# Approximation of Sum of Harmonic Progression

Aryan Phadke

**Abstract**

Background: The harmonic sequence and the infinite harmonic series have been a topic of great interest to mathematicians for many years. The sum of the infinite harmonic series has been linked to the Euler-Mascheroni constant. It has been demonstrated by Euler that, although the sum diverges, it can be expressed as the Euler-Mascheroni constant added to the natural log of infinity. By utilizing the Euler-Maclaurin method, we can extend the expression to approximate the sum of finite harmonic series with a fixed first term and a variable last term. However, natural extension is not possible for a variable value of the first term or the common difference of the reciprocals.

Aim: The aim of this paper is to create a formula that generates an approximation of the sum of a harmonic progression for a variable first term and common difference. The objective remains that the resultant formula is fundamentally similar to Euler's equation of the constant and the result using the Maclaurin method.

Method: The principle result of the paper is derived using approximation theory. The assertion that the graph of harmonic progression closely resembles the graph of y=1/x is key. The subsequent results come through a comparative view of Euler's expression and by using numerical manipulations on the Euler-Mascheroni Constant.

Results: We created a general formula that approximates the sum of harmonic progression with variable components. Its fundamental nature is apparent because we can derive the results of the Maclaurin method from our results.

*Keywords:* Approximation Theory, Harmonic Progression, Euler-Mascheroni Constant, Numerical Analysis, Harmonic Series

## 1. Introduction

### *1.1. Core Concepts*

#### *1.1.1. Arithmetic progression*

It is the sequence of numbers such that the difference between any two consecutive terms is equal.

If the first term of the progression is $\boldsymbol{a}$ with the common difference being $\boldsymbol{d}$, then the resultant arithmetic progression is as follows

*Email address:* aryanphadke7@gmail.com (Aryan Phadke)

*August 6, 2023*

$a,\ a+d,\ a+2d,\ a+3d,\ ............,\ a+(n-1)d$ where $\boldsymbol{n}$ is the number of terms.

For the purposes of this paper, we shall use the last term $\boldsymbol{L}$ as the variable instead of the number of terms. $L = a + (n-1)d$

*1.1.2. Harmonic progression*

It is the sequence of numbers such that each term is a reciprocal of the corresponding term of an arithmetic progression.

The general harmonic progression is as follows $\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ ............,\ \frac{1}{L}$.

*1.1.3. Harmonic series*

It is the special case of the sum of harmonic progression where the first term and common difference equal to unity.

$H(L)\ =\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ ........... + \frac{1}{L}$

To generalize the terms and the terminology, we shall make the following accommodations.

$$H(L,\ a,\ d)\ =\ \frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+ ........... + \frac{1}{L} \tag{0}$$

*1.1.4. Graph of Harmonic Progression*

[Figure 1 about here.]

*1.2. Background*

*1.2.1. Euler-Mascheroni constant*

It is the limiting difference between the harmonic series and the natural logarithm. It can be represented as follows $\gamma = \lim_{n\to\infty}[H(n,1,1) - \log(n)]$ . The approximate value of the constant is 0.5772156649...... [1]

By removing the limit, we get an expression for the infinite harmonic series. [2]

$$H(\infty,\ 1,\ 1)\ =\ \ln(\infty)\ +\ \gamma \tag{1}$$

*1.2.2. Approximation using Euler's constant*

With Equation (1) and by using Euler-Maclaurin method we can arrive at

$$H(L,\ 1,\ 1)\ =\ \ln(L)\ +\ \gamma\ +\ \frac{1}{2L} - \varepsilon_L$$

Where $0 \leq \varepsilon_L \leq \frac{1}{8L^2}$ [3]

As $\varepsilon_L$ is very small, we can ignore it and create an approximation for the partial sum of the harmonic series. The accuracy of which will increase as the value of $\boldsymbol{L}$ increases and as $\boldsymbol{L}$ reaches infinity we will arrive at Equation (1).

$$H(L,\ 1,\ 1)\ \approx\ \ln(L)\ +\ \frac{1}{2L} + \gamma \tag{2}$$

*1.3. Aims and Objectives*

**Aim:** To find a general formula that approximates the sum of a given harmonic progression.

**Objectives:**

1. The formula should be applicable for diverse values of $\boldsymbol{a}$, $\boldsymbol{d}$ and $\boldsymbol{L}$. This also includes non-integer values.
2. The formula should not require the use of discrete operators such as summation or series expansions. The purpose of this objective is to eliminate the need for any computation.
3. The formula should be fundamentally linked with the Euler's constant and the results of Euler-Maclaurin method.

## 2. Results

*2.1. General formula*

In view of approximation theory, we assert that the area of the graph of harmonic progression is approximately equal to the area under the curve of $y = \frac{1}{x}$.

By equating the areas we get an approximation represented as follows.

$$H(L,\ a,\ d) \ \approx\ f(L,\ a,\ d) \ =\ \frac{\ln\left(\dfrac{L}{a}\right)}{d} + \frac{1}{2a} + \frac{1}{2L} \tag{3}$$

*2.2. Error function and derivation of Equation (2)*

The next logical step is to introduce an error function that calculates the difference between *H(L, a, d)* and *f(L, a, d)*.

Let $E(L,\ a,\ d) \ =\ H(L,\ a,\ d) \ -\ f(L,\ a,\ d)$.

The error that can be calculated by this definition is of the infinite harmonic series i.e. $E(\infty,\ 1,\ 1) \ =\ H(\infty,\ 1,\ 1) \ -\ f(\infty,\ 1,\ 1)$ .

We can substitute the expression of *H(L, a, d)* and *f(L, a, d)* when $\boldsymbol{L}$ equals infinity from Equation (1) and Equation (3) respectively. By doing so we get

$$E(\infty,\ 1,\ 1) \ =\ \gamma - \frac{1}{2} \tag{4}$$

If the assertion that $E(L,\ 1,\ 1) \ \approx\ E(\infty,\ 1,\ 1)$ is made, we obtain Equation (2) as a direct consequent.

*2.3. Formula for a variable first and last term*

We have the error of infinite harmonic series from Equation (4). To generalize it for a variable first and last term, we can simply introduce another function.

Let $E(L,\ a,\ 1)\ =\ E(\infty,\ 1,\ 1)\ .\ g\ (L,\ a,\ 1)$.

The absolute value of g(L, a, 1) cannot be determined by algebraic manipulation, however we can find an approximation that is accurate and checks certain specific cases for the function.

By doing so we have the result $g(L,\ a,\ 1)\ \approx\ \left[\frac{1}{a^2} - \frac{1}{L^2}\right]$

Consequently,

$E(L,\ a,\ 1)\ \approx\ E(\infty,\ 1,\ 1)\ .\ \left[\frac{1}{a^2} - \frac{1}{L^2}\right]$

$$E(L,\ a,\ 1)\ \approx\ \left(\gamma - \frac{1}{2}\right) \cdot \left[\frac{1}{a^2} - \frac{1}{L^2}\right] \tag{5}$$

Thus, we can conclude the formula for sum of a harmonic progression with variable first and last term as follows

$$H(L,\ a,\ 1)\ =\ f(L,\ a,\ 1)\ +\ E(\infty,\ 1,\ 1)\ .\ g(L,\ a,\ 1)$$

$$H(L,\ a,\ 1)\ \approx\ \ln\left(\frac{L}{a}\right) + \frac{1}{2a} + \frac{1}{2L} + \left(\gamma - \frac{1}{2}\right) \cdot \left[\frac{1}{a^2} - \frac{1}{L^2}\right] \tag{6}$$

*2.4. Formula for a variable common difference*

We have the error for when the first term and the last term are variable from Equation (5). In line with our logic, to extend it for a variable common difference, we can introduce another function.

