^3−13·P^2+170·P−3·S^2·P+13·S^2−173,%);
       /·See Table 1 and (12).·/
```

(row4) $(13\,O-3\,O\,P)\,S^2+O\,P^3-13\,O\,P^2+170\,O\,P-173\,O$

(%o2) $-O\,(3\,P\,S^2-13\,S^2-P^3+13\,P^2-170\,P+173)$

(%o3) 0.

Likewise, since $0 \times e^{Pt}\cos(St) = 0$, this portion amounts to 0, too.

Portion containing $e^{Pt}\sin(St)$

```
(%i3) row5:O·S^3+(−3·O·P^2+26·O·P−170·O)·S;
      factor(%);
      ratsubst(0,3·P^2−26·P−S^2+170,%);
      /·See Table 1 and (15).·/
```

$$\text{(row5)}\quad O\,S^3+(-3\,O\,P^2+26\,O\,P-170\,O)\,S$$

$$\text{(\%o2)}\quad O\,S\,(S^2-3\,P^2+26\,P-170)$$

$$\text{(\%o3)}\quad 0$$

Likewise, since $0\times e^{Pt}\sin(St)=0$, this portion becomes 0, too. Hence, the third row equals row 3 + row 4 + row 5 = 0+0+0 = 0. Eventually, all the rows become 0, which means U2 − A2.V2 amounts to $(0,\,0,\,0)^{\mathrm{T}}$. (29) has thus been shown to satisfy (1) twice. Since (26) (or (27)) has been simplified to become (29), (26) has been shown to satisfy (1) repeatedly. This leads us to hold (26) is a real-valued solution of (1) with IV and pay attention to TF's, which (26) contains, again. What about the SM of (26)? It follows from (26) that we have

$$\begin{cases} \nu J_1(t)=\nu[\{x_{F1}(0)-(C_2+C_3)(\zeta+62)\}e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}] \\ \qquad\quad =\{\nu x_{F1}(0)-(\nu C_2+\nu C_3)(\zeta+62)\}e^{\alpha t}+(\nu C_2+\nu C_3)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}, \\ \nu J_2(t)=\nu[\{x_{F2}(0)-(C_2+C_3)(11\zeta-15)\}e^{\alpha t}+(C_2+C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}] \\ \qquad\quad =\{\nu x_{F2}(0)-(\nu C_2+\nu C_3)(11\zeta-15)\}e^{\alpha t}+(\nu C_2+\nu C_3)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}, \\ \nu J_3(t)=\nu[\{x_{F3}(0)-(C_2+C_3)(\zeta^2-5\zeta-\eta^2+28)\}e^{\alpha t} \\ \qquad\qquad +(C_2+C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}] \\ \qquad\quad =\{\nu x_{F3}(0)-(\nu C_2+\nu C_3)(\zeta^2-5\zeta-\eta^2+28)\}e^{\alpha t} \\ \qquad\qquad +(\nu C_2+\nu C_3)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}, \end{cases} \tag{30}$$

ν being a scalar . Rewriting (30) yields

$$\begin{cases} W_1(t)=\{x_{F7}(0)-(C_8+C_9)(\zeta+62)\}e^{\alpha t}+(C_8+C_9)e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t\}, \\ W_2(t)=\{x_{F8}(0)-(C_8+C_9)(11\zeta-15)\}e^{\alpha t}+(C_8+C_9)e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t\}, \\ W_3(t)=\{x_{F9}(0)-(C_8+C_9)(\zeta^2-5\zeta-\eta^2+28)\}e^{\alpha t} \\ \qquad\quad +(C_8+C_9)e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\}, \end{cases} \tag{31}$$

where C_i, $i=8,\,9$, are AC's, too. (31) being essentially the same as (26), sm's of (26) also satisfy (1) [60] . In other words, (26) satisfies (1) up to an sm.

[60] *Cf.* **2.4**.

1.3 Extracting $\Im$'s from GS

Now we extract $\Im$'s from (4), which has been inspired by and is deduced from (11) [61] .

$$\begin{aligned}
&\Im\{x_1(t)\}\\
&= \Im\{C_1(\alpha+62)e^{\alpha t}+C_2(\beta+62)e^{\beta t}+C_3(\gamma+62)e^{\gamma t}\}\\
&= \Im\{C_1(\alpha+62)e^{\alpha t}+C_2(\zeta+\eta i+62)e^{(\zeta+\eta i)t}+C_3(\zeta-\eta i+62)e^{(\zeta-\eta i)t}\}\\
&= \Im\{C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}(\zeta+\eta i+62)e^{\eta i t}+C_3e^{\zeta t}(\zeta-\eta i+62)e^{-\eta i t}\}\\
&= \Im\{C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}(\zeta+62+\eta i)e^{\eta i t}+C_3e^{\zeta t}(\zeta+62-\eta i)e^{-\eta i t}\}\\
&= \Im\{C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}(\zeta+62+\eta i)(\cos\eta t+i\sin\eta t)+C_3e^{\zeta t}(\zeta+62-\eta i)(\cos\eta t-i\sin\eta t)\}\\
&= \Im[C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}\{(\zeta+62)(\cos\eta t+i\sin\eta t)+\eta i(\cos\eta t+i\sin\eta t)\}\\
&\qquad +C_3e^{\zeta t}\{(\zeta+62)(\cos\eta t-i\sin\eta t)-\eta i(\cos\eta t-i\sin\eta t)\}]\\
&= \Im[C_1(\alpha+62)e^{\alpha t}+C_2e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t+\eta i\cos\eta t+(\zeta+62)i\sin\eta t\}\\
&\qquad +C_3e^{\zeta t}\{(\zeta+62)\cos\eta t-\eta\sin\eta t-\eta i\cos\eta t-(\zeta+62)i\sin\eta t\}]\\
&= C_2e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\}-C_3e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\}\\
&= (C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\}.
\end{aligned}$$

$$\begin{aligned}
&\Im\{x_2(t)\}\\
&= \Im\{C_1(11\alpha-15)e^{\alpha t}+C_2(11\beta-15)e^{\beta t}+C_3(11\gamma-15)e^{\gamma t}\}\\
&= \Im[C_1(11\alpha-15)e^{\alpha t}+C_2\{11(\zeta+\eta i)-15\}e^{(\zeta+\eta i)t}+C_3\{11(\zeta-\eta i)-15\}e^{(\zeta-\eta i)t}]\\
&= \Im\{C_1(11\alpha-15)e^{\alpha t}+C_2e^{\zeta t}(11\zeta+11\eta i-15)e^{\eta i t}+C_3e^{\zeta t}(11\zeta-11\eta i-15)e^{-\eta i t}\}\\
&= \Im\{C_1(11\alpha-15)e^{\alpha t}+C_2e^{\zeta t}(11\zeta-15+11\eta i)e^{\eta i t}+C_3e^{\zeta t}(11\zeta-15-11\eta i)e^{-\eta i t}\}\\
&= \Im\{C_1(11\alpha-15)e^{\alpha t}+C_2e^{\zeta t}(11\zeta-15+11\eta i)(\cos\eta t+i\sin\eta t)\\
&\qquad +C_3e^{\zeta t}(11\zeta-15-11\eta i)(\cos\eta t-i\sin\eta t)\}\\
&= \Im[C_1(11\alpha-15)e^{\alpha t}+C_2e^{\zeta t}\{(11\zeta-15)(\cos\eta t+i\sin\eta t)+11\eta i(\cos\eta t+i\sin\eta t)\}\\
&\qquad +C_3e^{\zeta t}\{(11\zeta-15)(\cos\eta t-i\sin\eta t)-11\eta i(\cos\eta t-i\sin\eta t)\}]\\
&= \Im[C_1(11\alpha-15)e^{\alpha t}\\
&\qquad +C_2e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t+11\eta i\cos\eta t+(11\zeta-15)i\sin\eta t\}\\
&\qquad +C_3e^{\zeta t}\{(11\zeta-15)\cos\eta t-11\eta\sin\eta t-11\eta i\cos\eta t-(11\zeta-15)i\sin\eta t\}]\\
&= C_2e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\}-C_3e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\}\\
&= (C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\}.
\end{aligned}$$

$$\begin{aligned}
&\Im\{x_3(t)\}\\
&= \Im\{C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2(\beta^2-5\beta+28)e^{\beta t}+C_3(\gamma^2-5\gamma+28)e^{\gamma t}\}\\
&= \Im[C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2\{(\zeta+\eta i)^2-5(\zeta+\eta i)+28\}e^{(\zeta+\eta i)t}\\
&\qquad +C_3\{(\zeta-\eta i)^2-5(\zeta-\eta i)+28\}e^{(\zeta-\eta i)t}]
\end{aligned}$$

[61] As in **1.2**, we replace β and γ by $\zeta+\eta i$ and $\zeta-\eta i$, respectively.

$$
\begin{aligned}
&= \Im[C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2e^{\zeta t}\{(\zeta+\eta i)^2-5(\zeta+\eta i)+28\}e^{\eta i t}\\
&\qquad +C_3e^{\zeta t}\{(\zeta-\eta i)^2-5(\zeta-\eta i)+28\}e^{-\eta i t}]\\
&= \Im\{C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2e^{\zeta t}(\zeta^2+2\zeta\eta i-\eta^2-5\zeta-5\eta i+28)e^{\eta i t}\\
&\qquad +C_3e^{\zeta t}(\zeta^2-2\zeta\eta i-\eta^2-5\zeta+5\eta i+28)e^{-\eta i t}\}\\
&= \Im[C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2e^{\zeta t}\{\zeta^2-5\zeta-\eta^2+28+\eta(2\zeta-5)i\}e^{\eta i t}\\
&\qquad +C_3e^{\zeta t}\{\zeta^2-5\zeta-\eta^2+28-\eta(2\zeta-5)i\}e^{-\eta i t}]\\
&= \Im[C_1(\alpha^2-5\alpha+28)e^{\alpha t}+C_2e^{\zeta t}\{\zeta^2-5\zeta-\eta^2+28+\eta(2\zeta-5)i\}(\cos\eta t+i\sin\eta t)\\
&\qquad +C_3e^{\zeta t}\{\zeta^2-5\zeta-\eta^2+28+\eta(5-2\zeta)i\}(\cos\eta t-i\sin\eta t)]\\
&= \Im[C_1(\alpha^2-5\alpha+28)e^{\alpha t}\\
&\qquad +C_2e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t-\eta(2\zeta-5)\sin\eta t\\
&\qquad\qquad +\eta(2\zeta-5)i\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)i\sin\eta t\}\\
&\qquad +C_3e^{\zeta t}\{(\zeta^2-5\zeta-\eta^2+28)\cos\eta t+\eta(5-2\zeta)\sin\eta t\\
&\qquad\qquad -\eta(2\zeta-5)i\cos\eta t-(\zeta^2-5\zeta-\eta^2+28)i\sin\eta t\}]\\
&= C_2e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}\\
&\quad -C_3e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}\\
&= (C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}.
\end{aligned}
$$

We have thus extracted the following real-valued functions from (4):

$$
\begin{cases}
X_1(t)=(C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\},\\
X_2(t)=(C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\},\\
X_3(t)=(C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}.
\end{cases}
\qquad (32)
$$ [62]

We check if (32) satisfies (1). Replacing $x_i(t)$, $i = 1, 2, 3$, in the LHS of (1) by $X_i(t)$, $i = 1, 2, 3$, respectively, one gets

$$
\frac{d}{dt}\begin{pmatrix}X_1(t)\\X_2(t)\\X_3(t)\end{pmatrix}=\begin{pmatrix}X_1'(t)\\X_2'(t)\\X_3'(t)\end{pmatrix}
$$

$$
=\begin{pmatrix}
[(C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\}]'\\
[(C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\}]'\\
[(C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}]'
\end{pmatrix}
\qquad (33)
$$

$$
=\begin{pmatrix}
(C_2-C_3)e^{\zeta t}\{2\eta(\zeta+31)\cos\eta t+(\zeta^2+62\zeta-\eta^2)\sin\eta t\}\\
(C_2-C_3)e^{\zeta t}\{\eta(22\zeta-15)\cos\eta t+(11\zeta^2-15\zeta-11\eta^2)\sin\eta t\}\\
(C_2-C_3)e^{\zeta t}\{\eta(3\zeta^2-10\zeta-\eta^2+28)\cos\eta t\\
+(\zeta^3-5\zeta^2-3\eta^2\zeta+28\zeta+5\eta^2)\sin\eta t\}
\end{pmatrix}.
\qquad (34)
$$

On the other hand, applying a similar procedure to its RHS, we get

[62] If we set $\eta = 0$, we just get $(X_1(t), X_2(t), X_3(t)) = (0, 0, 0)$. What else? See **2.3.1** and/or **2.3.2**.

$$
\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}
\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} \tag{35}
$$

$$
= \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}
\begin{pmatrix} (C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\} \\ (C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\} \\ (C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\} \end{pmatrix}
$$

$$
= \begin{pmatrix} 1\times(C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\} \\ +6\times(C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\} \\ +1\times(C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\} \\ -4\times(C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\} \\ +4\times(C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\} \\ +11\times(C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\} \\ -3\times(C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\} \\ -9\times(C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\} \\ +8\times(C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\} \end{pmatrix}
$$

$$
= \begin{pmatrix} (C_2-C_3)e^{\zeta t}\{2\eta(\zeta+31)\cos\eta t+(\zeta^2+62\zeta-\eta^2)\sin\eta t\} \\ (C_2-C_3)e^{\zeta t}\{\eta(22\zeta-15)\cos\eta t+(11\zeta^2-15\zeta-11\eta^2)\sin\eta t\} \\ (C_2-C_3)e^{\zeta t}\{2\eta(8\zeta-71)\cos\eta t+(8\zeta^2-142\zeta-8\eta^2+173)\sin\eta t\} \end{pmatrix}. \tag{36}
$$

Therefore,

$$
(34)-(36) = \begin{pmatrix} 0 \\ 0 \\ (C_2-C_3)e^{\zeta t}\eta(3\zeta^2-26\zeta-\eta^2+170)\cos\eta t \\ +(C_2-C_3)e^{\zeta t}(\zeta^3-13\zeta^2+170\zeta-3\eta^2\zeta+13\eta^2-173)\sin\eta t \end{pmatrix} \overset{63}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{37}
$$

Hence, we have (34) = (36), which means that we also have (33) = (35). So (32) satisfies (1), which we check as follows:

