Ecriture Détaillée des Equations de la Relativité Générale : Cas d'Une Métrique Diagonale

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Résumé

L'objet de cette note est l'écriture des équations d'Einstein (EE) de la relativité générale, sans la constante cosmologique, pour une variété de dimension 4, munie d'une métrique $g = (g_{ij})$ diagonale. On calcule les expressions des composantes des tenseurs de Ricci, de Riemann et la valeur de la courbure scalaire R. Par la suite, on donne en détail les expressions des (EE) dans les cas suivants :

- a - $g_{ii} = g_i = g_i(x_i)$;
- b - $g_1 = g_1(x_1 = t)$ et $g_i = g_i(t, x_i)$ avec $i = 2, 3, 4$;
- c - cas particulier de (b) avec $x_4 = z_0 = constante$.

Abstract

In this note, we study Einstein equations (EE) of general relativity considering a manifold $\mathcal{M}$ with a diagonal metric g_{ij}. We calculate the expression of the components of Ricci and Riemann tensors and the value of the scalar curvature R. Then we give the expression of the (EE) :

-a- for the case where $g_{ii} = g_i = g_i(x_i)$;
-b- for the case where $g_1 = g_1(x_1 = t)$ and $g_i = g_i(t, x_i)$ for $i = 2, 3, 4$;
-c- for the case (b) with $x_4 = z_0 = constant$.

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1 Introduction

Soit $\mathcal{M}$ une variété riemannienne munie d'une métrique g_{ij}. Les équations d'Einstein de la relativité générale, sans tenir de la constante cosmologique, s'écrivent :

$$R_{ij} - \frac{1}{2}R.g_{ij} = \frac{8\pi G}{c^4}T_{ij} \quad \text{pour } i = 1,4 \quad \text{et } j = 1,4 \tag{1.1}$$

Avec :
- c la vitesse de la lumière,
- G la constante universelle de la gravitation,
- T_{ij} le tenseur d'énergie-impulsion,
- R la courbure scalaire :

$$R = g^{ij}R_{ij} \tag{1.2}$$

- g^{ij} les éléments de la matrice inverse de la matrice (g_{ij}),
- R_{ij} le tenseur de Ricci avec :

$$R_{jk} = R_{kj} = \sum_{i=1}^{i=4} R^i_{jik} \quad \text{pour } j,k = 1,4 \tag{1.3}$$

et R^i_{jik} le tenseur de Riemann de la courbure :

$$R^i_{jik} = \frac{\partial \Gamma^i_{jk}}{\partial x_i} - \frac{\partial \Gamma^i_{ji}}{\partial x_k} + \sum_m (\Gamma^m_{jk}\Gamma^i_{mi} - \Gamma^m_{ji}\Gamma^i_{mk}) \quad \text{pour } j,i,k = 1,4 \tag{1.4}$$

- Γ^l_{ij} les coefficients de Christoffel du deuxième espèce :

$$\Gamma^l_{ij} = \Gamma^l_{ji} = \sum_{k=1}^{k=4} \frac{g^{lk}}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right) \quad \text{pour } l,i,j = 1,4 \tag{1.5}$$

L'objet de cette note est la recherche d'une métrique g telle que :

$$g = ds^2 = \sum_{i=1}^{i=4} g_{ii} dx_i^2 \tag{1.6}$$

ou encore :

$$g = (g_{ij}) = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{pmatrix} = \begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{pmatrix} \tag{1.7}$$

Les équations de la relativité générale sont données dans notre cas par :

$$R_{ii} - \frac{1}{2} R.g_i = \frac{8\pi G}{c^4} T_{ii} \quad \text{pour } i = 1,4 \tag{1.8}$$

$$R_{ij} = \frac{8\pi G}{c^4} T_{ij} \quad \text{pour } i = 1,4; j = 1,4; i \neq j \tag{1.9}$$

avec :

$$R = g^{ij} R_{ij} = g^{ii} R_{ii} = \sum_i \frac{R_{ii}}{g_{ii}} = \sum_i \frac{R_{ii}}{g_i} \tag{1.10}$$

$$R_{jk} = R^i_{jik} = \sum_i R^i_{jik} \tag{1.11}$$

$$R^i_{jik} = \frac{\partial \Gamma^i_{jk}}{\partial x_i} - \frac{\partial \Gamma^i_{ji}}{\partial x_k} + \sum_m (\Gamma^m_{jk}\Gamma^i_{mi} - \Gamma^m_{ji}\Gamma^i_{mk}) \tag{1.12}$$

$$\Gamma^l_{ij} = \sum_{k=1}^{k=4} \frac{g^{lk}}{2} \left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right) \tag{1.13}$$

On pose :

$$\lambda = \frac{8\pi G}{c^4} \tag{1.14}$$

2 Calcul des coefficients Γ^l_{ij}

Nous obtenons :

$$\Gamma^1_{11} = \frac{1}{2g_1}\frac{\partial g_1}{\partial x_1}; \; \Gamma^2_{11} = -\frac{1}{2g_2}\frac{\partial g_1}{\partial x_2}; \; \Gamma^3_{11} = -\frac{1}{2g_3}\frac{\partial g_1}{\partial x_3}; \; \Gamma^4_{11} = -\frac{1}{2g_4}\frac{\partial g_1}{\partial x_4} \tag{2.15}$$

$$\Gamma^1_{12} = \frac{1}{2g_1}\frac{\partial g_1}{\partial x_2}; \; \Gamma^2_{12} = \frac{1}{2g_2}\frac{\partial g_2}{\partial x_1}; \quad \Gamma^3_{12} = 0; \quad \Gamma^4_{12} = 0 \tag{2.16}$$

$$\Gamma^1_{13} = \frac{1}{2g_1}\frac{\partial g_1}{\partial x_3}; \quad \Gamma^2_{13} = 0; \quad \Gamma^3_{13} = \frac{1}{2g_3}\frac{\partial g_3}{\partial x_1}; \quad \Gamma^4_{13} = 0 \tag{2.17}$$

$$\Gamma^1_{14} = \frac{1}{2g_1}\frac{\partial g_1}{\partial x_4}; \quad \Gamma^2_{14} = 0; \quad \Gamma^3_{14} = 0; \quad \Gamma^4_{14} = \frac{1}{2g_4}\frac{\partial g_4}{\partial x_1} \tag{2.18}$$

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$$\Gamma^1_{21} = \frac{1}{2g_1}\frac{\partial g_1}{\partial x_2}; \quad \Gamma^2_{21} = \frac{1}{2g_2}\frac{\partial g_2}{\partial x_1}; \quad \Gamma^3_{21} = 0; \quad \Gamma^4_{21} = 0 \tag{2.19}$$

$$\Gamma^1_{22} = -\frac{1}{2g_1}\frac{\partial g_2}{\partial x_1}; \Gamma^2_{22} = \frac{1}{2g_2}\frac{\partial g_2}{\partial x_2}; \Gamma^3_{22} = -\frac{1}{2g_3}\frac{\partial g_2}{\partial x_3}; \Gamma^4_{22} = -\frac{1}{2g_4}\frac{\partial g_2}{\partial x_4} \tag{2.20}$$

$$\Gamma^1_{23} = 0; \quad \Gamma^2_{23} = \frac{1}{2g_2}\frac{\partial g_2}{\partial x_3}; \quad \Gamma^3_{23} = \frac{1}{2g_3}\frac{\partial g_3}{\partial x_2}; \quad \Gamma^4_{23} = 0 \tag{2.21}$$

$$\Gamma^1_{24} = 0; \quad \Gamma^2_{24} = \frac{1}{2g_2}\frac{\partial g_2}{\partial x_4}; \quad \Gamma^3_{24} = 0; \quad \Gamma^4_{24} = \frac{1}{2g_4}\frac{\partial g_4}{\partial x_2} \tag{2.22}$$

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$$\Gamma^1_{31} = \frac{1}{2g_1}\frac{\partial g_1}{\partial x_3}; \quad \Gamma^2_{31} = 0; \quad \Gamma^3_{31} = \frac{1}{2g_3}\frac{\partial g_3}{\partial x_1}; \quad \Gamma^4_{31} = 0 \tag{2.23}$$

$$\Gamma^1_{32} = 0; \quad \Gamma^2_{32} = \frac{1}{2g_2}\frac{\partial g_2}{\partial x_3}; \quad \Gamma^3_{32} = \frac{1}{2g_3}\frac{\partial g_3}{\partial x_2}; \quad \Gamma^4_{32} = 0 \tag{2.24}$$

$$\Gamma^1_{33} = -\frac{1}{2g_1}\frac{\partial g_3}{\partial x_1}; \Gamma^2_{33} = -\frac{1}{2g_2}\frac{\partial g_3}{\partial x_2}; \Gamma^3_{33} = \frac{1}{2g_3}\frac{\partial g_3}{\partial x_3}; \Gamma^4_{33} = -\frac{1}{2g_4}\frac{\partial g_3}{\partial x_4} \tag{2.25}$$

$$\Gamma^1_{34} = 0; \quad \Gamma^2_{34} = 0; \quad \Gamma^3_{34} = \frac{1}{2g_3}\frac{\partial g_3}{\partial x_4}; \quad \Gamma^4_{34} = \frac{1}{2g_4}\frac{\partial g_4}{\partial x_3} \tag{2.26}$$