Let$E(L,\ a,\ d)\ =\ E(\infty,\ 1,\ 1)\ .\ g(L,\ a,\ 1)\ .\ k(L,\ a,\ d)$.

The absolute value of *k(L, a, d)* largely remains an open problem, but we do have an approximation.

The function's dependency on $\boldsymbol{L}$ is very small (<2%) and therefore negligible. This claim can be supported by comparative analysis of errors at infinity.

The approximation is based on the assertion that $k(L,\ a,\ d)\ \approx\ k(a+d,\ a,\ d)$

The value of *k(a+d, a, d)* can be calculated manually.

By definition

$$k(L,\ a,\ d)\ =\ \frac{H(L,\ a,\ d)\ -\ f(L,\ a,\ d)}{E(\infty,\ 1,\ 1).g(L,\ a,\ 1)}$$

Because $k(L,\ a,\ d)\ \approx\ k(a+d,\ a,\ d)$ we can conclude that

$$k(L,\ a,\ d)\ \approx\ \frac{H(a+d,\ a,\ d)\ -\ f(a+d,\ a,\ d)}{E(a+d,\ a,\ 1)}$$

Substituting the expressions of these terms from Equation (0)Equations (3) and (5) we get the following formula

$$k(L,\ a,\ d)\ \approx\ \frac{\left[\dfrac{1}{2a}+\dfrac{1}{2(a+d)}-\dfrac{\ln\left(\dfrac{a+d}{a}\right)}{d}\right]}{\left(\gamma-\dfrac{1}{2}\right)\cdot\left(\dfrac{1}{a^2}-\dfrac{1}{(a+d)^2}\right)} \tag{7}$$

Thus, we can conclude the formula for sum of harmonic progression with variable first term, common difference and last term is as follows

$$H(L,\ a,\ d)\ \approx\ f(L,\ a,\ d)\ +\ \frac{g(L,\ a,\ 1)}{g(a+d,\ a,\ 1)}\ \times\ E(a+d,\ a,\ d)$$

$$H(L,\ a,\ d)\ \approx\ \frac{\ln\left(\dfrac{L}{a}\right)}{d}+\frac{1}{2a}+\frac{1}{2L}+\frac{(L^2-a^2)\,.\,(a+d)^2}{L^2.d.(2a+d)}\ \times\ \left[\frac{1}{2a}+\frac{1}{2(a+d)}-\frac{\ln\left(\dfrac{a+d}{a}\right)}{d}\right] \tag{8}$$

## 3. Discussion

### *3.1. General Formula*

The method to obtain Equation (3) is based in approximation theory. The principle assumption is that the area of the graph of harmonic progression is approximately equal to the area under the curve of $y = \frac{1}{x}$ .

This assertion is valid because

1. The terms of any harmonic progression are included in the curve.
2. The graphs become similar as the value of $\boldsymbol{d}$ decreases.

With this assumption in mind, the next step is to calculate each of these areas.

The area of the graph of harmonic progression [Ar(HP)] can be calculated by simple geometrical expressions when Figure 1 is converted into Figure 2

[Figure 2 about here.]

The area then becomes a summation of the areas of variable rectangles and triangles.

Area of any rectangle for a variable term $\boldsymbol{x}$ can be represented as $d \times \frac{1}{x+d}$. The sum of the areas of these rectangles will therefore be

$$\sum_{x=a}^{L-d} \frac{d}{x+d}\ =\ d \times H(L,\ a,\ d)\ -\ \frac{d}{a}$$

Similarly, area for any triangle for a variable term $\boldsymbol{x}$ can be represented as $\frac{1}{2} \times d \times \left(\frac{1}{x} - \frac{1}{x+d}\right)$. The sum of the areas of these triangles will therefore be

$$\sum_{x=a}^{L-d} \frac{d}{2} \times \left(\frac{1}{x} - \frac{1}{x+d}\right) \;=\; \frac{d}{2a} - \frac{d}{2L}$$

The total area will be the sum of these two i.e.

$$Ar\ (HP) \;=\; d \times \left[H(L,\ a,\ d) \;-\; \frac{1}{2a} - \frac{1}{2L}\right]$$

The area under the curve is simply the integral of the function i.e. $Ar\ \left(\frac{1}{x}\right) \;= \int_a^L \frac{1}{x}\,\mathrm{d}\,x$ .

Therefore, the area under the curve can be represented as follows

$$Ar\ \left(\frac{1}{x}\right) = \ln\left(\frac{L}{a}\right)$$

In line with our assumption $Ar\ (HP)\ \approx\ Ar\ \left(\frac{1}{x}\right)$and therefore

$$d \times \left(H(L,\ a,\ d)\ -\ \frac{1}{2a} - \frac{1}{2L}\right) \;\approx\; \ln\left(\frac{L}{a}\right)$$

Hence, we can conclude Equation (3).

$$H(L,\ a,\ d)\ \approx\ \frac{\ln\left(\frac{L}{a}\right)}{d} + \frac{1}{2a} + \frac{1}{2L}$$

*3.2. Error function and derivation of Equation (2)*

We begin by introducing an error function that is equal to the difference between the sum and the approximation of the sum of a given harmonic progression. i.e.

$E(L,\ a,\ d)\ =\ H(L,\ a,\ d)\ -\ f(L,\ a,\ d)$

Next we determine the expression for the error of the infinite harmonic series i.e.

$E(\infty,\ 1,\ 1)\ =\ H(\infty,\ 1,\ 1)\ -\ f(\infty,\ 1,\ 1)$

We can substitute these values from Equation (1) and Equation (3) and we will get

$$E(\infty,\ 1,\ 1)\ =\ [\ln(\infty)\ +\ \gamma]\ -\ \left[\ln(\infty) + \frac{1}{2} + \frac{1}{2\infty}\right]$$

Solving this we can conclude Equation (4).

To derive Equation (2) we have to make the assumption that the error in a partial sum is approximately equal to the error of the infinite harmonic series. i.e.

$E(L,\ 1,\ 1)\ \approx\ E(\infty,\ 1,\ 1)$

If made so the expression for the partial sum becomes

$H(L,\ 1,\ 1)\ \approx\ f(L,\ 1,\ 1)\ +\ E(\infty,\ 1,\ 1).$

We can substitute the values for these from Equation (3) and Equation (4). By doing so we get

$$H(L,\ 1,\ 1)\ \approx\ \left[\ln\left(\frac{L}{1}\right) + \frac{1}{2} + \frac{1}{2L}\right] + \left[\gamma - \frac{1}{2}\right]$$

It is apparent without any further manipulation that this results in Equation (2).

*August 6, 2023*

*3.3. Formula for a variable first and last term*

We have obtained the error in the infinite harmonic series. To extend it for a variable first and last term we can introduce a function that acts as a coefficient.

$E(L,\ a,\ 1)\ =\ E(\infty,\ 1,\ 1)\ \times\ g(L,\ a,\ 1)$

The absolute value of *g(L, a, 1)* cannot be determined. An approximation can simply be obtained by finding a function that follows certain necessary constraints. I shall now list the constraints and the reasons for them.