```
% open-axiom

(1) ->  )read bp349_part2_1_3_1.input

Y1:=matrix[[D((C2-C3)*exp(zeta*t)*(eta*cos(eta*t)
               +(zeta+62)*sin(eta*t)),t)],
            [D((C2-C3)*exp(zeta*t)*(11*eta*cos(eta*t)
               +(11*zeta-15)*sin(eta*t)),t)],
            [D((C2-C3)*exp(zeta*t)*(eta*(2*zeta-5)*cos(eta*t)
               +(zeta^2-5*zeta-eta^2+28)*sin(eta*t)),t)]]
--This computes (33) to yield (34).
```

[63] See (15) and (12).

```
(1)
[
  [
                        2                                      2   t zeta
        ((- C3 + C2)zeta  + (- 62C3 + 62C2)zeta + (C3 - C2)eta )%e
     *
        sin(eta t)
    +
                                                                t zeta
      ((- 2C3 + 2C2)eta zeta + (- 62C3 + 62C2)eta)cos(eta t)%e
    ]
  ,
  [
                            2                                        2
        ((- 11C3 + 11C2)zeta  + (15C3 - 15C2)zeta + (11C3 - 11C2)eta )
     *
          t zeta
        %e      sin(eta t)
    +
                                                                t zeta
      ((- 22C3 + 22C2)eta zeta + (15C3 - 15C2)eta)cos(eta t)%e
    ]
  ,
  [
                         3                  2
          (- C3 + C2)zeta  + (5C3 - 5C2)zeta
        +
                         2                                       2
          ((3C3 - 3C2)eta  - 28C3 + 28C2)zeta + (- 5C3 + 5C2)eta
     *
          t zeta
        %e      sin(eta t)
    +
                               2                                        3
          (- 3C3 + 3C2)eta zeta  + (10C3 - 10C2)eta zeta + (C3 - C2)eta
        +
          (- 28C3 + 28C2)eta
     *
                    t zeta
        cos(eta t)%e
    ]
  ]

                                       Type: Matrix Expression Integer
```

```
A1:=matrix[[1,6,1],[-4,4,11],[-3,-9,8]]
--This corresponds to matrix A.

