“*<u>Supplementary Notes pertaining to a Specific Quaternion Analogue of the Cauchy-Goursat Theorem</u>.*”

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I. <u>PRELIMINARY REMARKS</u>.

(<u>1</u>) An abstract thus pertaining to the author’s original paper has been provided under the heading, **‘<u>PREFACE</u>’**, as is evident from Page [<u>3</u>] of his submission.

(<u>2</u>) The author accordingly wishes to thank the following mathematician, namely – Mr. Stephen J. Crothers, B.A; Adv. Cert. Comp. Tech. [TAFE NSW]; Grad. Dip. Sc.; Grad. Dip. Tech.; Grad. Cert. Eng.; M. Astron., who refereed his original paper on 1st May 2019.

(<u>3</u>) Page [<u>2</u>] of this submission contains a copy of one (1) reference provided by the aforesaid mathematician.

II. <u>COPY OF AUTHOR’S ORIGINAL PAPER</u>.

For further details, the reader should accordingly refer to the remainder of this submission from Page [<u>3</u>] onwards.

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1st May, 2019

TO WHOM IT MAY CONCERN

In April this year, Mr. Stephen C. Pearson very kindly provided me with a copy of his sixteen page mathematical paper titled:

'Supplementary Notes Pertaining to a Specific Quaternion Analogue of the Cauchy-Goursat Theorem',

which he had completed on the 6th of March, 2019.

This paper is an addendum to Mr. Pearson's antecedent paper and concomitant monograph which I had previously refereed on the 4th of July, 2018.

Having examined the Supplementary Notes I find that Mr. Pearson has applied the Principle of Mathematical Induction to prove a generalisation of the Cauchy-Goursat Theorem to the quaternion hypercomplex case in keeping with his extensive scholarly monograph on analytic functions of quaternion hypercomplex variables, which I have also had the privilege to review some years ago.

Stephen J. Crothers

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6th March 2019.

FINAL DRAFT pending further assessment.

PREFACE.

The overall aim of this paper is to further generalise a specific quaternion analogue of the Cauchy - Goursat Theorem from complex variable analysis, bearing in mind that this particular notion had previously been enunciated in the author's paper [2] and concomitant monograph [3].

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Member of the London Mathematical Society.

6th March 2019.

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I. Enunciation of Three Theorems pertaining to the Integration of Quaternion Hypercomplex Functions.

Theorem TI-1.

Let there exist a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its component smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] \xi_n(t), \; \forall t \in [a_n, b_n] \; \& \; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

such that the corresponding endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m), \; \forall m \in \{1, 2, 3, \ldots\ldots, N-1\},$

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Hence, it may be shown that, if the quasi-complex function,

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic at every point interior to and on the simple closed contour, C, then the definite integral,

$$\int_C f(q)\,dq = 0,$$

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likewise exists, if and only if the contour, C, is sufficiently small so as to be entirely contained within the δ-neighbourhood of any fixed point, $q(t_0)$, located on the contour. This result shall likewise be referred to as the quaternion analogue of the Cauchy-Goursat Theorem.

* * * * *

PROOF:-

A proof of this theorem, which was originally designated as Theorem TII-25 in the author's paper [2], is accordingly provided on pages 236-241 thereof. Q.E.D.

Theorem TI-2.

Let there exist a contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + iy_n(t) + j\hat{x}_n(t) + k\hat{y}_n(t), \quad \forall t \in [a_n, b_n],$$

whereupon we make the additional stipulation that the endpoints thereof are subject to the condition,

$$q_{m+1}(a_{m+1}) = q_m(b_m), \ \forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\}$$

(N is some finite positive integer).

Subsequently, by constructing another contour, $-C = \bigcup_{n=1}^{N} -K_n$, where

$$q_{m+1}(-b_{m+1}) = q_m(-a_m), \ \forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\},$$

we can prove that the definite integral,

$$\int_{-C} f(q)\,dq = -\int_{C} f(q)\,dq,$$

provided that the functions, $f(q_n(t))\frac{d}{dt}[q_n(t)]$ and $f(q_n(-t))\frac{d}{dt}[q_n(-t)]$, are likewise integrable with respect to the real parameter 't'.