*3.3.1. Special case constraints*

In the special case of the infinite harmonic series where L is equal to infinity$E(L,\ 1,\ 1)\ =\ E(\infty,\ 1,\ 1)$. Therefore,

$g(\infty,\ 1,\ 1)\ =\ 1$.

In the special case where first term is equal to the last term equal to unity, $E(1,\ 1,\ 1)\ =\ 0$. Therefore,

$g(1,\ 1,\ 1)\ =\ 0$ .

In the case of the partial sum of harmonic series the value of the approximation should not break the limits identified by $\varepsilon_L$.

*3.3.2. The sum constraint*

Consider the sum of error functions of the following harmonic progressions.

$E(a,\ 1,\ 1)\ +\ E(L,\ a,\ 1)\ +\ E(\infty,\ L,\ 1)$

We know

$E(x,\ y,\ z)\ =\ H(x,\ y,\ z)\ -\ f(x,\ y,\ z)$ by definition.

We know from Equation (0) that

$H(a,\ 1,\ 1)\ +\ H(L,\ a,\ 1)\ +\ H(\infty,\ 1,\ 1)\ =\ H(\infty,\ 1,\ 1)\ +\ \frac{1}{a}+\frac{1}{L}$.

From Equation (3) we get

$f(a,\ 1,\ 1)\ =\ \ln\left(\frac{a}{1}\right)\ +\ \frac{1}{2}+\frac{1}{2a}$

$f(L,\ a,\ 1)\ =\ \ln\left(\frac{L}{a}\right)+\ \frac{1}{2a}+\frac{1}{2L}$

$f(\infty,\ 1,\ 1)\ = \ln\left(\frac{\infty}{L}\right)+\frac{1}{2L}+\frac{1}{2\infty}$ .

The sum of the three is

$$f(a,\ 1,\ 1)\ +\ f(L,\ a,\ 1)\ +\ f(\infty,\ L,\ 1)\ = \ln\left(\infty\right)+\frac{1}{2}\ +\frac{1}{a}+\frac{1}{L}$$

Thus we have

$$E(a,\ 1,\ 1)\ +\ E(L,\ a,\ 1)\ +\ E(\infty,\ 1,\ 1)\ =\ \gamma-\frac{1}{2}=E(\infty,\ 1,\ 1)$$

Because *g(x, y, z)* is simply the coefficient of *E(x, y, z)*, the sum of the coefficients must be one.

Therefore

$$g(a,\ 1,\ 1)\ +\ g(L,\ a,\ 1)\ +\ g(\infty,\ L,\ 1)\ =\ 1$$

*3.3.3. Case for g(L, a, 1)*

The function that best approximates *g(L, a, 1)* that also follows the constraints listed above was found to be

$$g(L,\ a,\ 1)\ \approx \left[\frac{1}{a^2} - \frac{1}{L^2}\right]$$

I shall now verify that the constraints are followed.

$$g(\infty,\ 1,\ 1)\ =\ \left[\frac{1}{1^2} - \frac{1}{\infty^2}\right]\ =\ 1$$

$$g(1,\ 1,\ 1)\ =\ \left[\frac{1}{1^2} - \frac{1}{1^2}\right]\ =\ 0$$

For a partial sum of harmonic series where the first term is one and the last term is L, $g(L,\ 1,\ 1)\ =\ \left[\frac{1}{1} - \frac{1}{L^2}\right]$.

The sum will be

$$H(L,\ 1,\ 1)\ \approx \ln{(L)} + \frac{1}{2} + \frac{1}{2L} + \left(\gamma - \frac{1}{2}\right) \times \left[\frac{1}{1} - \frac{1}{L^2}\right]$$

By comparing it with the expression obtained from Euler-Mclaurin method, we get

$$\varepsilon_L = \frac{\left(\gamma - \frac{1}{2}\right)}{L^2} \approx \frac{1}{13L^2}$$

Which is well within the limits $0 \leq \varepsilon_L \leq \frac{1}{8L^2}$ .

The final constraint is that of the sum of the coefficients must equal unity.

$$g(a,\ 1,\ 1)\ + g(L,\ a,\ 1)\ +\ g(\infty,\ L,\ 1)\ =\ \left[1 - \frac{1}{a^2}\right] + \left[\frac{1}{a^2} - \frac{1}{L^2}\right] + \left[\frac{1}{L^2} - \frac{1}{\infty^2}\right]\ =\ 1$$

Additionally, this approximation makes geometrical sense as it is simply the difference of the derivative of $\frac{1}{x}$meaning it is difference between of the slopes of the first and the last term.

With all this in mind, we can conclude that our hypothesis is viable and hence Equation (5) and Equation (6) are valid.

*3.4. Formula for a variable common difference*

We have obtained the error for a variable first and last terms. To extend it further for a variable common difference, we must introduce another function that acts as a coefficient.

$E(L,\ a,\ d)\ =\ E(L,\ a,\ 1)\ \times\ k(L,\ a,\ d)$

The absolute value of k*(L, a, d)* cannot be determined. Also, the method of finding a function that checks certain necessary constraints does not lend any fruitful results either.

The best method to find an approximation for k*(L, a, d)* is simply to manually calculate a portion it. Experimentally k*(L, a, d)* is dependent on ***a***, ***d*** and ***L***. However its dependency on ***L*** is negligibly small (<2%).

If this is true, and we eliminate $\boldsymbol{L}$, we can make the case that

$k(L,\ a,\ d)\ \approx\ k(a+d,\ a,\ d)$. The value of ***k(a+d, a, d)*** can be calculated by its definition.

$$k(L,\ a,\ d)\ \approx\ k(a+d,\ a,\ d)\ =\ \frac{H(a+d,\ a,\ d)\ -\ f(a+d,\ a,\ d)}{E(a+d,\ a,\ d)} \tag{9}$$

*3.4.1. The dependency on L*

I shall make the case here that although k*(L, a, d)* is dependent on $\boldsymbol{L}$, it is so negligibly.

Because k*(L, a, d)* is directly proportional to $\boldsymbol{L}$, it will have the largest effect on it when $\boldsymbol{L}$ is equal to infinity.

Consider the infinite harmonic progression where $a = d = x$. It is apparent that all its components are equal to the corresponding components of the harmonic series divided by $\boldsymbol{x}$. ex.

$E(\infty,\ x,\ x)\ =\ \frac{E(\infty,\,1,\,1)}{x}$.

We know by definition that

$k(\infty,\ x,\ x)\ =\ \frac{E(\infty,\,x,\,x)}{E(\infty,\,1,\,1)\ \times g(\infty,\,x,\,x)}$

Therefore,

$$k(\infty,\ x,\ x)\ =\ \frac{\dfrac{E(\infty,\ 1,\ 1)}{x}}{\dfrac{E(\infty,\ 1,\ 1)}{x^2}}\ =\ x$$

$\boldsymbol{L}$ will have the smallest effect on *k(L, a, d)* when $\boldsymbol{L}$ is equal to ***(a+d)***.

Consider the harmonic progression where $(a = d = x)$ and $(L = a + d = 2x)$. We can calculate *k(2x, x, x)* from Equation (9).

$$k(2x,\ x,\ x)\ =\ \frac{\left[\dfrac{1}{x}+\dfrac{1}{2x}\right]-\left[\dfrac{\ln\left(\dfrac{2x}{x}\right)}{2}-\dfrac{1}{2x}-\dfrac{1}{4x}\right]}{\left(\gamma-\dfrac{1}{2}\right)\times\left[\dfrac{1}{x^2}-\dfrac{1}{(2x)^2}\right]}$$

Therefore

$$k(2x,\ x,\ x)\ =\ \frac{x\ \left(\dfrac{3}{4}-\ln(2)\right)}{\dfrac{3}{4}\left(\gamma-\dfrac{1}{2}\right)}\approx 0.9817x$$

The difference between the maximum and minimum effect is $0.0182x$. In percent of $\boldsymbol{x}$, it would equal to <2%.