        + 1    6   1 +
        |            |
   (2)  |- 4   4   11|
        |            |
        +- 3  - 9  8 +

                                                       Type: Matrix Integer

Z1:=matrix[[(C2-C3)*exp(zeta*t)*(eta*cos(eta*t)+(zeta+62)*sin(eta*t))],
           [(C2-C3)*exp(zeta*t)*(11*eta*cos(eta*t)
             +(11*zeta-15)*sin(eta*t))],
           [(C2-C3)*exp(zeta*t)*(eta*(2*zeta-5)*cos(eta*t)
             +(zeta^2-5*zeta-eta^2+28)*sin(eta*t))]]
--This corresponds to (32).

   (3)
   [
     [
                                            t zeta
         ((- C3 + C2)zeta - 62C3 + 62C2)%e      sin(eta t)
       +
                                      t zeta
         (- C3 + C2)eta cos(eta t)%e
       ]


     ,


     [
                                                t zeta
         ((- 11C3 + 11C2)zeta + 15C3 - 15C2)%e      sin(eta t)
       +
                                          t zeta
         (- 11C3 + 11C2)eta cos(eta t)%e
       ]

     ,
```

```
      [
                                   2                                  2
            ((- C3 + C2)zeta  + (5C3 - 5C2)zeta + (C3 - C2)eta

              - 28C3 + 28C2)

         *
              t zeta
            %e      sin(eta t)
         +
                                                                    t zeta
          ((- 2C3 + 2C2)eta zeta + (5C3 - 5C2)eta)cos(eta t)%e
         ]
      ]

                                                  Type: Matrix Expression Integer

Y1-A1*Z1
--This computes (33)-(35) to give (34)-(36).

   (4)
   [[0], [0],
     [
                                    3                    2
              (- C3 + C2)zeta  + (13C3 - 13C2)zeta
            +
                                    2                                          2
              ((3C3 - 3C2)eta  - 170C3 + 170C2)zeta + (- 13C3 + 13C2)eta
            +
              173C3 - 173C2
         *
              t zeta
            %e      sin(eta t)
         +
                                         2                                     3
             (- 3C3 + 3C2)eta zeta  + (26C3 - 26C2)eta zeta + (C3 - C2)eta
            +
              (- 170C3 + 170C2)eta
         *
                        t zeta
            cos(eta t)%e
         ]
      ]

                                                  Type: Matrix Expression Integer.
```

Rewriting this last output, we have

$$(34)-(36) = (33)-(35) = \begin{pmatrix} 0 \\ 0 \\ \eta(C_2-C_3)(3\zeta^2-26\zeta-\eta^2+170)e^{\zeta t}\cos(\eta t) \\ +(C_2-C_3)(\zeta^3-13\zeta^2+170\zeta-3\eta^2\zeta+13\eta^2-173)e^{\zeta t}\sin(\eta t) \end{pmatrix} =^{64} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again, we have (34) = (36) and (33) = (35). So (32) has been shown to satisfy (1) twice. Next,