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$$\Gamma^1_{41} = \frac{1}{2g_1}\frac{\partial g_1}{\partial x_4}; \quad \Gamma^2_{41} = 0; \quad \Gamma^3_{41} = 0; \quad \Gamma^4_{41} = \frac{1}{2g_4}\frac{\partial g_4}{\partial x_1} \tag{2.27}$$

$$\Gamma^1_{42} = 0; \quad \Gamma^2_{42} = \frac{1}{2g_2}\frac{\partial g_2}{\partial x_4}; \quad \Gamma^3_{42} = 0; \quad \Gamma^4_{42} = \frac{1}{2g_4}\frac{\partial g_4}{\partial x_2} \tag{2.28}$$

$$\Gamma^1_{43} = 0; \quad \Gamma^2_{43} = 0; \quad \Gamma^3_{43} = \frac{1}{2g_3}\frac{\partial g_3}{\partial x_4}; \quad \Gamma^4_{43} = \frac{1}{2g_4}\frac{\partial g_4}{\partial x_3} \tag{2.29}$$

$$\Gamma^1_{44} = -\frac{1}{2g_1}\frac{\partial g_4}{\partial x_1}; \Gamma^2_{44} = -\frac{1}{2g_2}\frac{\partial g_4}{\partial x_2}; \Gamma^3_{44} = -\frac{1}{2g_3}\frac{\partial g_4}{\partial x_3}; \Gamma^4_{44} = \frac{1}{2g_4}\frac{\partial g_4}{\partial x_4} \tag{2.30}$$

3 Calcul des coefficients R_{ij}

On a :

$$R_{11} = \sum_{i=1}^{i=4} R^i_{1i1} \tag{3.31}$$

Utilisant :

$$R^i_{jik} = \frac{\partial \Gamma^i_{jk}}{\partial x_i} - \frac{\partial \Gamma^i_{ji}}{\partial x_k} + \sum_m (\Gamma^m_{jk}\Gamma^i_{mi} - \Gamma^m_{ji}\Gamma^i_{mk})$$

on a :

$$R^1_{111} = \frac{\partial \Gamma^1_{11}}{\partial x_1} - \frac{\partial \Gamma^1_{11}}{\partial x_1} + \sum_m (\Gamma^m_{11}\Gamma^1_{m1} - \Gamma^m_{11}\Gamma^1_{m1}) = 0 \tag{3.32}$$

$$R^2_{121} = \frac{\partial \Gamma^2_{11}}{\partial x_2} - \frac{\partial \Gamma^2_{12}}{\partial x_1} + \sum_m (\Gamma^m_{11}\Gamma^2_{m2} - \Gamma^m_{12}\Gamma^2_{m1})$$

$$\begin{aligned} R^2_{121} = \frac{\partial \Gamma^2_{11}}{\partial x_2} - \frac{\partial \Gamma^2_{12}}{\partial x_1} + \Gamma^m_{11}\Gamma^2_{m2} + \Gamma^m_{11}\Gamma^2_{m2} + \Gamma^m_{11}\Gamma^2_{m2} + \Gamma^m_{11}\Gamma^2_{m2} \\ -\Gamma^m_{12}\Gamma^2_{m1} - \Gamma^m_{12}\Gamma^2_{m1} - \Gamma^m_{12}\Gamma^2_{m1} - \Gamma^m_{12}\Gamma^2_{m1} \\ = -\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_2}\frac{\partial g_2}{\partial x_1}\right) - \frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_2}\frac{\partial g_1}{\partial x_2}\right) + \frac{1}{4g_1g_2}\left(\frac{\partial g_1}{\partial x_2}\right)^2 - \frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x_1}\right)^2 \\ + \frac{1}{4g_1g_2}\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} - \frac{1}{4g_2^2}\frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_2} - \frac{1}{4g_2g_3}\frac{\partial g_1}{\partial x_3}\frac{\partial g_2}{\partial x_3} - \frac{1}{4g_2g_4}\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_4} \end{aligned} \tag{3.33}$$

$$\begin{aligned} R^3_{131} = \frac{\partial \Gamma^3_{11}}{\partial x_3} - \frac{\partial \Gamma^3_{11}}{\partial x_1} + \sum_m (\Gamma^m_{11}\Gamma^3_{m3} - \Gamma^m_{13}\Gamma^3_{m1}) \\ = -\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_3}\frac{\partial g_3}{\partial x_1}\right) - \frac{1}{2}\frac{\partial}{\partial x_3}\left(\frac{1}{g_3}\frac{\partial g_1}{\partial x_3}\right) - \frac{1}{4g_1^2}\frac{\partial g_1}{\partial x_1}\frac{\partial g_1}{\partial x_3} - \frac{1}{4g_3^2}\left(\frac{\partial g_3}{\partial x_1}\right)^2 \\ + \frac{1}{4g_1g_3}\frac{\partial g_1}{\partial x_1}\frac{\partial g_3}{\partial x_1} - \frac{1}{4g_2g_3}\frac{\partial g_1}{\partial x_2}\frac{\partial g_3}{\partial x_2} - \frac{1}{4g_3^2}\frac{\partial g_1}{\partial x_3}\frac{\partial g_3}{\partial x_3} - \frac{1}{4g_3g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_3}{\partial x_4} \end{aligned} \tag{3.34}$$

$$\begin{aligned} R^4_{141} = \frac{\partial \Gamma^4_{11}}{\partial x_4} - \frac{\partial \Gamma^4_{41}}{\partial x_1} + \sum_m (\Gamma^m_{11}\Gamma^4_{m4} - \Gamma^m_{14}\Gamma^4_{m4}) \\ = -\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial x_1}\right) - \frac{1}{2}\frac{\partial}{\partial x_4}\left(\frac{1}{g_4}\frac{\partial g_1}{\partial x_4}\right) - \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_4^2}\frac{\partial g_4}{\partial x_1}\frac{\partial g_4}{\partial x_4} \\ + \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2g_4}\frac{\partial g_1}{\partial x_2}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_3g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_3} - \frac{1}{4g_4^2}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_4} \end{aligned} \tag{3.35}$$

D'où :

$$R_{11} = -\frac{\partial}{\partial x_1}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2g_2}\frac{\partial g_1}{\partial x_2}\right) + \frac{1}{4g_1g_2}\left(\frac{\partial g_1}{\partial x_2}\right)^2 - \frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x_1}\right)^2$$
$$+\frac{1}{4g_1g_2}\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} - \frac{1}{4g_2^2}\frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_2} - \frac{1}{4g_2g_3}\frac{\partial g_1}{\partial x_3}\frac{\partial g_2}{\partial x_3} - \frac{1}{4g_2g_4}\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_4}$$
$$-\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial x_1}\right) - \frac{1}{2}\frac{\partial}{\partial x_4}\left(\frac{1}{g_4}\frac{\partial g_1}{\partial x_4}\right) - \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_4^2}\frac{\partial g_4}{\partial x_1}\frac{\partial g_4}{\partial x_4}$$
$$+\frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2g_4}\frac{\partial g_1}{\partial x_2}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_3g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_3} - \frac{1}{4g_4^2}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_4}$$
$$-\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial x_1}\right) - \frac{1}{2}\frac{\partial}{\partial x_4}\left(\frac{1}{g_4}\frac{\partial g_1}{\partial x_4}\right) - \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_4^2}\frac{\partial g_4}{\partial x_1}\frac{\partial g_4}{\partial x_4}$$
$$+\frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2g_4}\frac{\partial g_1}{\partial x_2}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_3g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_3} - \frac{1}{4g_4^2}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_4} \quad (3.36)$$

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De même :

$$R_{22} = \sum_{i=1}^{i=4} R^i_{2i2} \quad (3.37)$$

On trouve :

$$R^1_{212} = \frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{21}}{\partial x_2} + \sum_m (\Gamma^m_{22}\Gamma^1_{m1} - \Gamma^m_{21}\Gamma^1_{m1})$$
$$= \frac{\partial \Gamma^1_{22}}{\partial x_2} - \frac{\partial \Gamma^1_{21}}{\partial x_2} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{21}\Gamma^1_{21} + \Gamma^3_{22}\Gamma^1_{31} + \Gamma^4_{22}\Gamma^1_{41}$$
$$-\Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{21}\Gamma^1_{31} - \Gamma^4_{21}\Gamma^1_{41}$$

$$= -\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_1}\frac{\partial g_1}{\partial x_2}\right) - \frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial x_1}\right) - \frac{1}{4g_1^2}\left(\frac{\partial g_1}{\partial x_2}\right)^2 + \frac{1}{4g_1g_2}\left(\frac{\partial g_2}{\partial x_1}\right)^2$$
$$-\frac{1}{4g_1^2}\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + \frac{1}{4g_1g_2}\frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_2} - \frac{1}{4g_1g_3}\frac{\partial g_2}{\partial x_3}\frac{\partial g_1}{\partial x_3} - \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_2}{\partial x_4} \quad (3.38)$$