*3.4.2. The Final general formula*

Thus, we can conclude that the effects of ***L*** on *k(L, a, d)* can be ignored, and we can proceed with Equation (9).

$$k(L,\ a,\ d)\ \approx\ k(a+d,\ a,\ d)\ =\ \frac{\dfrac{1}{2a}+\dfrac{1}{2(a+d)}-\dfrac{\ln\left(\dfrac{a+d}{a}\right)}{d}}{\left(\gamma-\dfrac{1}{2}\right)\times\left[\dfrac{1}{a^2}-\dfrac{1}{(a+d)^2}\right]}$$

By definition

$E(L,\ a,\ d)\ \approx\ E(\infty,\ 1,\ 1)\times g(L,\ a,\ 1)\times k(a+d,\ a,\ d)$

Therefore

$$E(L,\ a,\ d)\ \approx\ \left(\gamma-\frac{1}{2}\right)\times\left[\frac{1}{a^2}-\frac{1}{L^2}\right]\times\left[\frac{\dfrac{1}{2a}+\dfrac{1}{2(a+d)}-\dfrac{\ln\left(\dfrac{a+d}{a}\right)}{d}}{\left(\gamma-\dfrac{1}{2}\right)\times\left[\dfrac{1}{a^2}-\dfrac{1}{(a+d)^2}\right]}\right]$$

By compressing this equation we have

$$E(L,\ a,\ d)\ \approx\ \left[\frac{(L^2-a^2)\times(a+d)^2}{L^2d\times(2a+d)}\right]\times\left[\frac{1}{2a}+\frac{1}{2(a+d)}-\frac{\ln\left(\dfrac{a+d}{a}\right)}{d}\right]$$

Thus, we can conclude Equation (8) as the general formula for an approximation of sum of harmonic progression with variable first term, common difference and last term.

*3.5. Statistical Verification*

To verify the approximation and test its accuracy, we shall use five sample harmonic progressions of varying first term and common difference.

For each of them we will graph the sum and the approximation, calculate the absolute error and also calculate expected accuracy.

*3.5.1. Case I: a=d=1*

Consider a harmonic progression where $a\ =1\ ;\ d=1$

We will use Equation (6) to calculate the sum.

[Figure 3 about here.]

Next we will look at the maximum absolute error and lowest accuracy for the given harmonic progression.

$E(max) = H(L,\ 1,\ 1) - f(L,\ 1,\ 1) - E(\infty,\ 1,\ 1) \times g(L,\ 1,\ 1)$ .

Maximum error was found at

$L = 2$ ; $E(max) = -0.0011$

$Accuracy\ (min) = \frac{H(L,1,1)}{f(L,1,1)+E(\infty,1,1)\times g(L,1,1)} \times 100$

Minimum accuracy was found at

$L = 2$ ; $Accuracy\ (min) = 99.929\%$

When $L > 12$ ; $Accuracy > 99.999\%$.

*3.5.2. Case II: a>1, d=1*

Consider a harmonic progression where $a = 75$ ; $d = 1$

We will use Equation (6) to calculate the sum.

[Figure 4 about here.]

Next we will look at the absolute error and accuracy.

$E(max) = H(L,\ 75,\ 1) - f(L,\ 75,\ 1) - E(\infty,\ 1,\ 1) \times g(L,\ 75,\ 1)$

Maximum error in this case is found at $L = \infty$. It cannot be determined but for a rough idea, we shall calculate the error for the sum of the first 10000 terms

$E(10074,\ 75,\ 1) = 1.086 \times 10^{-6}$.

$Accuracy(min) = \frac{H(L,75,1)}{f(L,75,1)+E(\infty,1,1)\times g(L,75,1)}$

The minimum accuracy is also found at $L = \infty$. It cannot be determined but for a rough idea the accuracy of the sum of the first 10000 terms is

$Accuracy\ (10074,\ 75,\ 1) = 99.9998\%$.

*3.5.3. Case III: a=1, d>1*

Consider a harmonic progression where $a = 1$ ; $d = 75$

We will use Equation (8) to calculate the sum.

[Figure 5 about here.]

Next we will look at absolute error and accuracy.

$E(max) = H(L,\ 1,\ 75) - f(L,\ 1,\ 75) - E(\infty,\ 1,\ 1) \times g(L,\ 1,\ 1) \times k(L,\ 1,\ 75)$

Maximum error in this case is found at $L = \infty$. It cannot be determined but for a rough idea, we shall calculate the error for the sum of the first 10000 terms.

$E(749926,\ 1,\ 75) = 9.27 \times 10^{-4}$

$Accuracy\ (min) = \frac{H(L,1,75)}{f(L,1,75)+E(\infty,1,1)\times g(L,1,1)\times k(L,1,75)} \times 100$

The minimum accuracy is also found at $L = \infty$. It cannot be determined but for a rough idea the accuracy of the sum of the first 10000 terms is

$Accuracy\ (749926,\ 1,\ 75) = 99.918\%$.

*3.5.4. Case IV: a, d > 1*

Consider a harmonic progression where $a = 100$ ; $d = 10$

We will use Equation (8) to calculate the sum.

[Figure 6 about here.]

Next we will look at absolute error and accuracy.

$E(max) = H(L, 100, 10) - f(L, 100, 10) - E(\infty, 1, 1) \times g(L, 100, 1) \times k(L, 100, 10)$

Maximum error in this case is found at $L = \infty$. It cannot be determined but for a rough idea, we shall calculate the error for the sum of the first 10000 terms.

$E(100090, 100, 10) = 6.828 \times 10^{-8}$

$Accuracy\ (min) = \frac{H(L, 100, 10)}{f(L, 100, 10) + E(\infty, 1, 1) \times g(L, 100, 1) \times k(L, 100, 10)} \times 100$

The minimum accuracy is also found at $L = \infty$. It cannot be determined but for a rough idea the accuracy of the sum of the first 10000 terms is

$Accuracy\ (100090, 100, 10) = 99.9998\%$

*3.5.5. Case V: a, d are non-integers*

Consider a harmonic progression where $a = \frac{15}{2}$ ; $d = \frac{1}{4}$

We will use Equation (8) to calculate the sum.

[Figure 7 about here.]

Next we will look at absolute error and accuracy.

$E(max) = H\left(L, \frac{15}{2}, \frac{1}{4}\right) - f\left(L, \frac{15}{2}, \frac{1}{4}\right) - E(\infty, 1, 1) \times g\left(L, \frac{15}{2}, 1\right) \times k\left(L, \frac{15}{2}, \frac{1}{4}\right)$

Maximum error in this case is found at $L = \infty$. It cannot be determined but for a rough idea, we shall calculate the error for the sum of the first 10000 terms.