```
% wxmaxima
```

```
(%i4)  Z2:matrix([(C2−C3)·%e^(zeta·t)·(eta·cos(eta·t)+(zeta+62)·sin(eta·t))],
                 [(C2−C3)·%e^(zeta·t)·(11·eta·cos(eta·t)+(11·zeta−15)·sin(eta·t))],
                 [(C2−C3)·%e^(zeta·t)·(eta·(2·zeta−5)·cos(eta·t)
                  +(zeta^2−5·zeta−eta^2+28)·sin(eta·t))]);
       /·This corresponds to (32).·/
       Y2:diff(Z2,t);
       /·This computes (33) to yield (34).·/
       A2:matrix([1,6,1],[−4,4,11],[−3,−9,8]);
       /·This corresponds to matrix A.·/
       ratsimp(expand(Y2−A2.Z2));
       /·This computes (33)−(35) to give (34)−(36).·/
```

(Z2)
$$\begin{pmatrix} (C2-C3)(\sin(\eta t)(\zeta+62)+\eta\cos(\eta t))\%e^{\zeta t} \\ (C2-C3)(\sin(\eta t)(11\zeta-15)+11\eta\cos(\eta t))\%e^{\zeta t} \\ (C2-C3)(\sin(\eta t)(\zeta^2-5\zeta-\eta^2+28)+\eta\cos(\eta t)(2\zeta-5))\%e^{\zeta t} \end{pmatrix}$$

(Y2)
$$\begin{pmatrix} (C2-C3)\zeta(\sin(\eta t)(\zeta+62)+\eta\cos(\eta t))\%e^{\zeta t} \\ +(C2-C3)(\eta\cos(\eta t)(\zeta+62)-\eta^2\sin(\eta t))\%e^{\zeta t} \\ (C2-C3)\zeta(\sin(\eta t)(11\zeta-15)+11\eta\cos(\eta t))\%e^{\zeta t} \\ +(C2-C3)(\eta\cos(\eta t)(11\zeta-15)-11\eta^2\sin(\eta t))\%e^{\zeta t} \\ (C2-C3)\zeta(\sin(\eta t)(\zeta^2-5\zeta-\eta^2+28)+\eta\cos(\eta t)(2\zeta-5))\%e^{\zeta t} \\ +(C2-C3)(\eta\cos(\eta t)(\zeta^2-5\zeta-\eta^2+28)-\eta^2\sin(\eta t)(2\zeta-5))\%e^{\zeta t} \end{pmatrix}$$

(A2)
$$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}$$

[64] See (15), (12), and (37).

(%o4)

$$\begin{pmatrix} 0 \\ 0 \\ \Big((C2-C3)\sin(\eta t)\zeta^3+((13C3-13C2)\sin(\eta t)+(3C2-3C3)\eta\cos(\eta t))\zeta^2 \\ +\Big(((3C3-3C2)\eta^2-170C3+170C2)\sin(\eta t)+(26C3-26C2)\eta\cos(\eta t)\Big)\zeta \\ +((13C2-13C3)\eta^2+173C3-173C2)\sin(\eta t) \\ +((C3-C2)\eta^3+(170C2-170C3)\eta)\cos(\eta t)\Big)\%e^{\zeta t} \end{pmatrix}.$$

Likewise, rewriting (%o4), we have

$$(34)-(36)=(33)-(35)=\begin{pmatrix} 0 \\ 0 \\ \eta(C_2-C_3)(3\zeta^2-26\zeta-\eta^2+170)e^{\zeta t}\cos(\eta t) \\ +(C_2-C_3)(\zeta^3-13\zeta^2+170\zeta-3\eta^2\zeta+13\eta^2-173)e^{\zeta t}\sin(\eta t) \end{pmatrix} =^{65} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again and again, we have (34) = (36) and (33) = (35). So (32) has been shown to satisfy (1) repeatedly. This leads us to hold that (32) is another real-valued solution of (1) devoid of IV and repeatedly pay attention to TF's, which (32) contains. What about the SM of (32)? It follows from (32) that we have

$$\begin{cases} \xi X_1(t)=\xi(C_2-C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\} \\ \qquad\quad =(\xi C_2-\xi C_3)e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\}, \\ \xi X_2(t)=\xi(C_2-C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\} \\ \qquad\quad =(\xi C_2-\xi C_3)e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\}, \\ \xi X_3(t)=\xi(C_2-C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\} \\ \qquad\quad =(\xi C_2-\xi C_3)e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}, \end{cases} \tag{38}$$

ξ being a scalar . Rewriting (38) yields

$$\begin{cases} a_1(t)=(C_{10}-C_{11})e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\}, \\ a_2(t)=(C_{10}-C_{11})e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\}, \\ a_3(t)=(C_{10}-C_{11})e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}, \end{cases} \tag{39}$$

where C_{10} and C_{11} are AC's, too. (39) being essentially the same as (32), sm's of (32) also satisfy (1) [66] . In other words, (32) satisfies (1) up to an sm.

[65] Ditto.
[66] *Cf.* **2.4**.

1.3.1 Taking IV into consideration again

Letting IV be $(x_{G1}(0), x_{G2}(0), x_{G3}(0))$, we extract $\Im$'s from (24) as follows [67] :

$$\begin{cases} x_{G1}(0) = \Im\{x_1(0)\} = (C_2 - C_3)\eta, \\ x_{G2}(0) = \Im\{x_2(0)\} = 11(C_2 - C_3)\eta, \\ x_{G3}(0) = \Im\{x_3(0)\} = (C_2 - C_3)\eta(2\zeta - 5). \end{cases} \tag{40}$$

Eliminating $C_2 - C_3$ between (32) and (40), we get

$$\begin{cases} b_1(t) = \frac{x_{G1}(0)}{\eta} e^{\zeta t}\{\eta \cos \eta t + (\zeta + 62) \sin \eta t\}, \\ b_2(t) = \frac{x_{G2}(0)}{11\eta} e^{\zeta t}\{11\eta \cos \eta t + (11\zeta - 15) \sin \eta t\}, \\ b_3(t) = \frac{x_{G3}(0)}{\eta(2\zeta-5)} e^{\zeta t}\{\eta(2\zeta - 5) \cos \eta t + (\zeta^2 - 5\zeta - \eta^2 + 28) \sin \eta t\}. \end{cases} \tag{41}$$ [68]

Setting $t = 0$ in (41) gives

$$\begin{cases} b_1(0) = \frac{x_{G1}(0)}{\eta} e^{\zeta \cdot 0}\{\eta \cos \eta \cdot 0 + (\zeta + 62) \sin \eta \cdot 0\} = \frac{x_{G1}(0)}{\eta} e^0\{\eta \cos 0 + (\zeta + 62) \sin 0\} \\ \qquad = \frac{x_{G1}(0)}{\eta} \cdot 1 \cdot \{\eta \cdot 1 + (\zeta + 62) \cdot 0\} = \frac{x_{G1}(0)}{\eta} \cdot \eta = x_{G1}(0), \\ b_2(0) = \frac{x_{G2}(0)}{11\eta} e^{\zeta \cdot 0}\{11\eta \cos \eta \cdot 0 + (11\zeta - 15) \sin \eta \cdot 0\} = \frac{x_{G2}(0)}{11\eta} e^0\{11\eta \cos 0 + (11\zeta - 15) \sin 0\} \\ \qquad = \frac{x_{G2}(0)}{11\eta} \cdot 1 \cdot \{11\eta \cdot 1 + (11\zeta - 15) \cdot 0\} = \frac{x_{G2}(0)}{11\eta} \cdot 11\eta = x_{G2}(0), \\ b_3(0) = \frac{x_{G3}(0)}{\eta(2\zeta-5)} e^{\zeta \cdot 0}\{\eta(2\zeta - 5) \cos \eta \cdot 0 + (\zeta^2 - 5\zeta - \eta^2 + 28) \sin \eta \cdot 0\} \\ \qquad = \frac{x_{G3}(0)}{\eta(2\zeta-5)} e^0\{\eta(2\zeta - 5) \cos 0 + (\zeta^2 - 5\zeta - \eta^2 + 28) \sin 0\} \\ \qquad = \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot 1 \cdot \{\eta(2\zeta - 5) \cdot 1 + (\zeta^2 - 5\zeta - \eta^2 + 28) \cdot 0\} = \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot \eta(2\zeta - 5) = x_{G3}(0). \end{cases}$$

Having regot (and confirmed) the IV $(x_{G1}(0), x_{G2}(0), x_{G3}(0))$ (or $(x_{G4}(0),\ x_{G5}(0),\ x_{G6}(0))$) [69] , we verify that (41) satisfies (1) as shown in the following.

1.3.2 Checking solution with IV in a componentwise manner again

We substitute (41) into both sides of (1) and check each row, which is divided in two parts, or terms containing $e^{\zeta t} \cos \eta t$ and $e^{\zeta t} \sin \eta t$.

[67] Alternatively, setting $t = 0$ in (32), we get

$$\begin{cases} X_1(0) = x_{G4}(0) = (C_2 - C_3)e^{\zeta \cdot 0}\{\eta \cos \eta \cdot 0 + (\zeta + 62) \sin \eta \cdot 0\} = (C_2 - C_3)e^0\{\eta \cos 0 + (\zeta + 62) \sin 0\} \\ \qquad = (C_2 - C_3) \cdot 1 \cdot \{\eta \cdot 1 + (\zeta + 62) \cdot 0\} = (C_2 - C_3)\eta, \\ X_2(0) = x_{G5}(0) = (C_2 - C_3)e^{\zeta \cdot 0}\{11\eta \cos \eta \cdot 0 + (11\zeta - 15) \sin \eta \cdot 0\} \\ \qquad = (C_2 - C_3)e^0\{11\eta \cos 0 + (11\zeta - 15) \sin 0\} \\ \qquad = (C_2 - C_3) \cdot 1 \cdot \{11\eta \cdot 1 + (11\zeta - 15) \cdot 0\} = 11(C_2 - C_3)\eta, \\ X_3(0) = x_{G6}(0) = (C_2 - C_3)e^{\zeta \cdot 0}\{\eta(2\zeta - 5) \cos \eta \cdot 0 + (\zeta^2 - 5\zeta - \eta^2 + 28) \sin \eta \cdot 0\} \\ \qquad = (C_2 - C_3)e^0\{\eta(2\zeta - 5) \cos 0 + (\zeta^2 - 5\zeta - \eta^2 + 28) \sin 0\} \\ \qquad = (C_2 - C_3) \cdot 1 \cdot \{\eta(2\zeta - 5) \cdot 1 + (\zeta^2 - 5\zeta - \eta^2 + 28) \cdot 0\} = (C_2 - C_3)\eta(2\zeta - 5). \end{cases}$$

We see that these coincide with (40). *Cf.* footnote 45.

[68] Considering the case where $\zeta = 0$ and $\eta \neq 0$ leads us to 'efface' $e^{\zeta t}$, since $e^{0 \cdot t} = e^0 = 1$. What else? See **2.3.3**.

[69] See footnote 67.

Comparison of both sides of the first row

$e^{\zeta t}\cos\eta t$ part [70]

LHS $= \frac{x_{G1}(0)}{\eta}\cdot\{\zeta\eta+(\zeta+62)\eta\}$.

RHS $= 1\times\frac{x_{G1}(0)}{\eta}\cdot\eta+6\times\frac{x_{G2}(0)}{11\eta}\cdot 11\eta+1\times\frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot\eta(2\zeta-5)$.