Et :

$$R^2_{222} = 0 \quad (3.39)$$

Et :

$$R^3_{232} = \frac{\partial \Gamma^2_{22}}{\partial x_3} - \frac{\partial \Gamma^3_{23}}{\partial x_2} + \sum_m (\Gamma^m_{22}\Gamma^3_{m3} - \Gamma^m_{23}\Gamma^3_{m2})$$
$$= -\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_3}\frac{\partial g_3}{\partial x_2}\right) + \frac{1}{2}\frac{\partial}{\partial x_3}\left(\frac{1}{g_2}\frac{\partial g_2}{\partial x_2}\right) - \frac{1}{4g_3^2}\left(\frac{\partial g_3}{\partial x_2}\right)^2 - \frac{1}{4g_1g_3}\frac{\partial g_2}{\partial x_1}\frac{\partial g_3}{\partial x_1}$$
$$+\frac{1}{4g_2g_3}\left(\frac{\partial g_2}{\partial x_3}\right)^2 + \frac{1}{4g_2g_3}\frac{\partial g_2}{\partial x_2}\frac{\partial g_3}{\partial x_2} - \frac{1}{4g_3^2}\frac{\partial g_2}{\partial x_3}\frac{\partial g_3}{\partial x_2} - \frac{1}{4g_3g_4}\frac{\partial g_2}{\partial x_4}\frac{\partial g_3}{\partial x_4} \quad (3.40)$$

Et :

$$R^4_{242} = \frac{\partial \Gamma^4_{22}}{\partial x_4} - \frac{\partial \Gamma^4_{24}}{\partial x_2} + \sum_m (\Gamma^m_{22}\Gamma^4_{m4} - \Gamma^m_{24}\Gamma^2_{m2})$$

$$\begin{aligned} R^4_{242} = & -\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial x_2}\right) - \frac{1}{2}\frac{\partial}{\partial x_4}\left(\frac{1}{g_4}\frac{\partial g_2}{\partial x_4}\right) - \frac{1}{4g_1g_4}\frac{\partial g_2}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2^2}\frac{\partial g_2}{\partial x_4}\frac{\partial g_2}{\partial x_2} \\ & -\frac{1}{4g_3g_4}\frac{\partial g_2}{\partial x_3}\frac{\partial g_4}{\partial x_3} - \frac{1}{4g_2g_4}\frac{\partial g_2}{\partial x_4}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_4^2}\frac{\partial g_2}{\partial x_4}\frac{\partial g_4}{\partial x_4} \end{aligned} \tag{3.41}$$

D'où l'expression de R_{22} :

$$\begin{aligned} R_{22} = & -\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_1}\frac{\partial g_1}{\partial x_2}\right) - \frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial x_1}\right) - \frac{1}{4g_1^2}\left(\frac{\partial g_1}{\partial x_2}\right)^2 + \frac{1}{4g_1g_2}\left(\frac{\partial g_2}{\partial x_1}\right)^2 \\ & -\frac{1}{4g_1^2}\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + \frac{1}{4g_1g_2}\frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_2} - \frac{1}{4g_1g_3}\frac{\partial g_2}{\partial x_3}\frac{\partial g_1}{\partial x_3} - \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_2}{\partial x_4} \\ & -\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_3}\frac{\partial g_3}{\partial x_2}\right) + \frac{1}{2}\frac{\partial}{\partial x_3}\left(\frac{1}{g_2}\frac{\partial g_2}{\partial x_2}\right) - \frac{1}{4g_3^2}\left(\frac{\partial g_3}{\partial x_2}\right)^2 - \frac{1}{4g_1g_3}\frac{\partial g_2}{\partial x_1}\frac{\partial g_3}{\partial x_1} \\ & +\frac{1}{4g_2g_3}\left(\frac{\partial g_2}{\partial x_3}\right)^2 + \frac{1}{4g_2g_3}\frac{\partial g_2}{\partial x_2}\frac{\partial g_3}{\partial x_2} - \frac{1}{4g_3^2}\frac{\partial g_2}{\partial x_3}\frac{\partial g_3}{\partial x_2} - \frac{1}{4g_3g_4}\frac{\partial g_2}{\partial x_4}\frac{\partial g_3}{\partial x_4} \\ & -\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial x_2}\right) - \frac{1}{2}\frac{\partial}{\partial x_4}\left(\frac{1}{g_4}\frac{\partial g_2}{\partial x_4}\right) - \frac{1}{4g_1g_4}\frac{\partial g_2}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2^2}\frac{\partial g_2}{\partial x_4}\frac{\partial g_2}{\partial x_2} \\ & -\frac{1}{4g_3g_4}\frac{\partial g_2}{\partial x_3}\frac{\partial g_4}{\partial x_3} - \frac{1}{4g_2g_4}\frac{\partial g_2}{\partial x_4}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_4^2}\frac{\partial g_2}{\partial x_4}\frac{\partial g_4}{\partial x_4} \end{aligned} \tag{3.42}$$

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De même, on obtient l'expression de R_{33} :

$$\begin{aligned} R_{33} = & -\frac{1}{2}\frac{\partial}{\partial x_3}\left(\frac{1}{g_1}\frac{\partial g_1}{\partial x_3}\right) - \frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial x_1}\right) + \frac{1}{4g_1^2}\frac{\partial g_3}{\partial x_1}\frac{\partial g_1}{\partial x_3} - \frac{1}{4g_1^2}\frac{\partial g_1}{\partial x_1}\frac{\partial g_3}{\partial x_1} \\ & -\frac{1}{4g_1g_2}\frac{\partial g_1}{\partial x_2}\frac{\partial g_3}{\partial x_2} + \frac{1}{4g_1g_3}\frac{\partial g_3}{\partial x_1}\frac{\partial g_3}{\partial x_3} + \frac{1}{4g_1g_3}\frac{\partial g_1}{\partial x_3}\frac{\partial g_3}{\partial x_3} - \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_3}{\partial x_4} \\ & -\frac{1}{2}\frac{\partial}{\partial x_3}\left(\frac{1}{g_2}\frac{\partial g_2}{\partial x_3}\right) - \frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{1}{g_2}\frac{\partial g_3}{\partial x_2}\right) - \frac{1}{4g_1g_2}\frac{\partial g_2}{\partial x_1}\frac{\partial g_3}{\partial x_1} - \frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x_3}\right)^2 \\ & +\frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x_2}\right)^2 + \frac{1}{4g_2g_3}\left(\frac{\partial g_3}{\partial x_2}\right)^2 + \frac{1}{4g_2g_3}\frac{\partial g_3}{\partial x_3}\frac{\partial g_2}{\partial x_2} - \frac{1}{4g_2g_4}\frac{\partial g_3}{\partial x_4}\frac{\partial g_2}{\partial x_4} \\ & -\frac{\partial}{\partial x_3}\left(\frac{1}{2g_4}\frac{\partial g_4}{\partial x_3}\right) - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_4}\frac{\partial g_3}{\partial x_4}\right) - \frac{1}{4g_1g_4}\frac{\partial g_4}{\partial x_1}\frac{\partial g_3}{\partial x_1} - \frac{1}{4g_2g_4}\frac{\partial g_3}{\partial x_2}\frac{\partial g_4}{\partial x_2} \\ & +\frac{1}{4g_3g_4}\left(\frac{\partial g_3}{\partial x_4}\right)^2 + \frac{1}{4g_4^2}\frac{\partial g_3}{\partial x_4}\frac{\partial g_4}{\partial x_3} + \frac{1}{4g_3g_4}\frac{\partial g_3}{\partial x_3}\frac{\partial g_4}{\partial x_3} + \frac{1}{4g_4^2}\frac{\partial g_3}{\partial x_4}\frac{\partial g_4}{\partial x_4} \end{aligned} \tag{3.43}$$