$E\left(\frac{10029}{4}, \frac{15}{2}, \frac{1}{4}\right) = 3.85 \times 10^{-8}$

$$Accuracy\ (min) = \frac{H\left(L, \frac{15}{2}, \frac{1}{4}\right)}{f\left(L, \frac{15}{2}, \frac{1}{4}\right) + E(\infty, 1, 1) \times g\left(L, \frac{15}{2}, 1\right) \times k\left(L, \frac{15}{2}, \frac{1}{4}\right)} \times 100$$

The minimum accuracy is also found at $L = \infty$. It cannot be determined but for a rough idea the accuracy of the sum of the first 10000 terms is

$Accuracy\ \left(\frac{10029}{4}, \frac{15}{2}, \frac{1}{4}\right) = 99.999999\%$

## 4. Conclusion

In keeping with the principal aim of the paper, we were able to create a general formula to approximate the sum of a given harmonic progression. Majority of the contemporary approximations are only applicable for the special case of harmonic series. The resultant formula is applicable for diverse values of the first term and common difference, which include non-integer values.

One of our objectives was to construct a formula that doesn't depend on discrete operators such as summation. It is apparent that none of the resultant formulas are dependent on the use of discrete operators or series expansions.

The fundamental nature of the general form must be stressed. It shares a strong connection with original equation of Euler-Mascheroni constant and also derives the results found by the Euler-Maclaurin method for the harmonic series.

The statistical analysis also reveals positive results. The accuracy of the formula although will decrease as the last term grows, it will do so very slowly, and it is highly unlikely for small values of the common difference to drop below 99.99%. If only absolute error is relevant, the results are even more promising.

### *4.1. Application*

To calculate the sum of a harmonic progression when common difference is unity, Equation (6) should be preferred, in any other case Equation (8) should be used.

The general application of the formula is in any area that requires the sum of a given harmonic progression, such that a computation is not viable, or one that requires an algebraic approximation.

### *4.2. Future Research*

The problem that still requires further attention is perhaps finding a better approximation of $k(L, a, d)$ when d is very large.

A constraint based search for $k(L, a, d)$ is also an avenue that requires work.

The sum of other series where the degree of the terms is less than zero such as the finite Basel problem could potentially be approximated using similar methods.

## References


[1] N. J. A. Sloane, Sequence A001620 (Decimal expansion of Euler's constant (or the Euler-Mascheroni constant), gamma)".

[2] J. Havil, Gamma: Exploring Euler's Constant, Princeton University Press, 2003.

[3] D. M. Bressoud, A Radical Approach to Real Analysis, in: Classroom Resource Materials Series, Mathematical Association of America, 2007, pp. 137–138.

[4] J. C. Lagarias, Euler's constant: Euler's work and modern developments, Bulletin of the American Mathematical Society 50 (4) (2013) 556–556.

[5] T. J. Osler, Partial sums of series that cannot be an integer, The Mathematical Gazette 96 (2012) 515–519.

## List of Figures

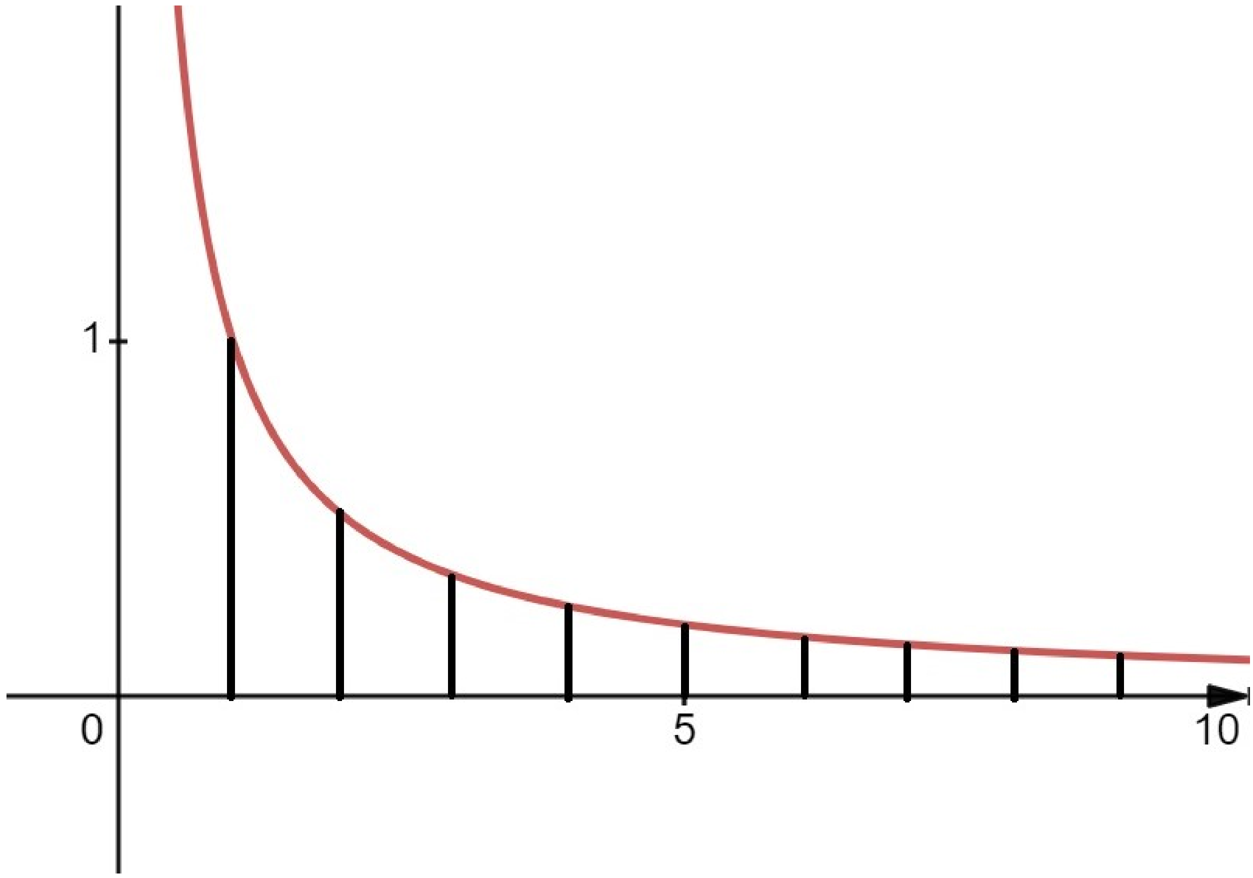


Figure 1: Graph of harmonic progression with a=d=1 superimposed with the function y=1/x

