Three, Four and N-Dimensional Swastikas & their Projections

Sascha Vongehr[a]

（风洒沙，博士）

College of Engineering and Applied Sciences, Nanjing University, Jiangsu, China

Abstract: Difficulties with generalizing the swastika shape for N dimensional spaces are discussed. While distilling the crucial general characteristics such as whether the number of arms is 2^N or $2N$, a three dimensional (3D) swastika is introduced and then a construction algorithm for any natural number N so that it reproduces the 1D, 2D, and 3D shapes. The 4D hyper swastika and internal surfaces inside its hypercube envelope are then presented for the first time.

Keywords: Higher Dimension Geometry; Hyper Swastika; Reclaiming of Symbols; Didactic Arts

1 Introduction

Classification belongs to the basis of science. Generalizing shapes to obtain concepts that are recognizable in spaces of any number of dimensions, such as N-dimensional cubes (point, line, square, cube, hypercubes), facilitates understanding of and classification in multidimensional topology and geometry, which are important to several branches of physics, for example for string theories, which are theories of multidimensional membranes (n-branes) and their shape transformations, the string being a one dimensional (1D) membrane (1-brane). Also design and art gain from such hyper shapes and their projections into sub-spaces, sculptures and paintings such as Dali's famous 1954 "Corpus Hypercubus" in the Metropolitan Museum New York. It depicts Dali's wife Gala looking up to Jesus Christ fixed at a cross that results from the unfolding of a hollow hypercube (Fig. 1a) [1].

[a] Corresponding author's electronic mail: vongehr8@yahoo.com vongehr@163.com

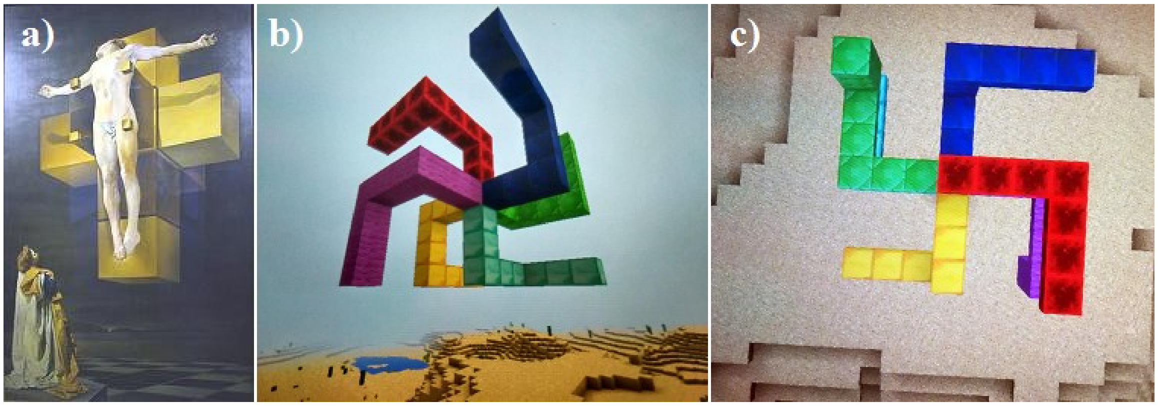

Fig. 1: a) Dali's "Corpus Hypercubus" is one of many examples for art inspired by higher dimensional shapes. b) A 3D swastika constructed on top of a mountain in a computer generated virtual reality (Minecraft), c) seen in its orthogonal projection when looking straight down the vertical (z-axis). The 2D swastika is not a part inside this shape, but only appears in "one sided" projections. Opposing arms have complementary colors; certainly also these can count as art inspired by generalization of shapes.

One very recognizable and widely used shape, in many cultures ancient and recent, is the four armed swastika, which is globally the most recognized symbol, beating the Christian cross and the Flag of the USA. Those who may feel that certain uses of this shape in its rich history prevent us from discussing its general geometry should consider that scientific description can belong to a "reclaiming of suppressive/discriminating" symbols or language, which is increasingly often demanded. In fact, this work completes previous work which reinterpreted the controversial 2D swastika as a mere "one-sided projection" of a richer, 3D shape (Fig. 1b) which does not contain the 2D shape anywhere except for "only in the shadows of certain rigid perspectives" (Fig. 1c) [[2]].

Since the four arms of the swastika usually lie either along the two coordinate axes or in the four quadrants of 2D flat space (Fig. 2a), generalizing may seem especially easy, and in two or more ways. However, the most obvious paths all soon encounter hurdles.