* * * * *

PROOF:-

A proof of this theorem, which was originally designated as Theorem TII-27 in the author's paper [2], is accordingly provided on pages 248 – 249 thereof.
Q.E.D.

Theorem TI-3.

Let there exist a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t),\ \forall t \in [a_n, b_n]\ \&\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

insofar as the corresponding endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m),\ \forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\}$,

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Hence, it may be shown that, if the quasi-complex function,

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$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic at every point interior to and on the simple closed contour, C, then the definite integral,

$$\int_C f(q)\,dq = 0,$$

likewise exists. This result shall accordingly be referred to as the 'generalised' quaternion analogue of the Cauchy-Goursat Theorem.

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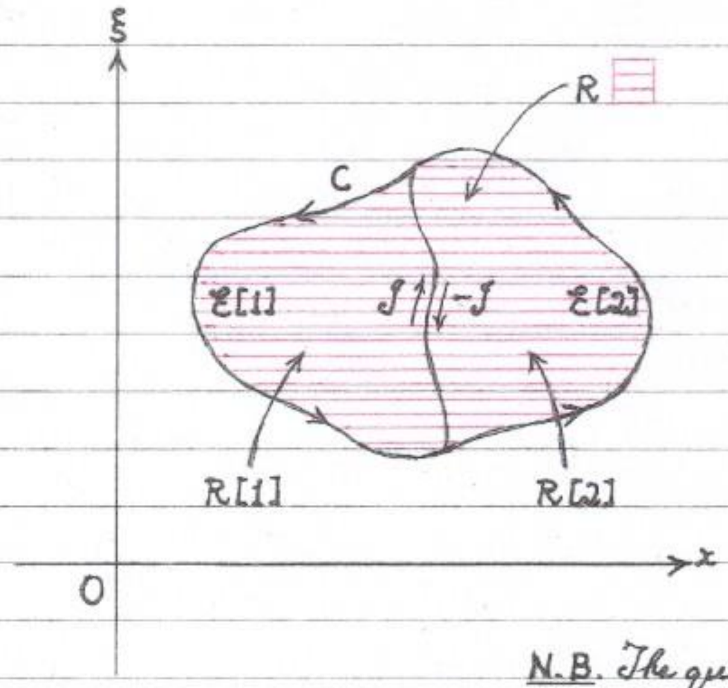
PROOF:-

In order to facilitate the proof of this particular theorem, we will invoke the principle of mathematical induction.



N.B. The quasi-complex plane,
$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\rangle$.

Fig. 1.

With reference to Fig. 1 depicted above, we initially observe that

(a) the region,

$$R = R[1] \cup R[2] = \bigcup_{n=1}^{2} R[n];$$

(b) the contour, which encloses the region, R,

$$C = \partial R = \mathcal{E}[1] \cup \mathcal{E}[2];$$

(c) the contour, which encloses the region, R[1],

$$\partial R[1] = \mathcal{E}[1] \cup \mathcal{J};$$

(d) the contour, which encloses the region, R[2],

$$\partial R[2] = \mathcal{E}[2] \cup -\mathcal{J}.$$

Moreover, in accordance with the established definitions and theorems thus pertaining to the integration of quatenion hypercomplex functions having been enunciated in the author's papers [1] & [2], we subsequently deduce that the definite integral,

$$\int_C f(q)\,dq = \int_{\partial R} f(q)\,dq = \int_{\mathcal{E}[1]} f(q)\,dq + \int_{\mathcal{E}[2]} f(q)\,dq$$

$$= \int_{\mathcal{E}[1]} f(q)\,dq + \int_{\mathcal{E}[2]} f(q)\,dq + \int_{\mathcal{J}} f(q)\,dq - \int_{\mathcal{J}} f(q)\,dq$$