*August 6, 2023*

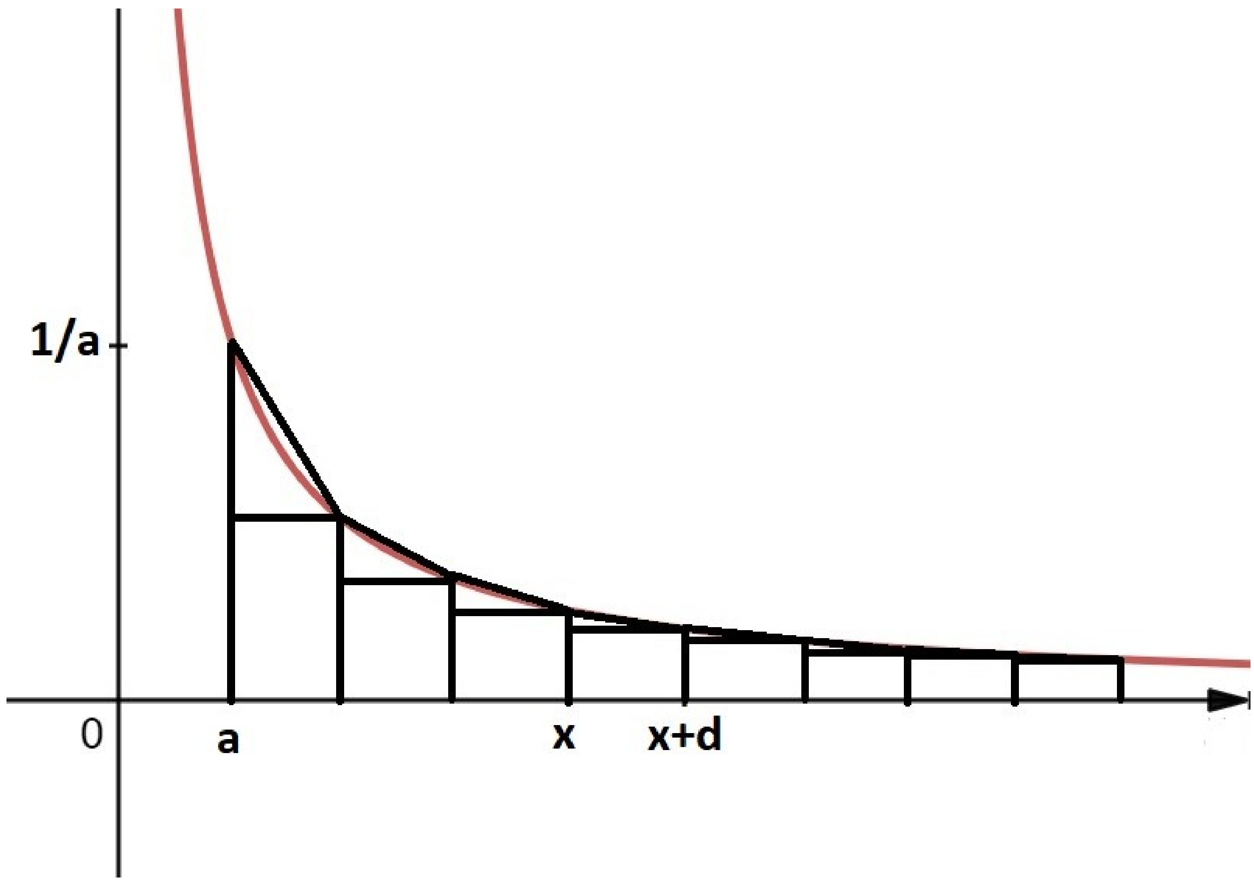


Figure 2: Graph of area of a harmonic progression with a variable term $x$

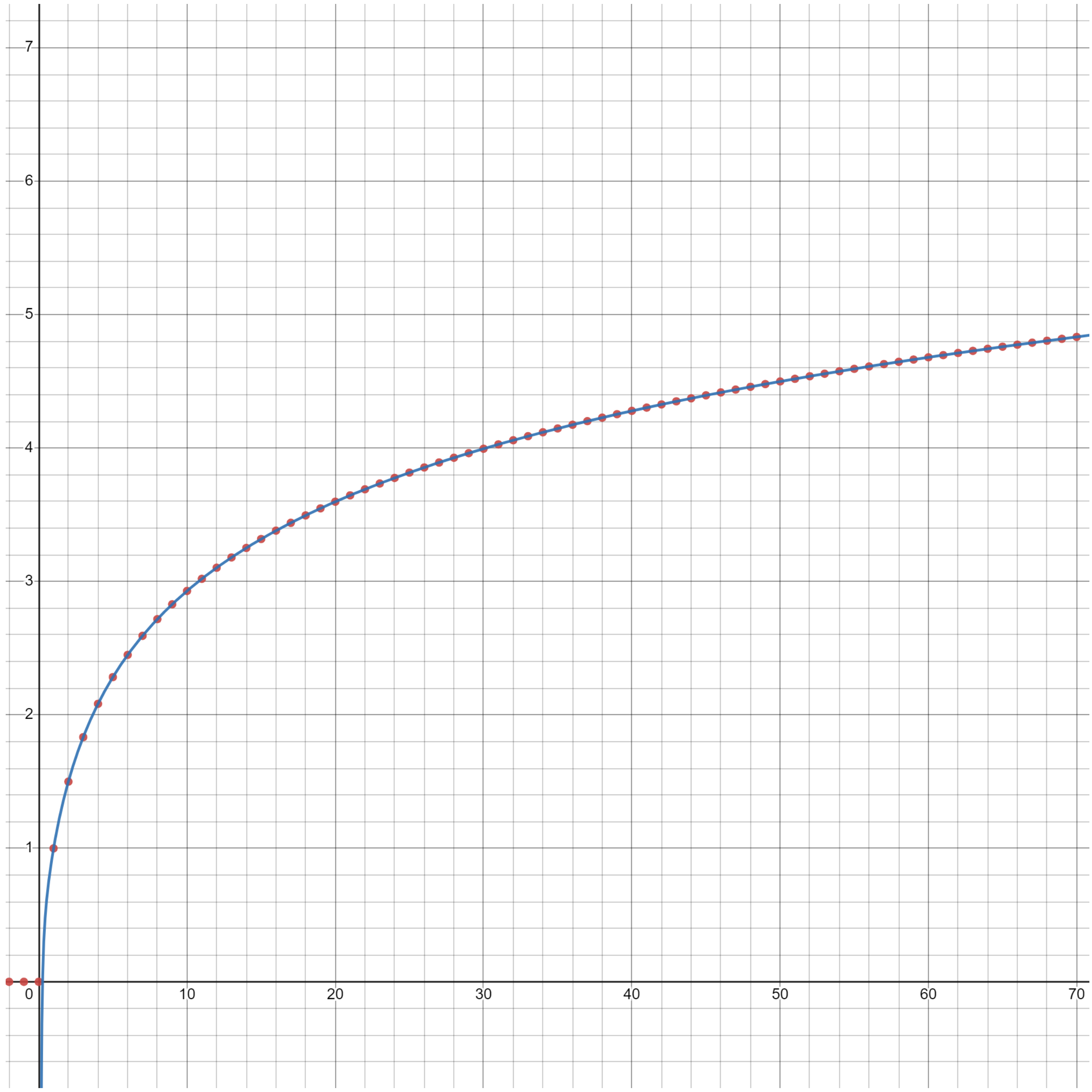


Figure 3: Graph of H(L, 1, 1) superimposed with Equation (6)