****

Calcul de $R_{44} = \sum_{i=1}^{i=4} R^i_{4i4}$

$$
\begin{aligned}
R^1_{414} = \frac{\partial \Gamma^1_{44}}{\partial x_1} - \frac{\partial \Gamma^1_{41}}{\partial x_4} + \sum_m (\Gamma^m_{44}\Gamma^1_{m1} - \Gamma^m_{41}\Gamma^1_{m4}) = -\frac{\partial}{\partial x_4}\left(\frac{1}{2g_1}\frac{\partial g_1}{\partial x_4}\right) \\
-\frac{\partial}{\partial x_1}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial x_1}\right) - \frac{1}{4g_1^2}\left(\frac{\partial g_1}{\partial x_4}\right)^2 - \frac{1}{4g_1^2}\frac{\partial g_4}{\partial x_1}\frac{\partial g_1}{\partial x_1} - \frac{1}{4g_1g_2}\frac{\partial g_1}{\partial x_2}\frac{\partial g_4}{\partial x_2} \\
-\frac{1}{4g_1g_3}\frac{\partial g_1}{\partial x_3}\frac{\partial g_4}{\partial x_4} + \frac{1}{4g_1g_4}\left(\frac{\partial g_4}{\partial x_1}\right)^2 + \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_4}
\end{aligned} \tag{3.44}
$$

$$
\begin{aligned}
R^2_{424} = \frac{\partial \Gamma^2_{44}}{\partial x_2} - \frac{\partial \Gamma^2_{42}}{\partial x_4} + \sum_m (\Gamma^m_{44}\Gamma^2_{m2} - \Gamma^m_{42}\Gamma^2_{m4}) = -\frac{\partial}{\partial x_4}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial x_4}\right) \\
-\frac{\partial}{\partial x_2}\left(\frac{1}{2g_2}\frac{\partial g_4}{\partial x_2}\right) - \frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x_4}\right)^2 - \frac{1}{4g_1g_2}\frac{\partial g_2}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2^2}\frac{\partial g_2}{\partial x_2}\frac{\partial g_4}{\partial x_2} \\
-\frac{1}{4g_2g_3}\frac{\partial g_2}{\partial x_3}\frac{\partial g_4}{\partial x_4} + \frac{1}{4g_2g_4}\frac{\partial g_4}{\partial x_2}\frac{\partial g_4}{\partial x_2} + \frac{1}{4g_2g_4}\frac{\partial g_2}{\partial x_4}\frac{\partial g_4}{\partial x_4}
\end{aligned} \tag{3.45}
$$

$$
\begin{aligned}
R^3_{434} = \frac{\partial \Gamma^3_{44}}{\partial x_3} - \frac{\partial \Gamma^3_{34}}{\partial x_4} + \sum_m (\Gamma^m_{44}\Gamma^3_{m3} - \Gamma^m_{34}\Gamma^3_{m4}) = -\frac{\partial}{\partial x_4}\left(\frac{1}{2g_3}\frac{\partial g_3}{\partial x_4}\right) \\
-\frac{\partial}{\partial x_3}\left(\frac{1}{2g_3}\frac{\partial g_4}{\partial x_4}\right) - \frac{1}{4g_3^2}\left(\frac{\partial g_3}{\partial x_4}\right)^2 - \frac{1}{4g_1g_3}\frac{\partial g_3}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2g_3}\frac{\partial g_3}{\partial x_2}\frac{\partial g_4}{\partial x_2} \\
-\frac{1}{4g_3^2}\frac{\partial g_3}{\partial x_3}\frac{\partial g_4}{\partial x_4} - \frac{1}{4g_3g_4}\frac{\partial g_3}{\partial x_4}\frac{\partial g_4}{\partial x_4} + \frac{1}{4g_4^2}\frac{\partial g_4}{\partial x_3}\frac{\partial g_4}{\partial x_4}
\end{aligned} \tag{3.46}
$$

et :

$$
R^4_{444} = 0 \tag{3.47}
$$

D'où :

$$
\begin{aligned}
R_{44} = &-\frac{\partial}{\partial x_4}\left(\frac{1}{2g_1}\frac{\partial g_1}{\partial x_4}\right) - \frac{\partial}{\partial x_1}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial x_1}\right) \\
&-\frac{1}{4g_1^2}\left(\frac{\partial g_1}{\partial x_4}\right)^2 - \frac{1}{4g_1^2}\frac{\partial g_4}{\partial x_1}\frac{\partial g_1}{\partial x_1} - \frac{1}{4g_1g_2}\frac{\partial g_1}{\partial x_2}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_1g_3}\frac{\partial g_1}{\partial x_3}\frac{\partial g_4}{\partial x_4} \\
&+\frac{1}{4g_1g_4}\left(\frac{\partial g_4}{\partial x_1}\right)^2 + \frac{1}{4g_1g_4}\frac{\partial g_1}{\partial x_4}\frac{\partial g_4}{\partial x_4} - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial x_4}\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2g_2}\frac{\partial g_4}{\partial x_2}\right) \\
&-\frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x_4}\right)^2 - \frac{1}{4g_1g_2}\frac{\partial g_2}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2^2}\frac{\partial g_2}{\partial x_2}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_2g_3}\frac{\partial g_2}{\partial x_3}\frac{\partial g_4}{\partial x_4} \\
&+\frac{1}{4g_2g_4}\frac{\partial g_4}{\partial x_2}\frac{\partial g_4}{\partial x_2} + \frac{1}{4g_2g_4}\frac{\partial g_2}{\partial x_4}\frac{\partial g_4}{\partial x_4} - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_3}\frac{\partial g_3}{\partial x_4}\right) - \frac{\partial}{\partial x_3}\left(\frac{1}{2g_3}\frac{\partial g_4}{\partial x_4}\right) \\
&-\frac{1}{4g_3^2}\left(\frac{\partial g_3}{\partial x_4}\right)^2 - \frac{1}{4g_1g_3}\frac{\partial g_3}{\partial x_1}\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_2g_3}\frac{\partial g_3}{\partial x_2}\frac{\partial g_4}{\partial x_2} - \frac{1}{4g_3^2}\frac{\partial g_3}{\partial x_3}\frac{\partial g_4}{\partial x_4} \\
&-\frac{1}{4g_3g_4}\frac{\partial g_3}{\partial x_4}\frac{\partial g_4}{\partial x_4} + \frac{1}{4g_4^2}\frac{\partial g_4}{\partial x_3}\frac{\partial g_4}{\partial x_4}
\end{aligned} \tag{3.48}
$$

****

Exprimons maintenant les coefficients $R_{12}, R_{13}, R_{14}, R_{23}, R_{24}$ et R_{34}. Nous obtenons successivement :

$$
R_{12} = \sum_{i=1}^{i=4} R^i_{1i2} = \sum_{i=1}^{i=4}\left[\frac{\partial \Gamma^i_{12}}{\partial x_i} - \frac{\partial \Gamma^i_{1i}}{\partial x_2} + \sum_m (\Gamma^m_{12}\Gamma^i_{mi} - \Gamma^m_{1i}\Gamma^i_{m2})\right] \tag{3.49}
$$

Par suite :

$$
\begin{aligned}
R^1_{112} = \frac{\partial \Gamma^1_{12}}{\partial x_1} - \frac{\partial \Gamma^1_{11}}{\partial x_2} &+ (\Gamma^1_{12}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{12}) + (\Gamma^2_{12}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{22}) \\
&+(\Gamma^3_{12}\Gamma^3_{31} - \Gamma^3_{11}\Gamma^1_{32}) + (\Gamma^4_{12}\Gamma^1_{41} - \Gamma^4_{11}\Gamma^1_{42}) = \\
&\frac{\partial}{\partial x_1}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_1}\right)
\end{aligned} \tag{3.50}
$$

$$
\begin{aligned}
R^2_{122} = \frac{\partial \Gamma^2_{12}}{\partial x_2} - \frac{\partial \Gamma^2_{12}}{\partial x_2} &+ (\Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12}) + (\Gamma^2_{12}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{22}) \\
&+(\Gamma^3_{12}\Gamma^2_{32} - \Gamma^3_{12}\Gamma^3_{32}) + (\Gamma^4_{12}\Gamma^2_{42} - \Gamma^4_{12}\Gamma^2_{42}) = 0
\end{aligned} \tag{3.51}
$$

$$
\begin{aligned}
R^3_{132} &= \frac{\partial \Gamma^3_{12}}{\partial x_3} - \frac{\partial \Gamma^3_{13}}{\partial x_2} + \sum_m (\Gamma^m_{12}\Gamma^3_{m3} - \Gamma^m_{13}\Gamma^3_{m2}) \\
&= -\frac{\partial}{\partial x_2}\left(\frac{1}{2g_3}.\frac{\partial g_3}{\partial x_1}\right) + \frac{1}{4g_1g_3}.\frac{\partial g_1}{\partial x_2}.\frac{\partial g_3}{\partial x_1} - \frac{1}{2g_3^2}.\frac{\partial g_3}{\partial x_2}.\frac{\partial g_3}{\partial x_1} \\
&\qquad + \frac{1}{4g_2g_3}.\frac{\partial g_2}{\partial x_1}.\frac{\partial g_3}{\partial x_2}
\end{aligned} \tag{3.52}
$$

$$\begin{aligned}R^4_{142} = \frac{\partial \Gamma^4_{12}}{\partial x_4} - \frac{\partial \Gamma^4_{14}}{\partial x_2} + \sum_m (\Gamma^m_{12}\Gamma^4_{m4} - \Gamma^m_{14}\Gamma^4_{m2}) \\ - \frac{\partial}{\partial x_2}\left(\frac{1}{2g_4}.\frac{\partial g_4}{\partial x_1}\right) + \frac{1}{4g_1g_4}.\frac{\partial g_1}{\partial x_2}.\frac{\partial g_4}{\partial x_1} - \frac{1}{4g_4^2}.\frac{\partial g_4}{\partial x_1}.\frac{\partial g_4}{\partial x_2} \\ + \frac{1}{4g_2g_4}.\frac{\partial g_2}{\partial x_1}.\frac{\partial g_4}{\partial x_2}\end{aligned} \tag{3.53}$$