$$= \int_{E[1]} f(q)\,dq + \int_{J} f(q)\,dq + \int_{E[2]} f(q)\,dq - \int_{J} f(q)\,dq$$

$$= \int_{E[1]} f(q)\,dq + \int_{J} f(q)\,dq + \int_{E[2]} f(q)\,dq + \int_{-J} f(q)\,dq,$$

bearing in mind the criteria specified in the preceding <u>Theorem TI-2</u>. Furthermore, in view of the aforesaid definitions and theorems, it likewise follows that the definite integrals,

$$\int_{\partial R[1]} f(q)\,dq = \int_{E[1]} f(q)\,dq + \int_{J} f(q)\,dq;$$

$$\int_{\partial R[2]} f(q)\,dq = \int_{E[2]} f(q)\,dq + \int_{-J} f(q)\,dq,$$

and hence the definite integral,

$$\int_{C} f(q)\,dq = \int_{\partial R} f(q)\,dq = \int_{\partial R[1]} f(q)\,dq + \int_{\partial R[2]} f(q)\,dq$$

$$= \sum_{n=1}^{2} \int_{\partial R[n]} f(q)\,dq \quad (1-1).$$

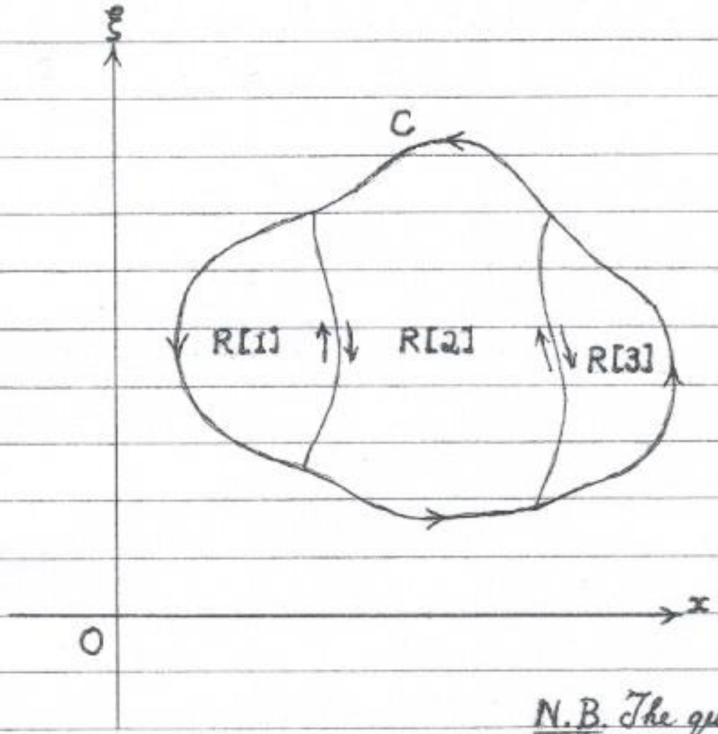


Fig. 2. N.B. The quasi-complex plane, $\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \rangle$.

With reference to Fig. 2 depicted above, we observe that the region, which is enclosed by the contour, C,

$$R = R[1] \cup R[2] \cup R[3] = \bigcup_{n=1}^{3} R[n] = R[*] \cup R[3],$$

insofar as the region,

$$R[*] = R[1] \cup R[2] = \bigcup_{n=1}^{2} R[n].$$

Furthermore, by letting ∂R; $\partial R[*]$; $\partial R[1]$; $\partial R[2]$ & $\partial R[3]$ respectively denote those contours, which enclose the regions, R; R[*]; R[1]; R[2] & R[3], we analogously deduce from Eq.(1-1) that the definite integral,