LHS − RHS

$= \frac{x_{G1}(0)}{\eta}\cdot\{\zeta\eta+(\zeta+62)\eta\}-\{1\times\frac{x_{G1}(0)}{\eta}\cdot\eta+6\times\frac{x_{G2}(0)}{11\eta}\cdot 11\eta+1\times\frac{x_{G3}(0)}{\eta(2\eta-5)}\cdot\eta(2\eta-5)\}$

$= 2x_{G1}(0)(\zeta+31)-\{x_{G1}(0)+6x_{G2}(0)+x_{G3}(0)\}$

= [71] $2(C_2-C_3)\eta(\zeta+31)-\{(C_2-C_3)\eta+6\cdot 11(C_2-C_3)\eta+(C_2-C_3)\eta(2\zeta-5)\}$

$= 2(C_2-C_3)\eta(\zeta+31)-2(C_2-C_3)\eta(\zeta+31)$

$= 0$.

$e^{\zeta t}\sin\eta t$ part

LHS $= \frac{x_{G1}(0)}{\eta}\cdot\{-\eta^2+\zeta(\zeta+62)\}$.

RHS $= 1\times\frac{x_{G1}(0)}{\eta}\cdot(\zeta+62)+6\times\frac{x_{G2}(0)}{11\eta}\cdot(11\zeta-15)+1\times\frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot(\zeta^2-5\zeta-\eta^2+28)$.

LHS − RHS $= \frac{x_{G1}(0)}{\eta}\cdot\{-\eta^2+\zeta(\zeta+62)\}$

$-\{1\times\frac{x_{G1}(0)}{\eta}\cdot(\zeta+62)+6\times\frac{x_{G2}(0)}{11\eta}\cdot(11\zeta-15)+1\times\frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot(\zeta^2-5\zeta-\eta^2+28)\}$

= [72] $(C_2-C_3)(\zeta^2+62\zeta-\eta^2)$

$-\{(C_2-C_3)(\zeta+62)+(C_2-C_3)(66\zeta-90)+(C_2-C_3)(\zeta^2-5\zeta-\eta^2+28)\}$

$= (C_2-C_3)(\zeta^2+62\zeta-\eta^2)-(C_2-C_3)(\zeta^2+62\zeta-\eta^2)$

$= 0$.

So we have

$$\text{the LHS of the first row} - \text{the RHS of the first row} = 0\cdot e^{\zeta t}\cos\eta t+0\cdot e^{\zeta t}\sin\eta t=0$$,

which means that the first row holds.

Comparison of both sides of the second row

$e^{\zeta t}\cos\eta t$ part

LHS $= \frac{x_{G2}(0)}{11\eta}\cdot\{11\zeta\eta+(11\zeta-15)\eta\}$.

RHS $= -4\times\frac{x_{G1}(0)}{\eta}\cdot\eta+4\times\frac{x_{G2}(0)}{11\eta}\cdot 11\eta+11\times\frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot\eta(2\zeta-5)$.

LHS − RHS $= \frac{x_{G2}(0)}{11\eta}\cdot\{11\zeta\eta+(11\zeta-15)\eta\}$

$-\{-4\times\frac{x_{G1}(0)}{\eta}\cdot\eta+4\times\frac{x_{G2}(0)}{11\eta}\cdot 11\eta+11\times\frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot\eta(2\zeta-5)\}$

[70] See footnote 48 for what '*X* part' means.

[71] We have eliminated $x_{Gi}(0)$, i = 1, 2, 3, between (40) and the above.

[72] Ditto.

$$= \frac{x_{G2}(0)}{11} \cdot (11\zeta + 11\zeta - 15) - \{-4x_{G1}(0) + 4x_{G2}(0) + 11x_{G3}(0)\}$$
$$\overset{73}{=} (C_2 - C_3)(22\zeta\eta - 15\eta)$$
$$- \{-4(C_2 - C_3)\eta + 44(C_2 - C_3)\eta + 11(C_2 - C_3)\eta(2\zeta - 5)\}$$
$$= (C_2 - C_3)(22\zeta\eta - 15\eta) - (C_2 - C_3)(22\zeta\eta - 15\eta)$$
$$= 0.$$

$e^{\zeta t} \sin \eta t$ part

$$\text{LHS} = \frac{x_{G2}(0)}{11\eta}\{-11\eta^2 + \zeta(11\zeta - 15)\}.$$
$$\text{RHS} = -4 \times \frac{x_{G1}(0)}{\eta} \cdot (\zeta + 62) + 4 \times \frac{x_{G2}(0)}{11\eta} \cdot (11\zeta - 15) + 11 \times \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot (\zeta^2 - 5\zeta - \eta^2 + 28).$$
$$\text{LHS} - \text{RHS} = \frac{x_{G2}(0)}{11\eta} \cdot \{-11\eta^2 + \zeta(11\zeta - 15)\}$$
$$- \{-4 \times \frac{x_{G1}(0)}{\eta} \cdot (\zeta + 62) + 4 \times \frac{x_{G2}(0)}{11\eta} \cdot (11\zeta - 15) + 11 \times \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot (\zeta^2 - 5\zeta - \eta^2 + 28)\}$$
$$\overset{74}{=} (C_2 - C_3)(11\zeta^2 - 15\zeta - 11\eta^2)$$
$$- \{-4(C_2 - C_3)(\zeta + 62) + 4(C_2 - C_3)(11\zeta - 15) + 11(C_2 - C_3)(\zeta^2 - 5\zeta - \eta^2 + 28)\}$$
$$= (C_2 - C_3)(11\zeta^2 - 15\zeta - 11\eta^2) - (C_2 - C_3)(11\zeta^2 - 15\zeta - 11\eta^2)$$
$$= 0.$$

So we have

$$\boxed{\text{the LHS of the second row} - \text{the RHS of the second row} = 0 \cdot e^{\zeta t}\cos \eta t + 0 \cdot e^{\zeta t} \sin \eta t = 0},$$

which means that the second row holds.

Comparison of both sides of the third row

$e^{\zeta t} \cos \eta t$ part

$$\text{LHS} = \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot \{\zeta\eta(2\zeta - 5) + \eta(\zeta^2 - 5\zeta - \eta^2 + 28)\}.$$
$$\text{RHS} = -3 \times \frac{x_{G1}(0)}{\eta} \cdot \eta - 9 \times \frac{x_{G2}(0)}{11\eta} \cdot 11\eta + 8 \times \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot \eta(2\zeta - 5).$$
$$\text{LHS} - \text{RHS} = \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot \{\zeta\eta(2\zeta - 5) + \eta(\zeta^2 - 5\zeta - \eta^2 + 28)\}$$
$$- \{-3 \times \frac{x_{G1}(0)}{\eta} \cdot \eta - 9 \times \frac{x_{G2}(0)}{11\eta} \cdot 11\eta + 8 \times \frac{x_{G3}(0)}{\eta(2\zeta-5)} \cdot \eta(2\zeta - 5)\}$$
$$\overset{75}{=} (C_2 - C_3)\{\zeta\eta(2\zeta - 5) + \eta(\zeta^2 - 5\zeta - \eta^2 + 28)\}$$
$$- \{-3(C_2 - C_3)\eta - 99(C_2 - C_3)\eta + 8(C_2 - C_3)\eta(2\zeta - 5)\}$$
$$= \eta(C_2 - C_3)(3\zeta^2 - 10\zeta - \eta^2 + 28) - \eta(C_2 - C_3)(-142 + 16\zeta)$$
$$= \eta(C_2 - C_3)(3\zeta^2 - 26\zeta - \eta^2 + 170)$$
$$\overset{76}{=} \eta(C_2 - C_3) \cdot 0$$
$$= 0.$$

[73] Ditto.
[74] Ditto.
[75] Ditto.
[76] See (15).

$\underline{e^{\zeta t}\sin\eta t \text{ part}}$

$$\text{LHS} = \frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot\{-(2\zeta-5)\eta^2+\zeta(\zeta^2-5\zeta-\eta^2+28)\}.$$

$$\text{RHS} = -3\times\frac{x_{G1}(0)}{\eta}\cdot(\zeta+62)-9\times\frac{x_{G2}(0)}{11\eta}\cdot(11\zeta-15)+8\times\frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot(\zeta^2-5\zeta-\eta^2+28).$$

$$\begin{aligned}
\text{LHS} - \text{RHS} &= \frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot\{-(2\zeta-5)\eta^2+\zeta(\zeta^2-5\zeta-\eta^2+28)\}\\
&\quad -\{-3\times\frac{x_{G1}(0)}{\eta}\cdot(\zeta+62)-9\times\frac{x_{G2}(0)}{11\eta}\cdot(11\zeta-15)\\
&\qquad +8\times\frac{x_{G3}(0)}{\eta(2\zeta-5)}\cdot(\zeta^2-5\zeta-\eta^2+28)\}\\
&=^{77}\ (C_2-C_3)\{-(2\zeta-5)\eta^2+\zeta(\zeta^2-5\zeta-\eta^2+28)\}\\
&\quad -\{-3(C_2-C_3)(\zeta+62)-9(C_2-C_3)(11\zeta-15)\\
&\qquad +8(C_2-C_3)(\zeta^2-5\zeta-\eta^2+28)\}\\
&=(C_2-C_3)(\zeta^3-5\zeta^2-3\eta^2\zeta+5\eta^2+28\zeta)-(C_2-C_3)(8\zeta^2-142\zeta-8\eta^2+173)\\
&=(C_2-C_3)(\zeta^3-13\zeta^2+170\zeta-3\eta^2\zeta+13\eta^2-173)\\
&=^{78}\ (C_2-C_3)\cdot 0\\
&=0.
\end{aligned}$$

So we have

$$\boxed{\text{the LHS of the third row} - \text{the RHS of the third row} = 0\cdot e^{\zeta t}\cos\eta t+0\cdot e^{\zeta t}\sin\eta t=0},$$

which means that the third row holds. (41) has thus been shown to satisfy (1).

Since we wish to make computational verification as usual, we prepare the following for later use:

Table 2. Some notational substitutions [79], [80]

$x_{G1}(0)$	Aa	$x_{G2}(0)$	Dd
C_2-C_3	Bb	$x_{G3}(0)$	Ee
η	Cc	ζ	Ff

According to this table, we rewrite (40) and (41) as

$$\begin{cases} Aa = BbCc,\\ Dd = 11BbCc,\\ Ee = BbCc(2Ff-5)\end{cases} \tag{42}$$

and

$$\begin{cases} c_1(t) = \frac{Aa}{Cc}\cdot e^{Fft}\{Cc\cos Cct+(Ff+62)\sin Cct\},\\ c_2(t) = \frac{Dd}{11Cc}\cdot e^{Fft}\{11Cc\cos Cct+(11Ff-15)\sin Cct\},\\ c_3(t) = \frac{Ee}{Cc(2Ff-5)}\cdot e^{Fft}\{Cc(2Ff-5)\cos Cct+(Ff^2-5Ff-Cc^2+28)\sin Cct\},\end{cases} \tag{43}$$

respectively. Eliminating Aa, Dd, and Ee between (42) and (43), one gets

[77] We have eliminated $x_{Gi}(0)$, $i = 1, 2, 3$, between (40) and the above.

[78] See (12).

[79] As in footnote 55, we avoid using the notations that have already been employed.

[80] For example, the pair ($x_{G1}(0)$, Aa) reads '$x_{G1}(0)$ corresponds to Aa and can thus be replaced by it'. *Cf.* footnote 56.

$$\begin{cases} d_1(t) = Bbe^{Ff\,t}\{Cc\cos Cc\,t + (Ff+62)\sin Cc\,t\}, \\ d_2(t) = Bbe^{Ff\,t}\{11Cc\cos Cc\,t + (11Ff-15)\sin Cc\,t\}, \\ d_3(t) = Bbe^{Ff\,t}\{Cc(2Ff-5)\cos Cc\,t + (Ff^2-5Ff-Cc^2+28)\sin Cc\,t\}. \end{cases} \tag{44}$$

(44) seeming a bit simpler than (43), we verify that (44) satisfies (1) as follows:

```
% open-axiom

 (1) -> )read bp349_part2_1_3_2_1.input

e1:=matrix[[D(Bb*exp(Ff*t)*(Cc*cos(Cc*t)+(Ff+62)*sin(Cc*t)),t)],
           [D(Bb*exp(Ff*t)*(11*Cc*cos(Cc*t)+(11*Ff-15)*sin(Cc*t)),t)],
           [D(Bb*exp(Ff*t)*(Cc*(2*Ff-5)*cos(Cc*t)
              +(Ff^2-5*Ff-Cc^2+28)*sin(Cc*t)),t)]]
--(44) is plugged into the LHS of (1) and subjected to differentiation.

   (1)
   [
     [
               2                   2   Ff t
        (Bb Ff  + 62Bb Ff - Bb Cc )%e    sin(Cc t)
      +
                                           Ff t
        (2Bb Cc Ff + 62Bb Cc)cos(Cc t)%e
      ]


     ,



     [
                2                     2   Ff t
        (11Bb Ff  - 15Bb Ff - 11Bb Cc )%e    sin(Cc t)
      +
                                             Ff t
        (22Bb Cc Ff - 15Bb Cc)cos(Cc t)%e
      ]


     ,



     [
              3          2             2                         2   Ff t
        (Bb Ff  - 5Bb Ff  + (- 3Bb Cc  + 28Bb)Ff + 5Bb Cc )%e    sin(Cc t)
```

```
        +
                  2                     3                      Ff t
          (3Bb Cc Ff  - 10Bb Cc Ff - Bb Cc  + 28Bb Cc)cos(Cc t)%e
        ]

      ]

                                          Type: Matrix Expression Integer

a1:=matrix[[1,6,1],[-4,4,11],[-3,-9,8]]
--This corresponds to matrix A.

        + 1    6   1 +
        |            |
   (2)  |- 4   4   11|
        |            |
        +- 3  - 9  8 +

                                                   Type: Matrix Integer

h1:=matrix[[Bb*exp(Ff*t)*(Cc*cos(Cc*t)+(Ff+62)*sin(Cc*t))],
           [Bb*exp(Ff*t)*(11*Cc*cos(Cc*t)+(11*Ff-15)*sin(Cc*t))],
           [Bb*exp(Ff*t)*(Cc*(2*Ff-5)*cos(Cc*t)
            +(Ff^2-5*Ff-Cc^2+28)*sin(Cc*t))]]
--This corresponds to (44).

   (3)
                    Ff t                             Ff t
   [[(Bb Ff + 62Bb)%e    sin(Cc t) + Bb Cc cos(Cc t)%e    ],

                     Ff t                                 Ff t
    [(11Bb Ff - 15Bb)%e    sin(Cc t) + 11Bb Cc cos(Cc t)%e    ],


      [
              2                   2          Ff t
          (Bb Ff  - 5Bb Ff - Bb Cc  + 28Bb)%e    sin(Cc t)
        +
                                         Ff t
          (2Bb Cc Ff - 5Bb Cc)cos(Cc t)%e
        ]

      ]

                                          Type: Matrix Expression Integer
```

```
e1-a1*h1

   (4)
   [[0], [0],

     [
              3          2           2                  2
       (Bb Ff  - 13Bb Ff  + (- 3Bb Cc  + 170Bb)Ff + 13Bb Cc  - 173Bb)

          *
                   Ff t
        sin(Cc t)%e

        +
                   2                  3                   Ff t
         (3Bb Cc Ff  - 26Bb Cc Ff - Bb Cc  + 170Bb Cc)cos(Cc t)%e
       ]

     ]

                                            Type: Matrix Expression Integer

factor(expand(Bb*Ff^3-13*Bb*Ff^2+(-3*Bb*Cc^2+170*Bb)*Ff+13*Bb*Cc^2-173*Bb))
--The coefficient of sin(Cc t)%e^(Ff t) in output (4) is expanded and
factored.

             3      2        2              2
   (5)  Bb(Ff  - 13Ff  + (- 3Cc  + 170)Ff + 13Cc  - 173)

                                          Type: Factored Polynomial Integer

factor(expand(3*Bb*Cc*Ff^2-26*Bb*Cc*Ff-Bb*Cc^3+170*Bb*Cc))
--The coefficient of cos(Cc t)%e^(Ff t) in output (4) is expanded and
factored.

                 2          2
   (6)  Bb Cc(3Ff  - 26Ff - Cc  + 170)

                                          Type: Factored Polynomial Integer.
```

It follows from **Table 2**, (12), and (15) that the last two outputs equal 0, which means `e1-a1*h1` equals $(0, 0, 0)^T$. (44) has thus been shown to satisfy (1). Since (41) (or (43)) has been simplified to become (44), (41) has been shown to satisfy (1) twice. Next,