*August 6, 2023*

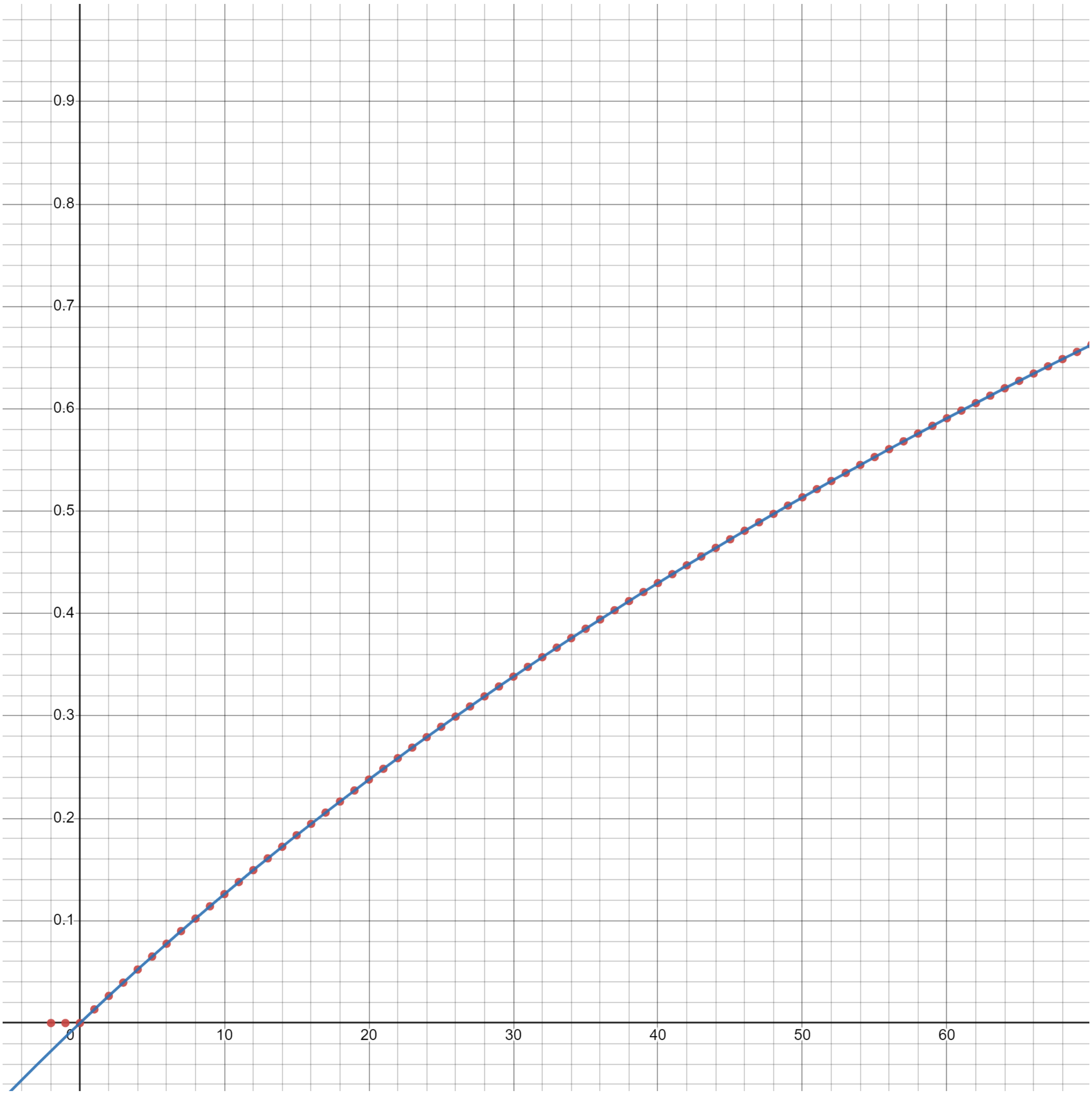

Figure 4: Graph of H(L, 75, 1) superimposed with Equation (6)

*August 6, 2023*

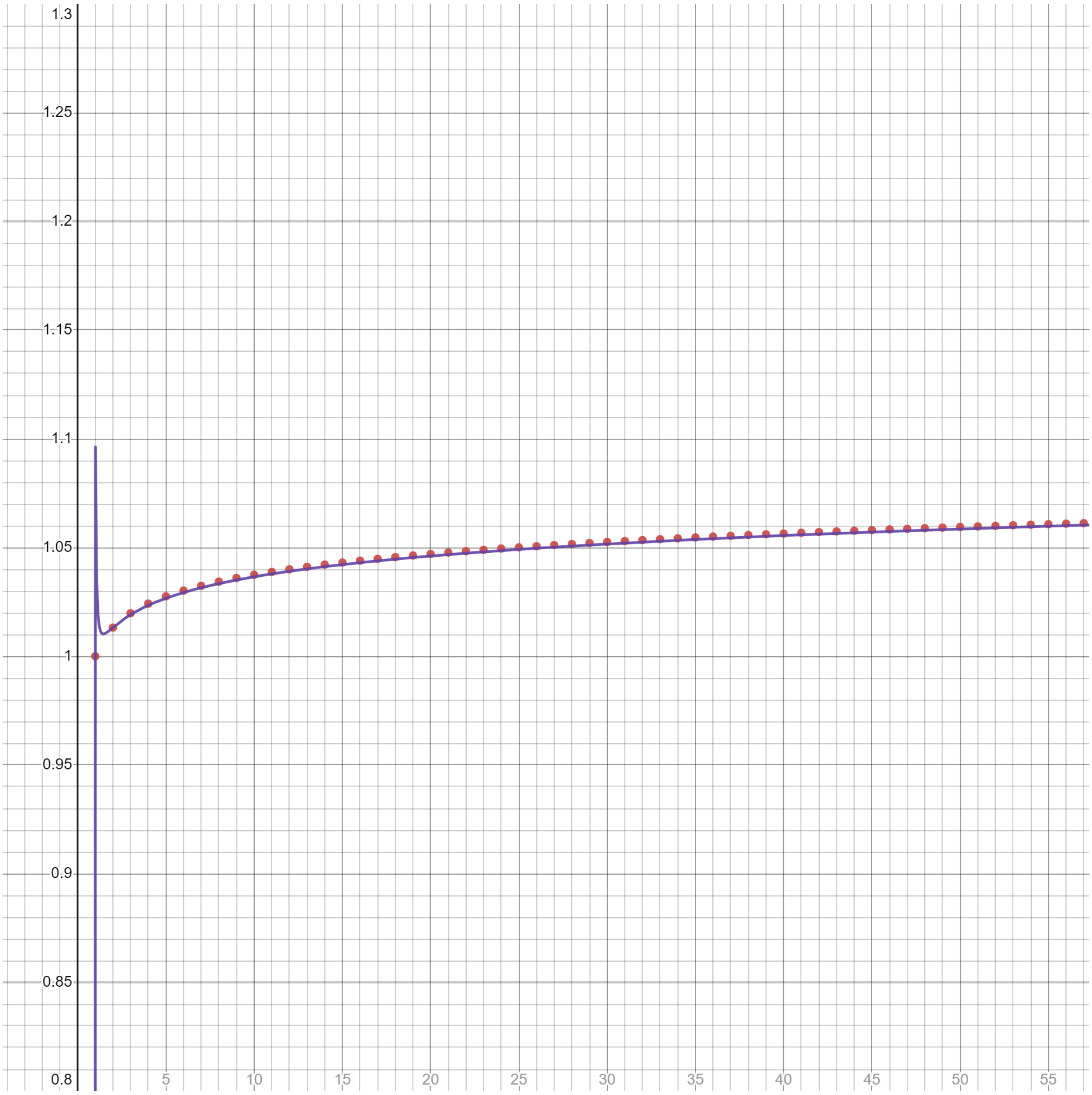


Figure 5: Graph of H(L, 1, 75) superimposed with Equation (8)