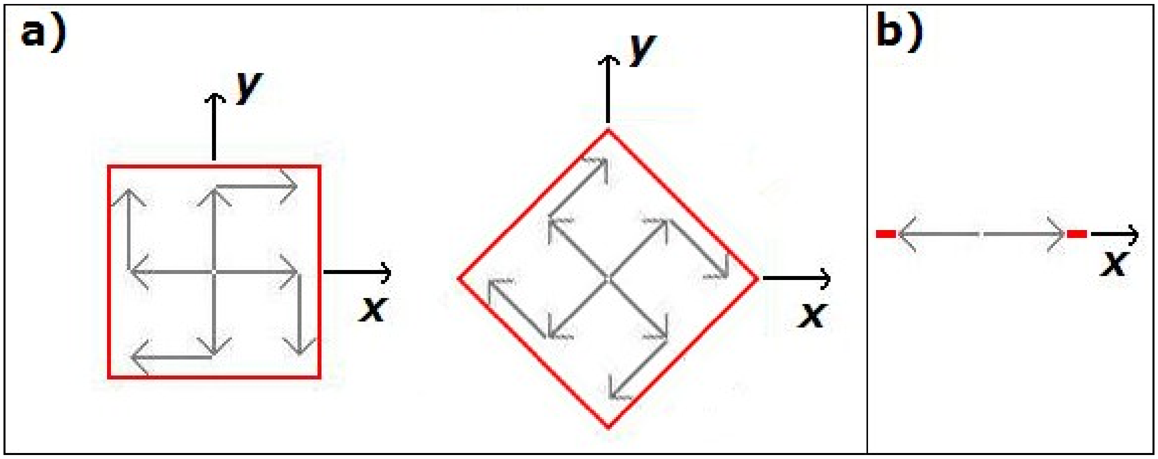

Fig. 2: a) The 2D swastika can be enveloped (red shape) in a hollow 2-square (2-cube) or 2-diamond (2-octahedron). b) The 1D swastika inside its 1D envelope, a 1-cube or 1-diamond, which are the same shape (similar to all 0D shapes being a point), just the endpoints of a hollowed out line remain.

In the following, Section 2 starts with the most general description of N-dimensional swastikas (N-swastika) consistent with the trivial 1D case (Fig. 2b). Difficulties with 2^N arms lead to the general requirement of $2N$ arms. Section 3 discusses more difficulties when trying to fix details of the general algorithm that would construct the desired shape in any number of dimensions. Necessary factors of negative one lead to over-complicated equations even in 2D. Section 4 presents the only general yet relatively simple, natural solution, clarifying the algorithm in 1D to 3D. Section 5 applies the general algorithm in order to present a 4D hyper-swastika for the first time.

2 Desired N Swastika Description from 1D, 2D, & 3D-Trial Cases

The usual 2D swastika in $N = 2$ dimensions has 4 arms, and each arm consists of two mutually orthogonal, equally long sections. It is a very "space-embracing/hugging" yet sturdy seeming shape. These characteristics should be preserved in higher dimensions. The 1D swastika (Fig. 2b), like all 1D shapes (e.g. 1-tetrahedra, 1-cubes, 1-octahedra), can only be a straight line. It can at most have two arms, and a single section for each of its two arms, because there is no orthogonal direction into which a second section on the same arm could reach into. Therefore, the basic requirements are:

Requirement 1: As many arms as there are dimensions (N)

Requirement 2: Each section is orthogonal to all others on the same arm.

The 3D swastika must therefore have arms with $N = 3$ sections, and either 6 or 8 arms, depending on whether the arms either always start with the first section along the main coordinate axes, or whether they instead point into the center of each octant. Generally speaking: Should the number of arms be 2^N, with their first sections pointing at the corners of a regular N-diamond (N-octahedron) standing on one corner, or $2N$ instead, with the symmetry of the corners of an N-square (N-cube)? Six-arm examples do not only look sturdier. Moreover, the directions that the second sections of each arm should proceed in are obvious; they must go along one of the main coordinate axes that are orthogonal to the first section of the arm, just like in the solution depicted in Fig. 1b.