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$$\int_C f(q)\,dq = \int_{\partial R} f(q)\,dq = \int_{\partial R[*]} f(q)\,dq + \int_{\partial R[3]} f(q)\,dq$$

$$= \int_{\partial R[1]} f(q)\,dq + \int_{\partial R[2]} f(q)\,dq + \int_{\partial R[3]} f(q)\,dq = \sum_{n=1}^{3} \int_{\partial R[n]} f(q)\,dq \quad (1\text{–}2).$$

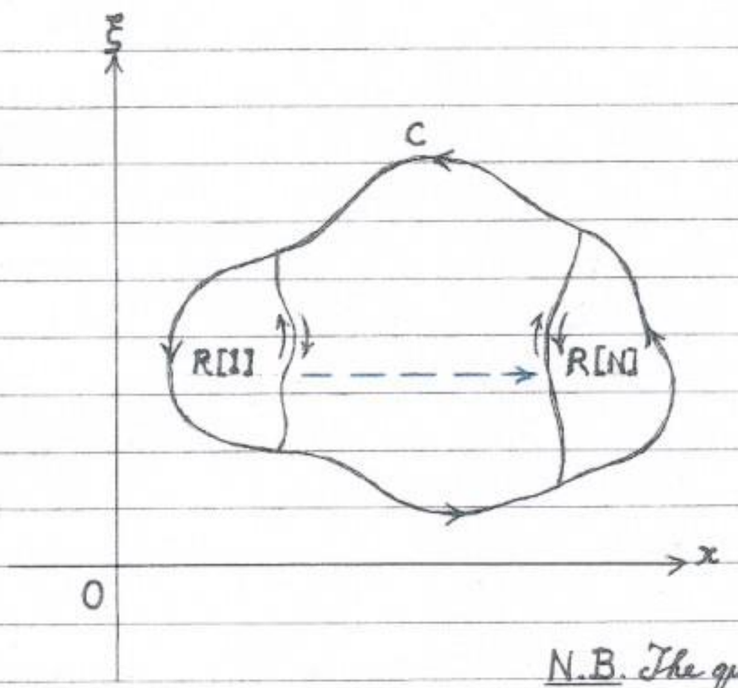

Fig. 3.

N.B. The quasi-complex plane, $\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle$.

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With reference to Fig. 3 depicted above, we observe that the region, which is enclosed by the contour, C,

$$R = R[1] \cup \ldots\ldots\ldots \cup R[N] = \bigcup_{n=1}^{N} R[n].$$

Furthermore, by letting ∂R; $\partial R[1]$; ....; $\partial R[N]$, respectively denote those contours, which enclose the regions, R; $R[1]$; ....; $R[N]$, we assert in view of the preceding Eqs. (1–1) & (1–2) that the definite integral,

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$$\int_C f(q)\,dq = \int_{\partial R} f(q)\,dq = \sum_{n=1}^{N} \int_{\partial R[n]} f(q)\,dq \quad (1\text{-}3).$$

Now in order to demonstrate the validity of this particular assertion, we observe from Fig. 4 depicted below that the region, which is enclosed by the contour, C,

$$R = R[1] \cup \ldots\ldots \cup R[N] \cup R[N+1] = \bigcup_{n=1}^{N+1} R[n] = \bigcup_{n=1}^{N} R[n] \cup R[N+1]$$