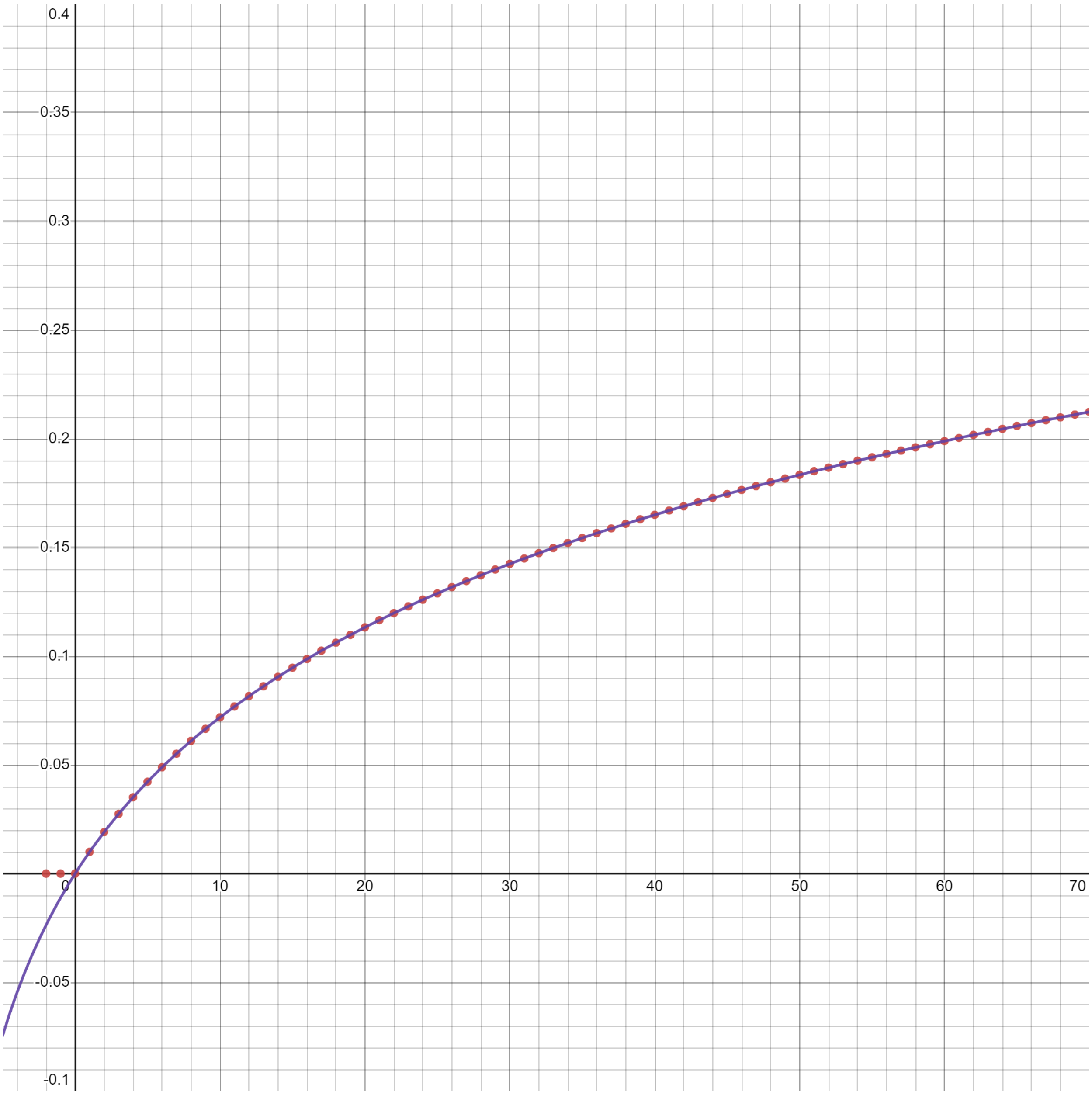


Figure 6: Graph of H(L, 100, 10) imposed with Equation (8)

*August 6, 2023*

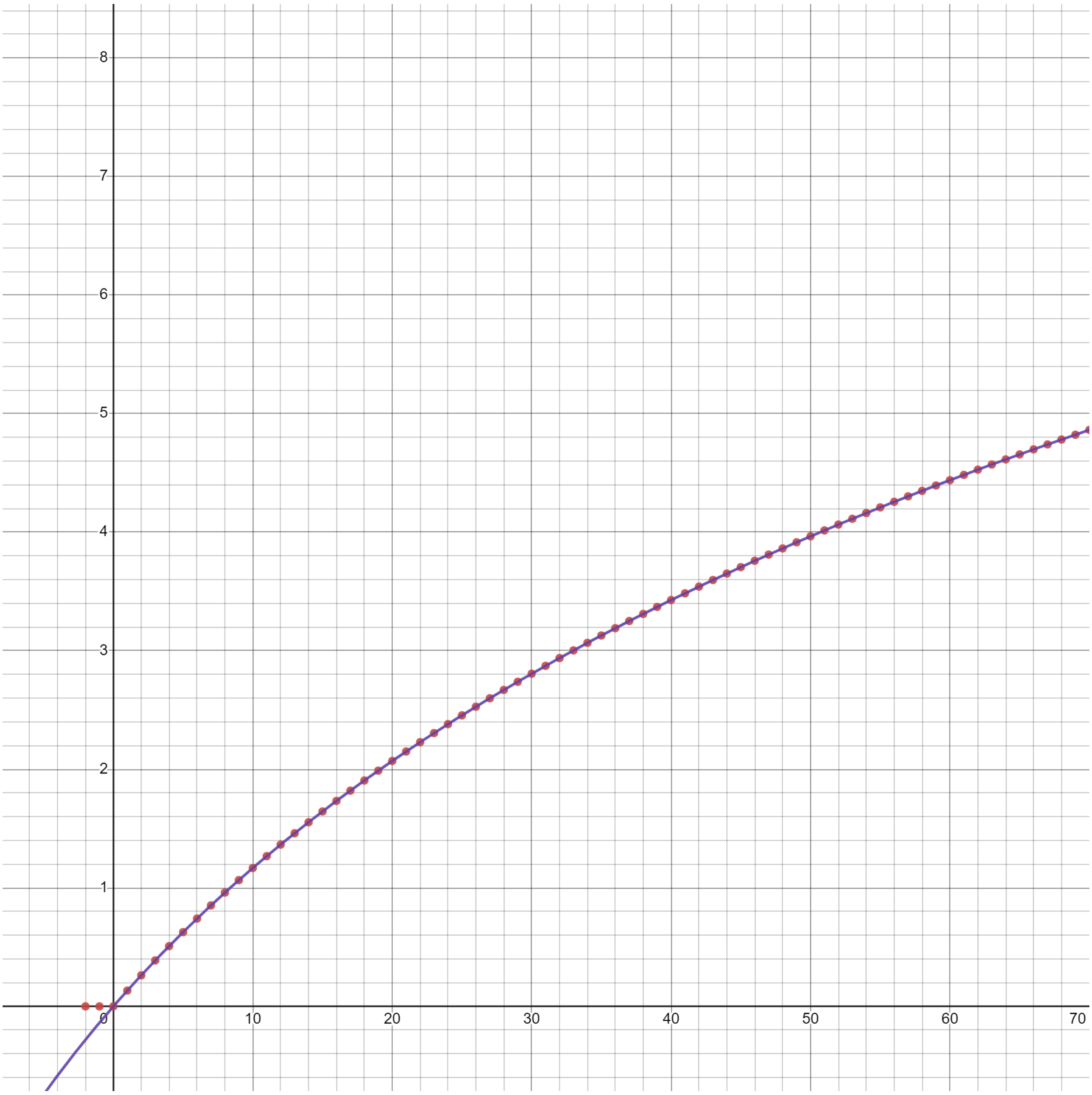


Figure 7: Graph of H(L, 15/2, 1/4) superimposed with Equation (8)

*August 6, 2023*