With 2^N arms, their paths are not well determined. If the first sections of each arm point into the middle of each side of the N-diamond, the eight arms of the 3D shape will point into the eight faces of an octahedron. An octahedron has 12 edges and six corners. There is no natural way in which they guide the directions of the second sections of the eight arms, especially not so that the prescription naturally applies to the 2D case, let alone being suggested by it! In 2D, the numbers of sides and corners are the same, both four. In 2D, there is also only one axis of rotation possible at any point, which allows saying that the second sections of each arm result from a 90 degree rotation of the first section. In 3D, the rotation axis could point anywhere on a circle around each of the eight arms' first sections. We can instead consider that the eight arms' first sections point into the corners of a cube, but this does also not reveal an obvious generalization for the rotations of the second and third sections of each arm. The main reason is that the number of edges (12) is not a multiple of the number of arms, and the number of faces (6) is less than eight. Trial drafts of possible 8-armed solutions look fragile and do not trigger associating the well recognized, sturdy 2D root shape. Therefore, our last formal requirement is:

Requirement 3: N-swastika have $2N$ arms, double the number of the dimensions. This is equivalent to demanding that the first sections of the arms are orthogonal to all other neighboring ones, which makes for the characteristic sturdy look.

3 Difficulties Finding Construction Algorithms for Odd/Even N

The shape presented in Fig. 1b was originally obtained with an algorithm that does not reproduce the 2D swastika. The algorithm starts with the first axis, the positive x-, or $+x_1$ axis. After going one positive step on it $[(-1)^{N-1} = (-1)^2 = +1]$, it turns to draw along the next orthogonal axis, the x_2 axis, but into the negative direction $[(-1)^{N-2} = -1]$, and so on, alternating positive and negative steps. After the x_N axis, the next axis is x_1 again. This procedure constructs the desired shapes for odd N, but for even N, such as 2D and 4D, the arms intersect (Fig. 3a).

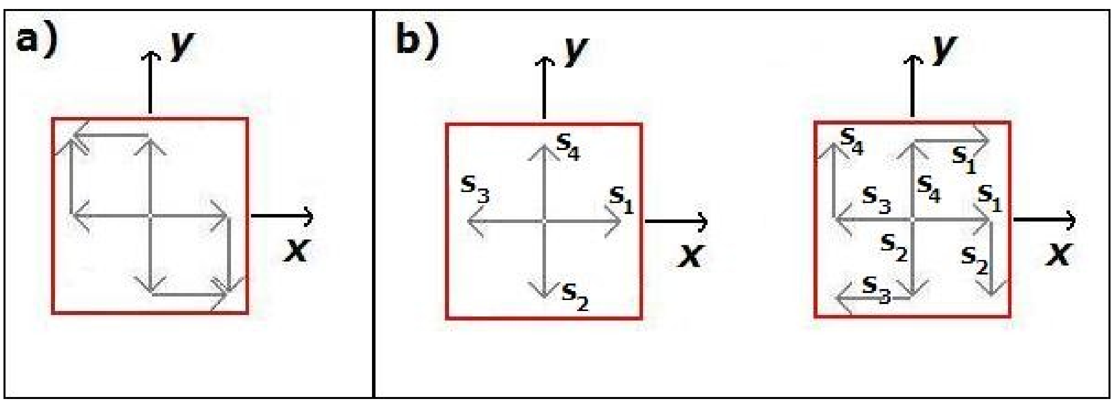

Fig. 3: a) Algorithms that step through all available dimensions for each arm work well for producing 3D solutions, but they do not even reproduce the 2D swastika; all arms meet in only two corners. b) The general algorithm first defines a new set of $2N$ vectors s_n and starts constructing all arms simultaneously with them, putting down all the first sections, then the second sections.

It is in general difficult to ensure that the arms avoid intersecting each other during construction and that in the final result each arm ideally extends far away from all others. When the algorithm during construction of an arm turns into an orthogonal direction along the next coordinate axis, a positive step is usually followed by a negative one. However, sometimes a positive step follows again, like with the upper arm of the 2D swastika, where a step increasing x follows after an already positive step along y. Straight forward algorithms for odd N lead to that with even N, all arms meet in either the corner with the coordinates (+1, –1, +1, …, –1) or the opposite corner. In 3D, there is really no alternative to the solution shown in Fig. 1b, but the original algorithm cannot produce the 2D version. We therefore try modifying that algorithm so that it keeps reproducing the same 3D shape. Changing equations for the signs leads to clumsy prescriptions where, for example, the algorithm takes into account whether numbers are odd or even. Equations with modulus values are more elegant, but such still feels too contrived; such is not naturally suggested by the 2D case.

4 The General Algorithm

A general as well as quite natural and easily remembered algorithm is suggested by observing that if we go around clockwise in the 2D swastika, the second section of each arm is precisely the first section of the previous arm, the same vector (Fig. 3b). This is also the case for the found 3D solution, if only we rotate it so that the arms are in the right sequence (Fig. 4).