$$= R[*] \cup R[N+1],$$

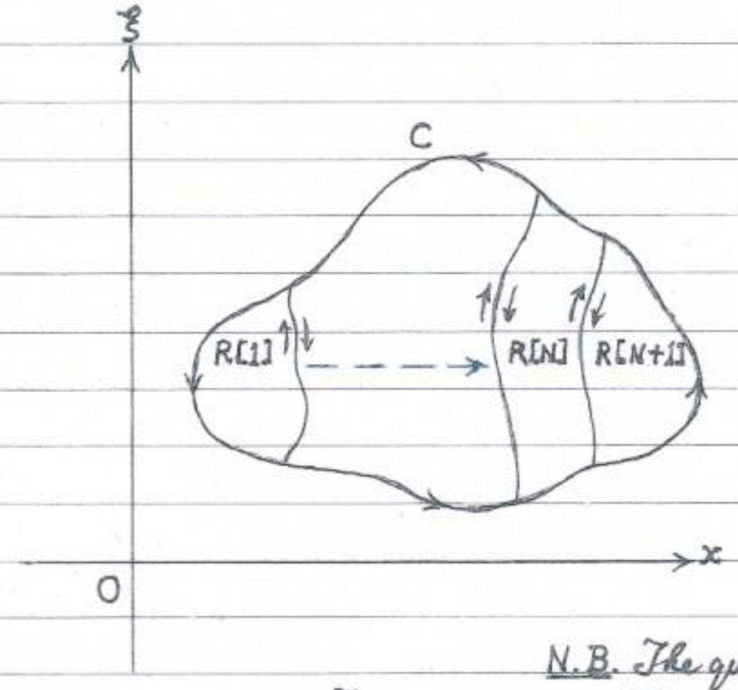

Fig. 4.

N.B. The quasi-complex plane,
$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \rangle$.

insofar as the region,

$$R[*] = \bigcup_{n=1}^{N} R[n].$$

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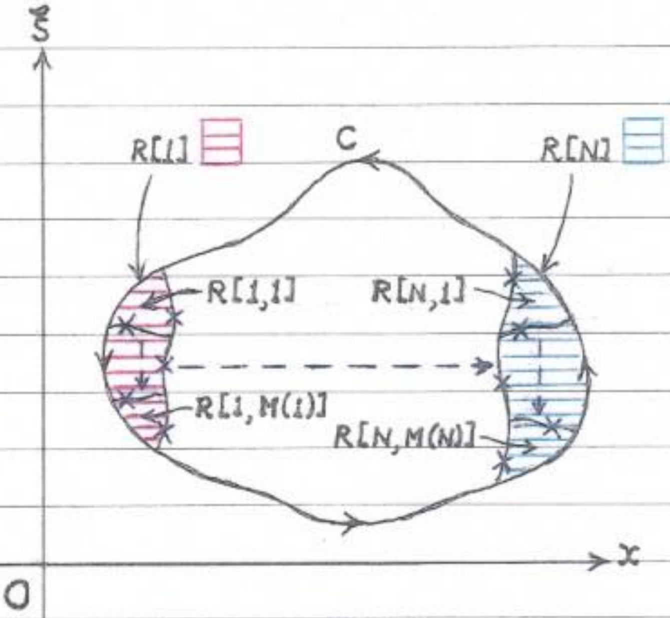
Furthermore, by letting ∂R; $\partial R[*]$; $\partial R[1]$; ....; $\partial R[N]$ & $\partial R[N+1]$ respectively denote those contours, which enclose the regions, R; $R[*]$; $R[1]$; ....; $R[N]$ & $R[N+1]$, we analogously deduce from Eqs. (1-1) & (1-3) that the definite integral,

$$\int_C f(q)\,dq = \int_{\partial R} f(q)\,dq = \int_{\partial R[*]} f(q)\,dq + \int_{\partial R[N+1]} f(q)\,dq$$

$$= \sum_{n=1}^{N} \int_{\partial R[n]} f(q)\,dq + \int_{\partial R[N+1]} f(q)\,dq = \sum_{n=1}^{N+1} \int_{\partial R[n]} f(q)\,dq, \text{ as anticipated.}$$



N.B.

Fig. 5. (a) The quasi-complex plane,
$\Pi = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle$.

(b) The symbol 'X' denotes integration along the contour in both directions (i.e. $\rightleftarrows$).

