# The Twin Power Conjecture

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**Abstract**

We consider a new conjecture regarding powers of integer numbers and more specifically, we are interesting in existence and finding pairs of integers: $n \geq 2$ and $m \geq 2$, such that $n^m = m^n$. We conjecture that $n = 2$, $m = 4$ and $n = 4$, $m = 2$ are the only integral solutions.

Next, we consider the corresponding generalizations for Hypercomplex Integers: Gaussian and Lipschitz Integers.

*Keywords*: integers; complex number; exponentiation; power; quaternion

## 1 Introduction

Exponentiation is a mathematical operation, written as $n^m$, involving two numbers, the base n and the exponent or power m. When m is a positive integer, exponentiation corresponds to repeated multiplication of the base: that is, $n^m$ is the product of multiplying m bases.

We consider a new conjecture regarding powers of integer numbers and more specifically, we are interesting in existence and finding pairs of integers: $n \geq 2$ and $m \geq 2$, such that $n^m = m^n$.

For $n = 2$ and $m = 4$ we have: $2^4 = 4^2 = 16$. So at least one such pair of powers does exist.

**Theorem 1.** *$n^m = m^n$ if and only if*

$$n/m = log(n)/log(m),\ n/m = log_m(n),\ m/n = log_n(m).$$

**Proof.** Indeed,

$$\log[n^m] = \log[m^n] \Leftrightarrow n \log(m) = m \log(n),$$

$$\log_m[n^m] = \log_m[m^n] \Leftrightarrow m \log_m(n) = n,$$

$$\log_n[n^m] = \log_n[m^n] \Leftrightarrow n \log_n(m) = m. \qquad \square$$

Is there another pair of integer $n \geq 2$ and integer $m \geq 2$ such that $n^m = m^n$ ?

Analysing the list of whole-number powers for $n = 1, ..., 10$ and $m = 1, ..., 10$ and monotonic domains of $f(x) = a^x - x^a$ we may expect that the answer is: no.

We conjecture that no other integer solutions exist.

## 2 The Twin Power Conjecture

Let us formulate our Conjecture(The Twin Power Conjecture).

**Conjecture 1(The Twin Power Conjecture).** There exist only one *pair of integer $n = 2$ and integer $m = 4$, so that $n^m = m^n$.*

So, $n = 2$, $m = 4$ and $n = 4$, $m = 2$ are the only integral solutions of the equation: $n^m = m^n$.

At least, it would be interesting to prove or disprove this conjecture and to develop general theory regarding existence and computation of such pairs of powers.

## 3 Generalization for Hypercomplex Integers

Similar to the situation, where despite no real solutions exists for the equation: $x^2 + 1 = 0$, however they do exist in the complex plane, we can expect, that our equation: $n^m = m^n$, may have more solutions in the complex plane, quaternionic, etc., hypercomplex algebras.

### 3.1 Generalization for Complex Gaussian Integers

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{ a + b\mathbf{i} \mid a, b \in \mathbf{Z} \}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2-dimensional integer lattice. The Gaussian integers form unique factorization domain: it is irreducible if and only if it is a prime(Gaussian primes). The field of Gaussian rationals consists of the complex numbers whose real and imaginary part are both rational(see, e.g., [1], [5]).

The norm of a Gaussian integer is its product with its conjugate:

$N(a + b\mathbf{i}) = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2.$

The norm is multiplicative, that is, one has:

$N(zw) = N(z)N(w), \; z, w \in \mathbf{Z}[\mathbf{i}].$

Since complex exponentiation is defined as $z^w = \exp(w \log(z))$, we can consider aforementioned equation for complex Gaussian integers, and more specifically, we are interesting in existence and finding pairs of Gaussian integers: $z, w \in \mathbf{Z}[\mathbf{i}]$, such that $z^w = w^z$.

**Theorem 2.** *$z^w = w^z$ if and only if*

$$z/w = \log(z)/\log(w),\; z/w = \log_w(z),\; w/z = \log_z(w).$$

**Proof.** Indeed,

$$\log[z^w] = \log[w^z] \Leftrightarrow z \log(w) = w \log(z),$$

$$\log_m[z^w] = \log_w[w^z] \Leftrightarrow w \log_w(z) = z,$$

$$\log_z[z^w] = \log_z[w^z] \Leftrightarrow z \log_z(w) = w. \qquad \square$$

**Conjecture 2**(**The Twin Power Conjecture**). There exist more than one *pair of Gaussian Integers z and w, so that* $z^w = w^z$.

### 3.2 Generalization for Quaternionic Lipschitz Integers

Similar integral subclass is well-known for quaternions: Lipschitz Integers(quaternions).

Quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers(see, e.g., [3], [4]).

The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$.

This norm makes it possible to define the distance $d(p, q) = \|p - q\|$, which makes **H** into a metric space.

Lipschitz Integer(quaternion) is defined as:

$$\mathbf{L} := \{ q: q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a \in \mathbf{Z},\ b \in \mathbf{Z}, c \in \mathbf{Z},\ d \in \mathbf{Z} \}.$$

Lipschitz Integer(quaternion) is a quaternion, whose components are all integers.

Since quaternion exponentiation is defined as $p^q = \exp(q \log(p))$, we can consider aforementioned equation for Lipschitz integers, and more specifically, we are interesting in existence and finding pairs of Lipschitz integers: $p, q \in \mathbf{L}$, such that $p^q = q^p$.

**Theorem 3.** *$p^q = q^p$ if and only if*

$$p/q = \log(p)/\log(q),\ p/q = \log_q(p),\ q/p = \log_p(q).$$

**Proof.** Indeed,

$$\log[p^q] = \log[q^p] \Leftrightarrow p \log(q) = q \log(p),$$

$$\log_q[p^q] = \log_q[q^p] \Leftrightarrow q \log_q(p) = p,$$

$$\log_p[p^q] = \log_p[q^p] \Leftrightarrow p \log_p(q) = q.$$

□

**Conjecture 3(The Twin Power Conjecture).** There exist more than one *pair of Lipschitz Integers p and q, so that* $p^q = q^p$.

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