D'où :

$$\begin{aligned}R_{12} = \sum_{i=1}^{i=4} R^i_{1i2} = \frac{\partial}{\partial x_1}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_1}\right) \\ - \frac{\partial}{\partial x_2}\left(\frac{1}{2g_3}.\frac{\partial g_3}{\partial x_1}\right) + \frac{1}{4g_1g_3}.\frac{\partial g_1}{\partial x_2}.\frac{\partial g_3}{\partial x_1} - \frac{1}{2g_3^2}.\frac{\partial g_3}{\partial x_2}.\frac{\partial g_3}{\partial x_1} \\ + \frac{1}{4g_2g_3}.\frac{\partial g_2}{\partial x_1}.\frac{\partial g_3}{\partial x_2} - \frac{\partial}{\partial x_2}\left(\frac{1}{2g_4}.\frac{\partial g_4}{\partial x_1}\right) + \frac{1}{4g_1g_4}.\frac{\partial g_1}{\partial x_2}.\frac{\partial g_4}{\partial x_1} \\ - \frac{1}{4g_4^2}.\frac{\partial g_4}{\partial x_1}.\frac{\partial g_4}{\partial x_2} + \frac{1}{4g_2g_4}.\frac{\partial g_2}{\partial x_1}.\frac{\partial g_4}{\partial x_2}\end{aligned} \tag{3.54}$$

****

$$\begin{aligned}R_{13} = \sum_{i=1}^{i=4} R^i_{1i3} = \sum_{i=1}^{i=4}\left[\frac{\partial \Gamma^i_{13}}{\partial x_i} - \frac{\partial \Gamma^i_{1i}}{\partial x_3} + \sum_m (\Gamma^m_{13}\Gamma^i_{mi} - \Gamma^m_{1i}\Gamma^i_{m3})\right] \\ = -\frac{\partial}{\partial x_1}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_3}\right) - \frac{\partial}{\partial x_3}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_1}\right) - \frac{\partial}{\partial x_3}\left(\frac{1}{2g_2}.\frac{\partial g_2}{\partial x_1}\right) \\ + \frac{1}{4g_1g_2}.\frac{\partial g_1}{\partial x_3}.\frac{\partial g_2}{\partial x_1} - \frac{1}{4g_2^2}.\frac{\partial g_2}{\partial x_1}.\frac{\partial g_2}{\partial x_3} + \frac{1}{4g_2g_3}.\frac{\partial g_2}{\partial x_3}.\frac{\partial g_3}{\partial x_1} \\ - \frac{\partial}{\partial x_3}\left(\frac{1}{2g_4}\frac{\partial g_4}{\partial x_1}\right) + \frac{1}{4g_1g_4}.\frac{\partial g_1}{\partial x_3}.\frac{\partial g_4}{\partial x_1} + \frac{1}{4g_3g_4}.\frac{\partial g_3}{\partial x_1}.\frac{\partial g_4}{\partial x_3} - \frac{1}{4g_4^2}\frac{\partial g_4}{\partial x_1}\frac{\partial g_4}{\partial x_3}\end{aligned} \tag{3.55}$$

****

$$\begin{aligned}R_{14} = \sum_{i=1}^{i=4} R^i_{1i4} = \sum_{i=1}^{i=4}\left[\frac{\partial \Gamma^i_{14}}{\partial x_i} - \frac{\partial \Gamma^i_{1i}}{\partial x_4} + \sum_m (\Gamma^m_{14}\Gamma^i_{mi} - \Gamma^m_{1i}\Gamma^i_{m4})\right] \\ = -\frac{\partial}{\partial x_1}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_4}\right) - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_1}\right) - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_2}.\frac{\partial g_2}{\partial x_1}\right) \\ + \frac{1}{4g_1g_2}.\frac{\partial g_1}{\partial x_4}.\frac{\partial g_2}{\partial x_1} - \frac{1}{4g_2^2}.\frac{\partial g_2}{\partial x_1}.\frac{\partial g_2}{\partial x_4} + \frac{1}{4g_2g_4}.\frac{\partial g_2}{\partial x_4}.\frac{\partial g_4}{\partial x_1} \\ - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_3}\frac{\partial g_3}{\partial x_1}\right) + \frac{1}{4g_1g_3}.\frac{\partial g_1}{\partial x_4}.\frac{\partial g_3}{\partial x_1} + \frac{1}{4g_3g_4}.\frac{\partial g_3}{\partial x_4}.\frac{\partial g_4}{\partial x_1}\end{aligned} \tag{3.56}$$

****

$$\begin{aligned}
R_{23} = \sum_{i=1}^{i=4} R^i_{2i3} = \sum_{i=1}^{i=4}\left[\frac{\partial\Gamma^i_{23}}{\partial x_i} - \frac{\partial\Gamma^i_{2i}}{\partial x_3} + \sum_m(\Gamma^m_{23}\Gamma^i_{mi} - \Gamma^m_{2i}\Gamma^i_{m3})\right] \\
= -\frac{\partial}{\partial x_3}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{2g_2}.\frac{\partial g_2}{\partial x_3}\right) - \frac{\partial}{\partial x_3}\left(\frac{1}{2g_2}.\frac{\partial g_2}{\partial x_2}\right) \\
-\frac{1}{4g_1^2}.\frac{\partial g_1}{\partial x_2}.\frac{\partial g_1}{\partial x_3} + \frac{1}{4g_1g_2}.\frac{\partial g_1}{\partial x_2}.\frac{\partial g_2}{\partial x_3} + \frac{1}{4g_1g_3}.\frac{\partial g_1}{\partial x_3}.\frac{\partial g_3}{\partial x_2} \\
-\frac{\partial}{\partial x_3}\left(\frac{1}{2g_4}\frac{\partial g_4}{\partial x_2}\right) + \frac{1}{4g_2g_4}.\frac{\partial g_2}{\partial x_3}.\frac{\partial g_4}{\partial x_2} + \frac{1}{4g_3g_4}.\frac{\partial g_3}{\partial x_2}.\frac{\partial g_4}{\partial x_3} \\
-\frac{1}{4g_4^2}.\frac{\partial g_4}{\partial x_2}.\frac{\partial g_4}{\partial x_3}
\end{aligned} \tag{3.57}$$

****

$$\begin{aligned}
R_{24} = \sum_{i=1}^{i=4} R^i_{2i4} = \sum_{i=1}^{i=4}\left[\frac{\partial\Gamma^i_{24}}{\partial x_i} - \frac{\partial\Gamma^i_{2i}}{\partial x_4} + \sum_m(\Gamma^m_{24}\Gamma^i_{mi} - \Gamma^m_{2i}\Gamma^i_{m4})\right] \\
= -\frac{\partial}{\partial x_4}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{2g_2}.\frac{\partial g_2}{\partial x_4}\right) - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_2}.\frac{\partial g_2}{\partial x_2}\right) \\
-\frac{1}{4g_1^2}.\frac{\partial g_1}{\partial x_2}.\frac{\partial g_1}{\partial x_4} + \frac{1}{4g_2^2}.\frac{\partial g_2}{\partial x_1}.\frac{\partial g_2}{\partial x_4} + \frac{1}{4g_1g_4}.\frac{\partial g_1}{\partial x_4}.\frac{\partial g_4}{\partial x_2} \\
-\frac{\partial}{\partial x_4}\left(\frac{1}{2g_3}\frac{\partial g_3}{\partial x_2}\right) + \frac{1}{4g_2g_3}.\frac{\partial g_2}{\partial x_4}.\frac{\partial g_3}{\partial x_2} - \frac{1}{4g_3^2}.\frac{\partial g_3}{\partial x_2}.\frac{\partial g_3}{\partial x_3} \\
+\frac{1}{4g_3g_4}.\frac{\partial g_3}{\partial x_4}.\frac{\partial g_4}{\partial x_2}
\end{aligned} \tag{3.58}$$