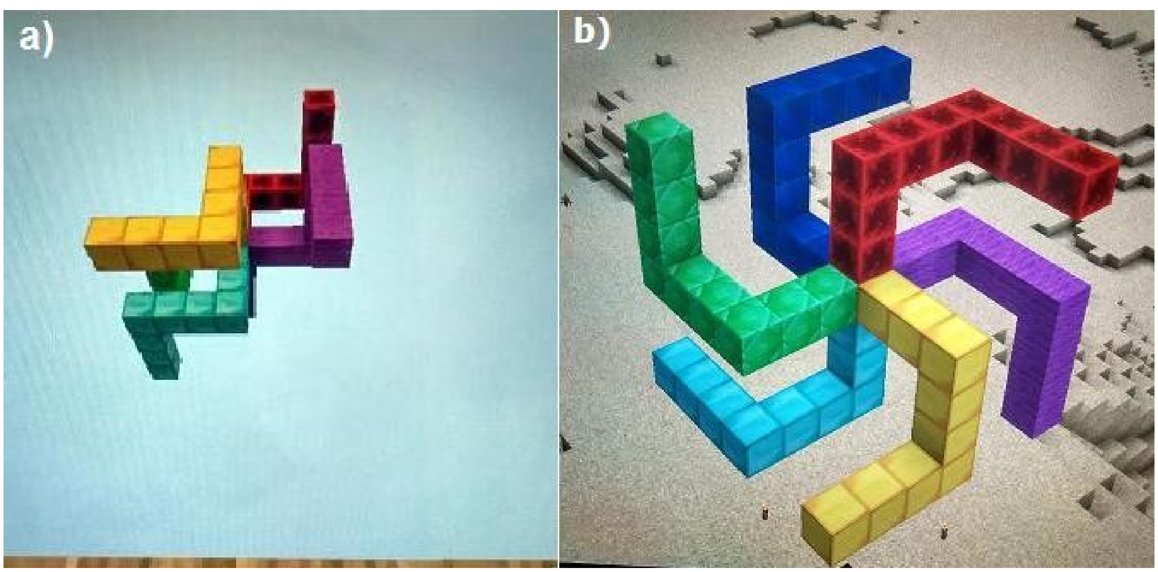

Fig. 4: a) The 3D swastika constructed on top of a mountain in a virtual reality seen when climbing the mountain coming from the north (positive y-axis), and b) for an observer hovering in the positive (1, 1, 1) octant, the positive x-axis being along the 1st section of the green arm. In this orientation, the 2nd section is along the y (= x_2) axis, the 3rd along x_3, and the 2nd arm (blue) starts with the 1st arms' 2nd section, and so on.

Hence, the general algorithm starts with constructing an ordered set of $2N$ basic arm sections s_n, the "*set of sections*", as follows: The first is the first basis vector, $s_1 = b_1$, for example $b_1 = (1, 0)$ in 2D. The second is the negative of the second basis vector, $s_2 = -b_2$, that is $s_2 = (0, -1)$ in 2D. We go on like this, alternating between adding the next basis vector positively or negatively to the set. When the basis vectors have all been used, we continue with the negative of b_1, i.e. $s_{N+1} = -b_1$, $s_{N+2} = b_2$, until s_{2N}. Differently put, we use up the basis (alternating signs) and then add as many vectors $s_{N+n} = -s_n$. These $2N$ vectors s_n are the first sections of the $2N$ arms. In 1D, the algorithm is trivial and already finished with just $s_1 = (1)$ and $s_2 = (-1)$. In 2D, the vectors shown in Fig. 3b (left) result clockwise in sequence, the set being $\{s_1 = b_1, s_2 = -b_2, s_3 = -b_1, s_4 = b_2\}$. The 3-swastika has the sections $s_1 = (1,0,0)$, $s_2 = (0,-1,0)$, $s_3 = (0,0,1)$, and $s_{4,5,6} = -s_{1,2,3}$.

The algorithm continues by attaching to each section s_n the next section s_{n+1}, counting cyclically, s_1 following s_{2N}. The second and later sections are all added like this. Each round starts at the first arm and attaches to the stump s_n the vector s_{n+1} as the next section. It then proceeds with the next arm, always drawing from the end of the arm as it was already painted in the previous round. After the N^{th} round, all arms each have N sections. Each arm A_n is a sequence of sections s_n, $s_{(n+1)}$, … . The 2D example renders the generalization obvious: $A_1 = s_{1\text{-}2}$, $A_2 = s_{2\text{-}3}$, $A_3 = s_{3\text{-}4}$, and $A_4 = s_{4\text{-}1}$. Since $s_3 = -s_1$ etc., A_3 and A_4 are $-A_1$ and $-A_2$, respectively. In 3D, the first three arms are $s_{1\text{-}2\text{-}3}$, $s_{2\text{-}3\text{-}4}$, and $s_{3\text{-}4\text{-}5}$, and the last three arms $A_{4,5,6}$ are all equal to $-A_{1,2,3}$, respectively.