With reference to Fig. 5 depicted above, we observe that

(a) the region, which is enclosed by the contour, C,

$R = R[1] \cup \ldots\ldots\ldots \cup R[N] = \bigcup_{n=1}^{N} R[n];$

(b) each constituent region thereof,

$R[n] = R[n,1] \cup \ldots\ldots\ldots\ldots \cup R[n, M(n)] = \bigcup_{m=1}^{M(n)} R[n,m], \ \forall n \in \{1, \ldots\ldots\ldots, N\}.$

Furthermore, by letting

(a) $\partial R; \partial R[1]; \ldots\ldots\ldots; \partial R[N]$ respectively denote those contours, which enclose the regions, $R; R[1]; \ldots\ldots\ldots\ldots; R[N]$;

(b) $\partial R[1,1]; \ldots\ldots\ldots; \partial R[N, M(N)]$ respectively denote those contours, which enclose the regions, $R[1,1]; \ldots\ldots\ldots\ldots; R[N, M(N)]$,

we analogously deduce from Eq. (1-3) that the definite integral,

$$\int_C f(q)\,dq = \int_{\partial R} f(q)\,dq = \sum_{n=1}^{N} \int_{\partial R[n]} f(q)\,dq = \sum_{n=1}^{N} \sum_{m=1}^{M(n)} \int_{\partial R[n,m]} f(q)\,dq.$$

Finally, by increasing the magnitude of the positive integers, $N; M(1); \ldots\ldots\ldots \ldots; M(N)$, it therefore follows from Fig. 5 that the respective sizes of the contours, $\partial R[1,1]; \ldots\ldots\ldots\ldots; \partial R[N, M(N)]$, will inevitably be reduced and hence, in accordance with Theorem TI-1, if each of these contours is sufficiently small so as to be entirely contained within the δ-neighbourhood of any fixed point located on the contour, then every concomitant definite integral,

$$\int_{\partial R[n,m]} f(q)\,dq = 0 \quad [\ast],$$

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thereby implying, after making the appropriate algebraic substitutions, that the definite integral,

$$\int_C f(q)\,dq = \sum_{n=1}^{N} \sum_{m=1}^{M(n)} 0 = \sum_{n=1}^{N} 0 = 0, \text{ as required. } \underline{\text{Q.E.D.}}$$

[∗] N.B.

Since the function, $f(q)$, is analytic at every point interior to and on the simple closed contour, $C = \partial R$, as specified in the preamble to this proof, it must therefore be analytic at every point interior to and on each simple closed contour, $\partial R[n,m]$, as is evident from Fig. 5.

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II. Additional Remarks.

With reference to the contents of the previous section, the following points should be noted, namely -

(1) In Section II of the author's paper [2], the proofs of Theorems TII-28 & TII-29 quote Theorem TII-25 as a prerequisite result. Subsequently, for the purposes of proving these particular theorems, Theorem TI-3 should preferably be quoted instead of its aforesaid antecedent.

(2) Section II of the author's monograph [3] provides a correlation of specific formulae pertaining to the definite integration of quaternion hypercomplex functions. Once again, for the purposes of providing this particular correlation, Theorem TI-3 should preferably be quoted instead of its aforesaid antecedent.

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III. BIBLIOGRAPHY.

[1] S.C. Pearson; An Introduction to Functions of a Quaternion Hypercomplex Variable [31st March 1984; 161 handwritten foolscap pages]. [1*]

[2] S.C. Pearson; A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions [5th March 2001; 316 handwritten foolscap pages]. [1*]

[3] S.C. Pearson; Correlation of Specific Results having been enunciated in Various Expository Articles and Papers - Re:- Mathematical Paper, thus entitled - "A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions." [14th June 2018; 22 handwritten A4 pages]. [2*]

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N.B.

[1*] Copies of this paper may be obtained free of charge from the following web site address:-

Underscore

http://vixra.org/author/stephen c pearson.

[2*] Copies of this unpublished monograph may be obtained from the author upon request via his email address, namely -

scpearson1952@outlook.com.

[N.B. THIS PAGE ACCORDINGLY SIGNIFIES THE END OF THE AUTHOR’S ORIGINAL PAPER.]

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