****

$$\begin{aligned}
R_{34} = \sum_{i=1}^{i=4} R^i_{3i4} = \sum_{i=1}^{i=4}\left[\frac{\partial\Gamma^i_{34}}{\partial x_i} - \frac{\partial\Gamma^i_{3i}}{\partial x_4} + \sum_m(\Gamma^m_{34}\Gamma^i_{mi} - \Gamma^m_{3i}\Gamma^i_{m4})\right] \\
= -\frac{\partial}{\partial x_4}\left(\frac{1}{2g_1}.\frac{\partial g_1}{\partial x_3}\right) - \frac{1}{4g_1^2}.\frac{\partial g_1}{\partial x_3}.\frac{\partial g_1}{\partial x_4} + \frac{1}{4g_1g_3}.\frac{\partial g_1}{\partial x_3}.\frac{\partial g_3}{\partial x_4} \\
+\frac{1}{4g_1g_4}.\frac{\partial g_1}{\partial x_4}.\frac{\partial g_4}{\partial x_3} - \frac{\partial}{\partial x_4}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial x_3}\right) - \frac{1}{4g_2^2}.\frac{\partial g_2}{\partial x_3}.\frac{\partial g_2}{\partial x_4} \\
+\frac{1}{4g_2g_3}.\frac{\partial g_2}{\partial x_3}.\frac{\partial g_3}{\partial x_4} + \frac{1}{4g_2g_4}.\frac{\partial g_2}{\partial x_4}.\frac{\partial g_4}{\partial x_3} - \frac{\partial}{\partial x_3}\left(\frac{1}{2g_3}.\frac{\partial g_3}{\partial x_4}\right) \\
-\frac{\partial}{\partial x_4}\left(\frac{1}{2g_3}.\frac{\partial g_3}{\partial x_3}\right)
\end{aligned} \tag{3.59}$$

4 Recherche de $g = (g_{ii}(x_i))$

On cherche la métrique g telle que les composantes :

$$g_i(x) = g_i(x_i) \tag{4.60}$$

Alors, on a les éléments suivants :

$$R_{11} = 0 \tag{4.61}$$

$$R_{22} = 0 \tag{4.62}$$

$$R_{33} = \frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x_2}\right)^2 + \frac{1}{4g_2g_3}\frac{\partial g_2}{\partial x_2}\frac{\partial g_3}{\partial x_3} \tag{4.63}$$

$$R_{44} = -\frac{1}{4g_3^2}\frac{\partial g_3}{\partial x_3}\frac{\partial g_4}{\partial x_4} \tag{4.64}$$

$$R_{12} = R_{13} = R_{14} = R_{23} = R_{24} = R_{34} = 0 \tag{4.65}$$

Ce qui donne :

$$R = \sum_i \frac{R_{ii}}{g_i} = \frac{1}{4g_2^2g_3}\left(\frac{\partial g_2}{\partial x_2}\right)^2 + \frac{1}{4g_2g_3^2}\frac{\partial g_2}{\partial x_2}\frac{\partial g_3}{\partial x_3} - \frac{1}{4g_3^2g_4}\frac{\partial g_3}{\partial x_3}\frac{\partial g_4}{\partial x_4} \tag{4.66}$$

On peut écrire maintenant :

$$R = \frac{1}{4g_2^2g_3}\left(\frac{dg_2}{dx_2}\right)^2 + \frac{1}{4g_2g_3^2}\frac{dg_2}{dx_2}\frac{dg_3}{dx_3} - \frac{1}{4g_3^2g_4}\frac{dg_3}{dx_3}\frac{dg_4}{dx_4} \tag{4.67}$$

D'où :

$$-\frac{1}{2}R.g_1 = \lambda T_{11} \tag{4.68}$$

$$-\frac{1}{2}R.g_2 = \lambda T_{22} \tag{4.69}$$

$$R_{33} - \frac{1}{2}R.g_3 = \lambda T_{33} \tag{4.70}$$

$$R_{44} - \frac{1}{2}R.g_4 = \lambda T_{44} \tag{4.71}$$

$$R_{ij} = \lambda T_{ij} = 0 \quad i \neq j \tag{4.72}$$

4.1 Milieu Vide

Dans ce cas $T_{ij} = 0$, ce qui donne :

$$-\tfrac{1}{2}R.g_1 = 0 \Longrightarrow R = 0 \quad \text{ou} \quad g_1 = 0 \; or \; g_1 \neq 0 \tag{4.73}$$

$$-\tfrac{1}{2}Rg_2 = 0 \Longrightarrow R = 0 \quad \text{ou} \quad g_2 = 0 \; or \; g_2 \neq 0 \tag{4.74}$$

$$R_{33} - \tfrac{1}{2}R.g_3 = 0 \tag{4.75}$$

$$R_{44} - \tfrac{1}{2}R.g_4 = 0 \tag{4.76}$$

L'équation (4.72) est vérifié pour $i \neq j$ et donne :

$$0 = 0 \tag{4.77}$$

De ce qui précède, on a les conditions :

$$R = 0 \tag{4.78}$$
$$R_{33} = 0 \tag{4.79}$$
$$R_{44} = 0 \tag{4.80}$$

D'où :

$$R_{33} = 0 \Longrightarrow \frac{dg_2}{dx_2}\left(\frac{1}{g_2}\frac{dg_2}{dx_2} + \frac{1}{g_3}\frac{dg_3}{dx_3}\right) = 0$$

$$\Longrightarrow \begin{cases} \dfrac{dg_2}{dx_2} = 0 \Longrightarrow g_2 = C_2 = constante \\ \text{ou } \underbrace{\dfrac{1}{g_2}\dfrac{dg_2}{dx_2}}_{f(x_2)} = -\underbrace{\dfrac{1}{g_3}\dfrac{dg_3}{dx_3}}_{h(x_3)} = C = constante \end{cases} \tag{4.81}$$

1. Si on considère le cas $g_2 = C_2 = constante$, l'expression $R = 0$ donne :

$$\frac{dg_3}{dx_3}\frac{dg_4}{dx_4} = 0 \Longrightarrow \frac{dg_3}{dx_3} = 0 \text{ ou } \frac{dg_4}{dx_4} = 0 \tag{4.82}$$

soit :

$$g_3 = constante = C_3 \tag{4.83}$$
$$\text{ou} \quad g_4 = constante = C_4 \tag{4.84}$$

On a alors :

$$g_2 = C_2$$
$$g_3 = \text{constante} = C_3$$
$$g_4 \text{ fonction indéterminée a priori}$$
$$g_1 \text{ fonction indéterminée}$$

La métrique g s'écrit :

$$\boxed{g = ds^2 = g_1(x_1)dx_1^2 + C_2dx_2^2 + C_3dx_3^2 + g_4(x_4)dx_4^2} \tag{4.85}$$

2. On a aussi le cas : $g_4 = \text{constante} = C_4$, ainsi que g_1 et g_3 des fonctions indéterminées. La métrique g s'écrit dans ce cas :

$$\boxed{g = ds^2 = g_1(x_1)dx_1^2 + C_2dx_2^2 + g_3(x_3)dx_3^2 + C_4dx_4^2} \tag{4.86}$$

3. Si on considère le deuxième cas de l'équation (4.81), on a alors :

$$g_2 = g_2(x_2) = \alpha_2 e^{Cx_2} \tag{4.87}$$

et :

$$g_3 = g_3(x_3) = \alpha_3 e^{-Cx_3} \tag{4.88}$$

$R_{44} = 0$ implique que :

$$\frac{dg_4}{dx_4} = 0 \Longrightarrow g_4 = C_4 = constante \tag{4.89}$$

La fonction g_1 est indéterminée.

La métrique g s'écrit alors :

$$\boxed{g = ds^2 = g_1(x_1)dx_1^2 + \alpha_2 e^{Cx_2}dx_2^2 + \alpha_3 e^{-Cx_3}dx_3^2 + C_4 dx_4^2} \tag{4.90}$$

5 Recherche de $g = g_1(x_1)dx_1^2 + \sum_{i=2}^{i=4} g_i(x_i, x_1)dx_i^2$

Dans cette section, on va écrire en détail les équations de la relativité générale que vérifient les éléments $g_1(x_1), g_i(x_i, x_1), i = 2, 3, 4$. Pour simplifier les écritures, on pose :

$$x_1 = t \tag{5.91}$$
$$x_2 = x \tag{5.92}$$
$$x_3 = y \tag{5.93}$$
$$x_4 = z \tag{5.94}$$

La métrique s'écrit alors :

$$ds^2 = g = g_1(t)dt^2 + g_2(x,t)dx^2 + g_3(y,t)dy^2 + g_4(z,t)dz^2 \tag{5.95}$$

Les éléments du tenseur de Ricci sont successivement :

$$\begin{aligned} R_{11} = & -\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right) - \frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2 + \frac{1}{4g_1g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} \\ & -\frac{\partial}{\partial t}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial t}\right) - \frac{1}{2g_4^2}\frac{\partial g_4}{\partial t}\frac{\partial g_4}{\partial z} + \frac{1}{2g_1g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_4}{\partial t} \end{aligned} \tag{5.96}$$

****

$$\begin{aligned} R_{22} = & -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right) + \frac{1}{4g_1g_2}\left(\frac{\partial g_2}{\partial t}\right)^2 - \frac{1}{4g_1^2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} \\ & -\frac{1}{4g_1g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t} - \frac{1}{4g_1g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t} \end{aligned} \tag{5.97}$$