```
% wxmaxima
(%i4)   h2:matrix([Bb·exp(Ff·t)·(Cc·cos(Cc·t)+(Ff+62)·sin(Cc·t))],
                  [Bb·exp(Ff·t)·(11·Cc·cos(Cc·t)+(11·Ff−15)·sin(Cc·t))],
                  [Bb·exp(Ff·t)·(Cc·(2·Ff−5)·cos(Cc·t)+(Ff^2−5·Ff−Cc^2+28)·sin(Cc·t))]);
        /·This corresponds to (44).·/
        e2:diff(h2,t);
        /·(44) is plugged into the LHS of (1) and subjected to differentiation.·/
        a2:matrix([1,6,1],[−4,4,11],[−3,−9,8]);
        /·This corresponds to matrix A.·/
        ratsimp(expand(e2−a2.h2));
```

(h2) $$\begin{pmatrix} Bb\%e^{Ff\,t}((Ff+62)\sin(Cc\,t)+Cc\cos(Cc\,t)) \\ Bb\%e^{Ff\,t}((11Ff-15)\sin(Cc\,t)+11Cc\cos(Cc\,t)) \\ Bb\%e^{Ff\,t}((Ff^2-5Ff-Cc^2+28)\sin(Cc\,t)+Cc(2Ff-5)\cos(Cc\,t)) \end{pmatrix}$$

(e2) $$\begin{pmatrix} BbFf\%e^{Ff\,t}((Ff+62)\sin(Cc\,t)+Cc\cos(Cc\,t)) \\ +Bb\%e^{Ff\,t}(Cc\,(Ff+62)\cos(Cc\,t)-Cc^2\sin(Cc\,t)) \\ BbFf\%e^{Ff\,t}((11Ff-15)\sin(Cc\,t)+11Cc\cos(Cc\,t)) \\ +Bb\%e^{Ff\,t}(Cc\,(11Ff-15)\cos(Cc\,t)-11Cc^2\sin(Cc\,t)) \\ BbFf\%e^{Ff\,t}((Ff^2-5Ff-Cc^2+28)\sin(Cc\,t)+Cc(2Ff-5)\cos(Cc\,t)) \\ +Bb\%e^{Ff\,t}(Cc\,(Ff^2-5Ff-Cc^2+28)\cos(Cc\,t)-Cc^2(2Ff-5)\sin(Cc\,t)) \end{pmatrix}$$

(a2) $$\begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}$$

(%o4) $$\begin{pmatrix} 0 \\ 0 \\ (BbFf^3-13BbFf^2+(170Bb-3BbCc^2)Ff+13BbCc^2-173Bb)\%e^{Ff\,t}\sin(Cc\,t) \\ +(3BbCcFf^2-26BbCcFf-BbCc^3+170BbCc)\%e^{Ff\,t}\cos(Cc\,t) \end{pmatrix}.$$

Likewise, factoring the coefficients of $\%e^{Ff\,t}\sin(Cc\,t)$ and $\%e^{Ff\,t}\cos(Cc\,t)$ yields $Bb\{Ff^3-13Ff^2+(-3Cc^2+170)Ff+13Cc^2-173\}$ and $BbCc(3Ff^2-26Ff-Cc^2+170)$, respectively, both of which amount to 0 [81] . (44) has thus been shown to satisfy (1) twice. Since (41) (or (43)) has been simplified to become (44), (41) has been shown to satisfy (1) repeatedly. This leads us to hold that (41) is another real-valued solution of (1) with IV and repeatedly pay attention to TF's, which (41) contains. What about the SM of (41)? It follows from (41) that we have

$$\begin{cases} \mathrm{o}b_1(t) = \frac{\mathrm{o}x_{G1}(0)}{\eta}e^{\zeta t}\{\eta\cos\eta t+(\zeta+62)\sin\eta t\}, \\ \mathrm{o}b_2(t) = \frac{\mathrm{o}x_{G2}(0)}{11\eta}e^{\zeta t}\{11\eta\cos\eta t+(11\zeta-15)\sin\eta t\}, \\ \mathrm{o}b_3(t) = \frac{\mathrm{o}x_{G3}(0)}{\eta(2\zeta-5)}e^{\zeta t}\{\eta(2\zeta-5)\cos\eta t+(\zeta^2-5\zeta-\eta^2+28)\sin\eta t\}, \end{cases} \tag{45}$$

[81] See **Table 2**, (12), and (15).

o being a scalar . Rewriting (45) yields

$$\begin{cases} i_1(t) = \frac{x_{G7}(0)}{\eta} e^{\zeta t}\{\eta \cos \eta t + (\zeta + 62) \sin \eta t\}, \\ i_2(t) = \frac{x_{G8}(0)}{11\eta} e^{\zeta t}\{11\eta \cos \eta t + (11\zeta - 15) \sin \eta t\}, \\ i_3(t) = \frac{x_{G9}(0)}{\eta(2\zeta - 5)} e^{\zeta t}\{\eta(2\zeta - 5) \cos \eta t + (\zeta^2 - 5\zeta - \eta^2 + 28) \sin \eta t\}. \end{cases} \tag{46}$$

(46) being essentially the same as (41), sm's of (41) also satisfy (1). In other words, (41) satisfies (1) up to an sm.

Taken together, taking > 1 eigenvalue/eigenvector into consideration has led us to deal with TF's like $\cos \eta t$ and $\sin \eta t$, which we didn't encounter in [1] [82] . With regard to AC's such as C_1, C_2, and so on, we can dispense with them only when we introduce IV into $\mathfrak{I}$'s extracted from GS, which is seen in (41) compared with (16), (26), and (32). Behavior of $\|x(t)\|$ will be discussed elsewhere.

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2 Appendix

2.1 Some detailed computations

We roughly factor (2) into $-(\lambda - 1.1)(\lambda^2 - 12\lambda + 157)$. Then, it follows from QF that the roots of

[82] *Cf.* footnote 40.

the equation $A(\lambda) = 0$ approximate $(1.1, 6+11i, 6-11i)$ [83] . Using Polynomial Roots Calculator and wxMaxima , we try to get more precise values as follows:

Result

The roots of the polynomial $x^2 + 1$ are:

$$x_1 = i$$

$$x_2 = -i$$

Result

The roots of the polynomial $-x^3 + 13x^2 - 170x + 173$ are:

$$x_1 = 1.10275$$

$$x_2 = 5.94862 + 11.02244 * i$$

$$x_3 = 5.94862 - 11.02244 * i \,.$$

Next,

```
% wxmaxima
(%i3)   eq1:-x^3+13·x^2-170·x+173=0;
        fpprec:9;
        bfloat(roots:allroots(eq1));

(eq1)   -x^3+13 x^2-170 x+173=0
(fpprec) 9
(%o3)   [x=1.10275166b0,x=1.10224389b1 %i+5.94862417b0,x=5.94862417b0-1.10224389b1 %i] .
```

[83] In what follows, the variable λ is replaced by x, which won't affect our computations.

Now using Giac and SageMath , we solve the equation $x^3 - 13x^2 + 170x - 173 = 0$, which we anticipate to give *almost* identical results [84] .

```
% giac -v

// Using locale /usr/share/locale/
// ja_JP.utf8
// /usr/share/locale/
// giac
// UTF-8
// Maximum number of parallel threads 4
// (c) 2001, 2018 B. Parisse & others
1.5.0

% giac

Homepage http://www-fourier.ujf-grenoble.fr/~parisse/giac.html
Released under the GPL license 3.0 or above
See http://www.gnu.org for license details
May contain BSD licensed software parts (lapack, atlas, tinymt)
-------------------------------------------------
Press CTRL and D simultaneously to finish session
Type ?commandname for help
0>> i^2
-1
// Time 0
1>> eq2:=x^3-13*x^2+170*x-173
x^3-13*x^2+170*x-173
// Time 0
2>> proot(eq2)
[1.1027516612,5.9486241694-11.0224389371*i,5.9486241694+11.0224389371*i]
// Time 0.01.
```

Next,

```
% sage

SageMath version 8.7, Release Date: 2019-03-23
Using Python 2.7.16. Type "help()" for help.

sage: I^2
-1
sage: eq3=x^3-13*x^2+170*x-173==0
```

[84] According to footnote 3, results we obtain should be identical. Why *almost*, then? Because precision differs among software we employ.

```
sage: eq3.roots(x,ring=CC,multiplicities=False)
[1.10275166120159,
 5.94862416939920 - 11.0224389370911*I,
 5.94862416939920 + 11.0224389370911*I].
```

Sure enough, solving $-x^3+13x^2-170x+173=0$ and $x^3-13x^2+170x-173=0$ gave *almost* the same results [85] .

2.2 Computing $\det(A)$ and $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

We compute $\det A$.

$$\begin{aligned}\det A &= 1\times\begin{vmatrix}4 & 11\\ -9 & 8\end{vmatrix} - 6\times\begin{vmatrix}-4 & 11\\ -3 & 8\end{vmatrix} + 1\times\begin{vmatrix}-4 & 4\\ -3 & -9\end{vmatrix}\\ &= 1\times\{4\cdot 8-11\cdot(-9)\}-6\times\{-4\cdot 8-11\cdot(-3)\}+1\times\{-4\cdot(-9)-4\cdot(-3)\}\\ &= 1\cdot 131-6\cdot 1+1\cdot 48 = 131-6+48 = 173,\end{aligned}$$

which we verify using GNU Octave and Maxima :

```
% octave -W
GNU Octave, version 5.1.0
Copyright (C) 2019 John W. Eaton and others.
This is free software; see the source code for copying conditions.
There is ABSOLUTELY NO WARRANTY; not even for MERCHANTABILITY or
FITNESS FOR A PARTICULAR PURPOSE.  For details, type 'warranty'.

Octave was configured for "x86_64-pc-linux-gnu".

Additional information about Octave is available at https://www.octave.org.

octave:1> det([1,6,1;-4,4,11;-3,-9,8])
ans =  173.
```

Next,

```
% maxima
;;; Loading #P"/usr/lib/ecl-16.1.3/sb-bsd-sockets.fas"
;;; Loading #P"/usr/lib/ecl-16.1.3/sockets.fas"
;;; Loading #P"/usr/lib/ecl-16.1.3/defsystem.fas"
;;; Loading #P"/usr/lib/ecl-16.1.3/cmp.fas"
Maxima 5.42.2 http://maxima.sourceforge.net
using Lisp ECL 16.1.3
Distributed under the GNU Public License. See the file COPYING.
```

[85] See footnote 84.