****

$$R_{33} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right) - \frac{1}{4g_1^2}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t} + \frac{1}{4g_1g_3}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}$$
$$-\frac{1}{4g_1g_2}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t} + \frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x}\right)^2 + \frac{1}{4g_2g_3}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x} \tag{5.98}$$

****

$$R_{44} = -\frac{\partial}{\partial t}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial t}\right) - \frac{1}{4g_1^2}\frac{\partial g_4}{\partial t}\frac{\partial g_1}{\partial t}$$
$$+\frac{1}{4g_1g_4}\left(\frac{\partial g_4}{\partial t}\right)^2 - \frac{1}{4g_1g_2}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t} + \frac{1}{4g_3^2}\frac{\partial g_3}{\partial y}.\frac{\partial g_4}{\partial z}$$
$$-\frac{1}{4g_1g_3}\frac{\partial g_3}{\partial t}\frac{\partial g_4}{\partial t} \tag{5.99}$$

****

Et :

$$R_{12} = 0 \tag{5.100}$$
$$R_{13} = 0 \tag{5.101}$$
$$R_{14} = 0 \tag{5.102}$$
$$R_{23} = 0 \tag{5.103}$$
$$R_{24} = 0 \tag{5.104}$$
$$R_{34} = 0 \tag{5.105}$$

****

5.1 Calcul de la courbure scalaire

Le rayon de courbure scalaire R est :

$$\begin{aligned}
R = \sum_{i=1}^{i=4}\frac{R_{ii}}{g_i} &= -\frac{1}{g_1}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right) - \frac{1}{4g_1g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2 + \frac{1}{4g_1^2g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} \\
&-\frac{1}{g_1}\frac{\partial}{\partial t}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial t}\right) - \frac{1}{2g_1g_4^2}\frac{\partial g_4}{\partial t}\frac{\partial g_4}{\partial z} + \frac{1}{2g_1^2g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_4}{\partial t} \\
&+ - \frac{1}{2g_2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right) + \frac{1}{4g_1g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2 - \frac{1}{4g_1^2g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} \\
&-\frac{1}{4g_1g_2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t} - \frac{1}{4g_1g_2g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t} - \frac{1}{2g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right) \\
&-\frac{1}{4g_1^2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t} + \frac{1}{4g_1g_3^2}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y} - \frac{1}{4g_1g_2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t} \\
&+\frac{1}{4g_2^2g_3}\left(\frac{\partial g_2}{\partial x}\right)^2 + \frac{1}{4g_2g_3^2}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x} - \frac{1}{g_4^2}\frac{\partial}{\partial t}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial t}\right) \\
&-\frac{1}{4g_1^2g_4}\frac{\partial g_4}{\partial t}\frac{\partial g_1}{\partial t} + \frac{1}{4g_1g_4^2}\left(\frac{\partial g_4}{\partial t}\right)^2 - \frac{1}{4g_1g_2g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t} \\
&+\frac{1}{4g_3^2g_4}\frac{\partial g_3}{\partial y}.\frac{\partial g_4}{\partial z} - \frac{1}{4g_1g_3g_4}\frac{\partial g_3}{\partial t}\frac{\partial g_4}{\partial t}
\end{aligned} \tag{5.106}$$

5.2 Ecriture des Equations Différentielles

Dans le milieu vide, on a les équations :

$$R_{11} - \frac{1}{2}R.g_1 = 0 \tag{5.107}$$

$$R_{22} - \frac{1}{2}R.g_2 = 0 \tag{5.108}$$

$$R_{33} - \frac{1}{2}R.g_3 = 0 \tag{5.109}$$

$$R_{44} - \frac{1}{2}R.g_4 = 0 \tag{5.110}$$

$$R_{ij} = 0 = 0 \quad i \neq j \tag{5.111}$$

Ce qui donne :

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\left(\frac{1}{g_2}\frac{\partial g_2}{\partial t}\right)-\frac{1}{2g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2+\frac{1}{2g_1g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-2\frac{\partial}{\partial t}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial t}\right)-\frac{1}{g_4^2}\frac{\partial g_4}{\partial t}\frac{\partial g_4}{\partial z}+\frac{1}{g_1g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_4}{\partial t}\\
&+\frac{1}{g_1^2}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right)-\frac{1}{4g_1^2g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2-\frac{1}{4g_1^3g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{g_1^2}\frac{\partial}{\partial t}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial t}\right)+\frac{1}{2g_1^2g_4^2}\frac{\partial g_4}{\partial t}\frac{\partial g_4}{\partial z}-\frac{1}{2g_1^3g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_4}{\partial t}\\
&+\frac{1}{2g_1g_2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right)+\frac{1}{4g_1^2g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2+\frac{1}{4g_1^3g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{4g_1^2g_2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_1^2g_2g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}+\frac{1}{2g_1g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)\\
&+\frac{1}{4g_1^3g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}-\frac{1}{4g_1^2g_3^2}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}+\frac{1}{4g_1^2g_2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&-\frac{1}{4g_1g_2^2g_3}\left(\frac{\partial g_2}{\partial x}\right)^2-\frac{1}{4g_1g_2g_3^2}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}+\frac{1}{g_1g_4^2}\frac{\partial}{\partial t}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial t}\right)\\
&+\frac{1}{4g_1^3g_4}\frac{\partial g_4}{\partial t}\frac{\partial g_1}{\partial t}-\frac{1}{4g_1^2g_4^2}\left(\frac{\partial g_4}{\partial t}\right)^2+\frac{1}{4g_1^2g_2g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}\\
&-\frac{1}{4g_1g_3^2g_4}\frac{\partial g_3}{\partial y}.\frac{\partial g_4}{\partial z}+\frac{1}{4g_1^2g_3g_4}\frac{\partial g_3}{\partial t}\frac{\partial g_4}{\partial t}=0
\end{aligned}
\tag{5.112}
$$

Et :

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right)+\frac{1}{2g_1g_2}\left(\frac{\partial g_2}{\partial t}\right)^2-\frac{1}{2g_1^2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-\frac{1}{2g_1g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}-\frac{1}{2g_1g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}\\
&+\frac{1}{g_1g_2}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right)+\frac{1}{4g_1g_2^3}\left(\frac{\partial g_2}{\partial t}\right)^2-\frac{1}{4g_1^2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{g_1g_2}\frac{\partial}{\partial t}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial t}\right)+\frac{1}{2g_1g_2g_4^2}\frac{\partial g_4}{\partial t}\frac{\partial g_4}{\partial z}-\frac{1}{2g_1^2g_2g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_4}{\partial t}\\
&+\frac{1}{2g_2^2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right)-\frac{1}{4g_1g_2^3}\left(\frac{\partial g_2}{\partial t}\right)^2+\frac{1}{4g_1^2g_2^2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{4g_1g_2^2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_1g_2^2g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}+\frac{1}{2g_2g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)\\
&+\frac{1}{4g_1^2g_2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}-\frac{1}{4g_1g_2g_3^2}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}+\frac{1}{4g_1g_2^2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&-\frac{1}{4g_2^3g_3}\left(\frac{\partial g_2}{\partial x}\right)^2-\frac{1}{4g_2^2g_3^2}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}+\frac{1}{g_2g_4^2}\frac{\partial}{\partial t}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial t}\right)\\
&+\frac{1}{4g_1^2g_2g_4}\frac{\partial g_4}{\partial t}\frac{\partial g_1}{\partial t}-\frac{1}{4g_1g_2g_4^2}\left(\frac{\partial g_4}{\partial t}\right)^2+\frac{1}{4g_1g_2^2g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}\\
&-\frac{1}{4g_2g_3^2g_4}\frac{\partial g_3}{\partial y}.\frac{\partial g_4}{\partial z}+\frac{1}{4g_1g_2g_3g_4}\frac{\partial g_3}{\partial t}\frac{\partial g_4}{\partial t}=0
\end{aligned}
\tag{5.113}
$$

Et :

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)-\frac{1}{2g_1^2}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{2g_1g_3}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}\\
&-\frac{1}{2g_1g_2}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x}\right)^2+\frac{1}{2g_2g_3}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}\\
&-\frac{1}{g_1g_3}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right)-\frac{1}{4g_1g_2^2g_3}\left(\frac{\partial g_2}{\partial t}\right)^2+\frac{1}{4g_1^2g_2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-\frac{1}{g_1g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial t}\right)-\frac{1}{2g_1g_3g_4^2}\frac{\partial g_4}{\partial t}\frac{\partial g_4}{\partial z}+\frac{1}{2g_1^2g_3g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_4}{\partial t}\\
&+-\frac{1}{2g_2g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right)+\frac{1}{4g_1g_2^2g_3}\left(\frac{\partial g_2}{\partial t}\right)^2-\frac{1}{4g_1^2g_2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-\frac{1}{4g_1g_2g_3^2}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}-\frac{1}{4g_1g_2g_3g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}-\frac{1}{2g_3^2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)\\
&-\frac{1}{4g_1^2g_3^2}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_1g_3^3}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}-\frac{1}{4g_1g_2g_3^2}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&+\frac{1}{4g_2^2g_3^2}\left(\frac{\partial g_2}{\partial x}\right)^2+\frac{1}{4g_2g_3^3}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}-\frac{1}{g_3g_4^2}\frac{\partial}{\partial t}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial t}\right)\\
&-\frac{1}{4g_1^2g_3g_4}\frac{\partial g_4}{\partial t}\frac{\partial g_1}{\partial t}+\frac{1}{4g_1g_3g_4^2}\left(\frac{\partial g_4}{\partial t}\right)^2-\frac{1}{4g_1g_2g_3g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}\\
&+\frac{1}{4g_3^3g_4}\frac{\partial g_3}{\partial y}.\frac{\partial g_4}{\partial z}-\frac{1}{4g_1g_3^2g_4}\frac{\partial g_3}{\partial t}\frac{\partial g_4}{\partial t}=0
\end{aligned}
\tag{5.114}
$$

Finalement :

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_4}{\partial t}\right)-\frac{1}{2g_1^2}\frac{\partial g_4}{\partial t}\frac{\partial g_1}{\partial t}\\
&+\frac{1}{2g_1g_4}\left(\frac{\partial g_4}{\partial t}\right)^2-\frac{1}{2g_1g_2}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}+\frac{1}{2g_3^2}\frac{\partial g_3}{\partial y}.\frac{\partial g_4}{\partial z}\\
&-\frac{1}{2g_1g_3}\frac{\partial g_3}{\partial t}\frac{\partial g_4}{\partial t}\\
&-\frac{1}{g_1g_4}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right)-\frac{1}{4g_1g_2^2g_4}\left(\frac{\partial g_2}{\partial t}\right)^2+\frac{1}{4g_1^2g_2g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-\frac{1}{g_1g_4}\frac{\partial}{\partial t}\left(\frac{1}{g_4}\frac{\partial g_4}{\partial t}\right)-\frac{1}{2g_1g_4^3}\frac{\partial g_4}{\partial t}\frac{\partial g_4}{\partial z}+\frac{1}{2g_1^2g_4^2}\frac{\partial g_1}{\partial t}\frac{\partial g_4}{\partial t}\\
&+\frac{1}{2g_2g_4}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right)+\frac{1}{4g_1g_2^2g_4}\left(\frac{\partial g_2}{\partial t}\right)^2-\frac{1}{4g_1^2g_2g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-\frac{1}{4g_1g_2g_3g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}-\frac{1}{4g_1g_2g_4^2}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}-\frac{1}{2g_3g_4}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)\\
&-\frac{1}{4g_1^2g_3g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_1g_3^2g_4}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}-\frac{1}{4g_1g_2g_3g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&+\frac{1}{4g_2^2g_3g_4}\left(\frac{\partial g_2}{\partial x}\right)^2+\frac{1}{4g_2g_3^2g_4}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}-\frac{1}{g_4^2}\frac{\partial}{\partial t}\left(\frac{1}{2g_1}\frac{\partial g_4}{\partial t}\right)\\
&-\frac{1}{4g_1^2g_4^2}\frac{\partial g_4}{\partial t}\frac{\partial g_1}{\partial t}+\frac{1}{4g_1g_4^3}\left(\frac{\partial g_4}{\partial t}\right)^2-\frac{1}{4g_1g_2g_4^2}\frac{\partial g_2}{\partial t}\frac{\partial g_4}{\partial t}\\
&+\frac{1}{4g_3^2g_4^2}\frac{\partial g_3}{\partial y}.\frac{\partial g_4}{\partial z}-\frac{1}{4g_1g_3g_4^2}\frac{\partial g_3}{\partial t}\frac{\partial g_4}{\partial t}=0
\end{aligned}
\tag{5.115}
$$

6 Recherche de la métrique sur le plan équipotentiel $z = Cte$

On se place sur le plan $z = z_0 = constante$ donc $\dfrac{\partial g_4}{\partial z} = 0$. On a alors les équations suivantes :

$$\begin{aligned}
&-\frac{\partial}{\partial t}\left(\frac{1}{g_2}\frac{\partial g_2}{\partial t}\right) - \frac{1}{2g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2 + \frac{1}{2g_1g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} + \frac{1}{g_1^2}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right)\\
&-\frac{1}{4g_1^2g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2 - \frac{1}{4g_1^3g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} + \frac{1}{2g_1g_2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right) + \frac{1}{4g_1^2g_2^2}\left(\frac{\partial g_2}{\partial t}\right)^2\\
&+\frac{1}{4g_1^3g_2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} + \frac{1}{4g_1^2g_2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t} + \frac{1}{2g_1g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right) + \frac{1}{4g_1^3g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}\\
&-\frac{1}{4g_1^2g_3^2}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y} + \frac{1}{4g_1^2g_2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t} - \frac{1}{4g_1g_2^2g_3}\left(\frac{\partial g_2}{\partial x}\right)^2\\
&-\frac{1}{4g_1g_2g_3^2}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x} = 0
\end{aligned} \tag{6.116}$$

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$$\begin{aligned}
&-\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right) + \frac{1}{2g_1g_2}\left(\frac{\partial g_2}{\partial t}\right)^2 - \frac{1}{2g_1^2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t} - \frac{1}{2g_1g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&+\frac{1}{g_1g_2}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right) + \frac{1}{4g_1g_2^3}\left(\frac{\partial g_2}{\partial t}\right)^2 - \frac{1}{4g_1^2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{2g_2^2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right) - \frac{1}{4g_1g_2^3}\left(\frac{\partial g_2}{\partial t}\right)^2 + \frac{1}{4g_1^2g_2^2}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{4g_1g_2^2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t} + \frac{1}{2g_2g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)\\
&+\frac{1}{4g_1^2g_2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t} - \frac{1}{4g_1g_2g_3^2}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y} + \frac{1}{4g_1g_2^2g_3}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&-\frac{1}{4g_2^3g_3}\left(\frac{\partial g_2}{\partial x}\right)^2 - \frac{1}{4g_2^2g_3^2}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x} = 0
\end{aligned} \tag{6.117}$$

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\begin{aligned}
&-\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)-\frac{1}{2g_1^2}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{2g_1g_3}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}\\
&-\frac{1}{2g_1g_2}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_2^2}\left(\frac{\partial g_2}{\partial x}\right)^2+\frac{1}{2g_2g_3}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}\\
&-\frac{1}{g_1g_3}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right)-\frac{1}{4g_1g_2^2g_3}\left(\frac{\partial g_2}{\partial t}\right)^2+\frac{1}{4g_1^2g_2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{2g_2g_3}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right)+\frac{1}{4g_1g_2^2g_3}\left(\frac{\partial g_2}{\partial t}\right)^2-\frac{1}{4g_1^2g_2g_3}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-\frac{1}{4g_1g_2g_3^2}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}-\frac{1}{2g_3^2}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)\\
&-\frac{1}{4g_1^2g_3^2}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_1g_3^3}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}-\frac{1}{4g_1g_2g_3^2}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&+\frac{1}{4g_2^2g_3^2}\left(\frac{\partial g_2}{\partial x}\right)^2+\frac{1}{4g_2g_3^3}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}=0
\end{aligned}
\tag{6.118}
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\begin{aligned}
&-\frac{1}{g_1g_4}\frac{\partial}{\partial t}\left(\frac{1}{2g_2}\frac{\partial g_2}{\partial t}\right)-\frac{1}{4g_1g_2^2g_4}\left(\frac{\partial g_2}{\partial t}\right)^2+\frac{1}{4g_1^2g_2g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&+\frac{1}{2g_2g_4}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial t}\right)+\frac{1}{4g_1g_2^2g_4}\left(\frac{\partial g_2}{\partial t}\right)^2-\frac{1}{4g_1^2g_2g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_2}{\partial t}\\
&-\frac{1}{4g_1g_2g_3g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}-\frac{1}{2g_3g_4}\frac{\partial}{\partial t}\left(\frac{1}{g_1}\frac{\partial g_3}{\partial t}\right)\\
&-\frac{1}{4g_1^2g_3g_4}\frac{\partial g_1}{\partial t}\frac{\partial g_3}{\partial t}+\frac{1}{4g_1g_3^2g_4}\frac{\partial g_3}{\partial t}\frac{\partial g_3}{\partial y}-\frac{1}{4g_1g_2g_3g_4}\frac{\partial g_2}{\partial t}\frac{\partial g_3}{\partial t}\\
&+\frac{1}{4g_2^2g_3g_4}\left(\frac{\partial g_2}{\partial x}\right)^2+\frac{1}{4g_2g_3^2g_4}\frac{\partial g_3}{\partial y}\frac{\partial g_2}{\partial x}=0
\end{aligned}
\tag{6.119}
$$

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