```
(%i1) determinant(matrix[[1,6,1],[-4,4,11],[-3,-9,8]]);
(%o1)                              173.
```

We have thus checked that $\det A = 173$. By the way, since $\alpha\beta\gamma = 173$ [86] [6], and $\alpha\beta+\beta\gamma+\gamma\alpha = 170$ [87], $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \frac{170}{173}$, a rational number [88].

2.3 Answering questions raised in footnotes 31, 36, 62, and 68

2.3.1 The case where $\zeta = \eta = 0$

In a sense, this is a 'funny' case, since substituting $\zeta = \eta = 0$ into (12) and (15) yields $-173 = 0$ and $170 = 0$, respectively! [89] Of course, both are unequivocally wrong. Then, considering this case is entirely nonsensical? Not always, unfortunately. For example, substituting $\zeta = \eta = 0$ into (16), one gets

$$\begin{cases} F_1(t) = C_1(\alpha+62)e^{\alpha t} + 62(C_2+C_3), & (47)\\ F_2(t) = C_1(11\alpha-15)e^{\alpha t} - 15(C_2+C_3), & (48)\\ F_3(t) = C_1(\alpha^2-5\alpha+28)e^{\alpha t} + 28(C_2+C_3). & (49)\end{cases}$$

$1\times(47)+6\times(48)+1\times(49)$ gives $F_1(t)+6F_2(t)+F_3(t) = C_1\alpha(\alpha+62)e^{\alpha t}$, the RHS of which reminds us of **2.2** and **2.3** in [1], if we replace the coefficient of $e^{\alpha t}$ by, *e.g.*, $\sqrt{A_1^2+A_2^2+A_3^2}$ and $B(\alpha+62)$, respectively. Moreover, substituting $\zeta = \eta = 0$ into (26), one gets

$$\begin{cases} J_1(t) = \{x_{F1}(0) - 62(C_2+C_3)\}e^{\alpha t} + 62(C_2+C_3), & (50)\\ J_2(t) = \{x_{F2}(0) + 15(C_2+C_3)\}e^{\alpha t} - 15(C_2+C_3), & (51)\\ J_3(t) = \{x_{F3}(0) - 28(C_2+C_3)\}e^{\alpha t} + 28(C_2+C_3). & (52)\end{cases}$$

Setting $t = 0$ in $(50)-(52)$ yields $(J_1(0), J_2(0), J_3(0)) = (x_{F1}(0), x_{F2}(0), x_{F3}(0))$, which makes us aware of the very existence (and importance) of IV. And $1\times(50)+6\times(51)+1\times(52)$ gives $J_1(t)+6J_2(t)+J_3(t) = \{x_{F1}(0)+6x_{F2}(0)+x_{F3}(0)\}e^{\alpha t}$, the RHS of which becomes related to $(47)-(49)$ as soon as we replace the coefficient of $e^{\alpha t}$ by $C_1\alpha(\alpha+62)$.

2.3.2 The case where $\zeta \neq 0$, $\eta = 0$

In this case, we are able to 'forget' $\mathbb{C}$ and focus on $\mathbb{R}$, since $\alpha \in \mathbb{R}$, and $\beta = \gamma = \zeta \in \mathbb{R}$ [90]. However, substituting $\eta = 0$ into (12) yields $\zeta^3 - 13\zeta^2 + 170\zeta - 173 = 0$, which is essentially the same as (3) [91]. We thus immediately recall $\alpha \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{C}$ [92], though we have set $\beta = \gamma = \zeta \in \mathbb{R}$. In this sense, $\mathbb{C}$ is 'recapitulated'.

[86] *Cf.* footnote 12.

[87] *Cf.* footnote 7.

[88] $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ *themselves* are complex numbers, however.

[89] On the other hand, doing so into (13) and (14) just makes us confirm that $\pm 0\times$something$= 0$, which is not so hilarious. *Cf.* footnote 32.

[90] See, *e.g.*, footnote 12.

[91] Substituting $\eta = 0$ into (13) and (14) again yields $\pm 0\times$something$= 0$, the trivial. Doing so into (15) gives the equation $3\zeta^2 - 26\zeta + 170 = 0$. Using QF, we get the roots $\frac{13\pm\sqrt{13^2-3\cdot170}}{3} = 4.3\ldots \pm 6.1\ldots i \in \mathbb{C}$.

[92] See, *e.g.*, **2.1**.

2.3.3 The case where $\zeta = 0$, $\eta \neq 0$

As in footnote 68, setting $\zeta = 0$ in (41) 'obliterates' $e^{\zeta t}$'s of it to yield

$$\begin{cases} b_1(t) = \frac{x_{G1}(0)}{\eta}(\eta \cos \eta t + 62 \sin \eta t), & (53) \\ b_2(t) = \frac{x_{G2}(0)}{11\eta}(11\eta \cos \eta t - 15 \sin \eta t), & (54) \\ b_3(t) = \frac{x_{G3}(0)}{5\eta}\{5\eta \cos \eta t + (\eta^2 - 28) \sin \eta t\}. & (55) \end{cases}$$

Viewing TF's in (53) – (55) as derived from $e^{\eta i t}$ [93], we can 'live' in $\mathbb{C}$. However, substituting $\zeta = 0$ into (12) and (15) gives the equations $13\eta^2 - 173 = 0$ and $-\eta^2 + 170 = 0$, respectively [94]. At any rate, we get real roots as soon as we solve them. Moreover, we can remember $\alpha \in \mathbb{R}$ [95], if we wish. In this sense, $\mathbb{R}$ is 'recapitulated'.

2.4 SM of eigenvector yields eigenvector again [7]

We touch on a (typical) example:

Example 2.4.1. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So an eigenvector and its corresponding eigenvalue of the matrix are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and 2, respectively. We then consider $\begin{pmatrix} 0 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, an sm of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It follows from $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ that $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ is another eigenvector whose eigenvalue is 2.

2.5 Are C_i, $i = 1, 2, 3$, *really* arbitrary?

Using matrix notation, we express (25) as

$$\begin{pmatrix} x_{F1}(0) \\ x_{F2}(0) \\ x_{F3}(0) \end{pmatrix} = \begin{pmatrix} \alpha + 62 & \zeta + 62 & \zeta + 62 \\ 11\alpha - 15 & 11\zeta - 15 & 11\zeta - 15 \\ \alpha^2 - 5\alpha + 28 & \zeta^2 - 5\zeta - \eta^2 + 28 & \zeta^2 - 5\zeta - \eta^2 + 28 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \Pi \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

Due to rule of Sarrus, we have $\det \Pi = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$ [96]. The second column of Π being the same as the third one, we consider its special case where $a_{12} = a_{13}$, $a_{22} = a_{23}$, and $a_{32} = a_{33}$. Now $\det \Pi$ becomes $a_{11}a_{23}a_{33} + a_{13}a_{23}a_{31} + a_{13}a_{21}a_{33} - a_{31}a_{23}a_{13} - a_{33}a_{23}a_{11} - a_{33}a_{21}a_{13} = a_{11}a_{23}a_{33} + a_{13}a_{23}a_{31} + a_{13}a_{21}a_{33} - a_{13}a_{23}a_{31} - a_{11}a_{23}a_{33} - a_{13}a_{21}a_{33} = 0$. Since $\det\Pi = 0$, the matrix Π is singular and thus Π^{-1} doesn't exist. So we cannot determine (C_1, C_2, C_3) uniquely by computing

[93] Remember EF. *Cf.* footnote 35.

[94] Doing so into (13) and (14) gives $\eta(-\eta^2 + 170) = 0$ and $-\eta(-\eta^2 + 170) = 0$, respectively. Since $\eta \neq 0$, upon dividing both sides of them by η, we get $\pm(-\eta^2 + 170) = 0$.

[95] See, *e.g.*, footnote 12.

[96] See p11 for what a_{ij}, where $1 \leq i, j \leq 3$, stands for.

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \Pi^{-1} \begin{pmatrix} x_{F1}(0) \\ x_{F2}(0) \\ x_{F3}(0) \end{pmatrix},$$

even if the IV $(x_{F1}(0), x_{F2}(0), x_{F3}(0))$ is explicitly given. Hence, C_i, $i = 1, 2, 3$, are *really* arbitrary, at least until we reach (25).