Drawing the 3D and 4D shapes needs the coordinates of the sections' endpoints along the arms. The 2-swastika's first arm has coordinates $[s_1, s_1+s_2] = [(1,0), (1,-1)]$, the second $[s_2, s_2+s_3] = [(0,-1), (-1,-1)]$. The last N arms have always the negative of the first N arms' coordinates. With the 3D sections and arms as stated above, the coordinates of the first N arms are:

$$[s_1, s_1+s_2, s_1+s_2+s_3] = [(1, 0, 0), (1,-1, 0), (1,-1, 1)]$$
$$[s_2, s_2+s_3, s_2+s_3+s_4] = [(0,-1, 0), (0,-1, 1), (-1,-1, 1)]$$
$$[s_3, s_3+s_4, s_3+s_4+s_5] = [(0, 0, 1), (-1, 0, 1), (-1, 1, 1)]$$

Thinking of an observer “looking north” onto the coordinate origin, the front face of the enveloping cube is a large square that has all x_2 coordinates equal –1. We then draw the back face ($x_2 = 1$) as a smaller square into the larger one, slightly shifted into the outer square’s lower left corner. This avoids that the middle will become too cluttered. It seems as if we stand in that corner of the cube, gazing back and up (Fig. 5a).

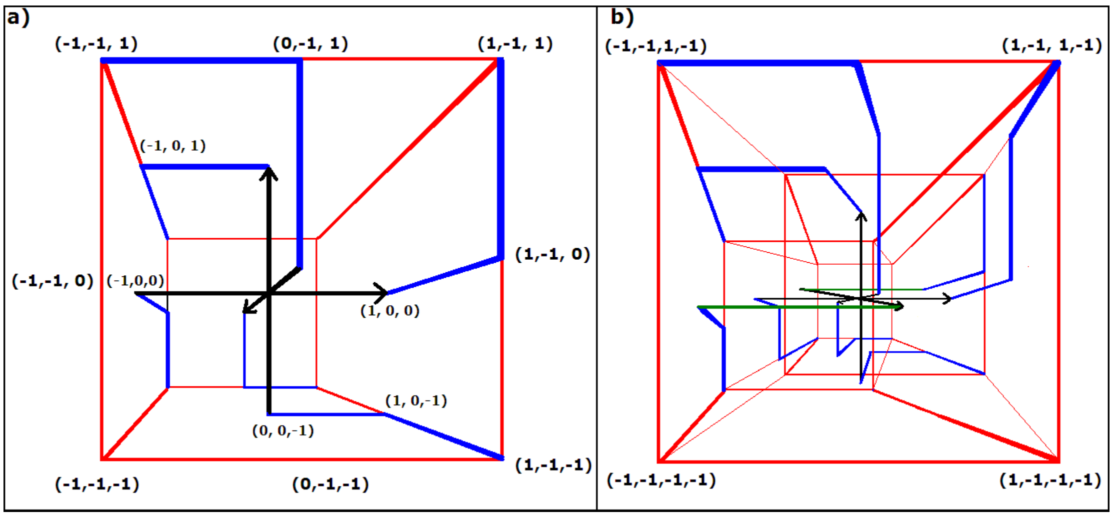

Fig. 5: a) The 3D swastika in its enveloping cube (red), which is drawn with a small square in a larger square for the front and back faces, then connecting corners before drawing arms (first sections black, later sections blue). b) The 4D hyper swastika in an equivalent construction, starting with the enveloping hypercube “faces” being a small cube inside a larger cube. For clarity, the 2nd sections of the 4th and 8^{th} arms are green.

5 The 4D Hyper Swastika (4-Swastika)

The 4D hyper swastika is now presented in almost precisely the same way. The sections are $s_1 = (1,0,0,0)$, $s_2 = (0,–1,0,0)$, $s_3 = (0,0,1,0)$, $s_4 = (0,0,0,–1)$, and $s_{5,6,7,8} = –s_{1,2,3,4}$. The arms are $A_1 = s_{1\text{-}2\text{-}3\text{-}4}$, $A_2 = s_{2\text{-}3\text{-}4\text{-}5}$, and so on. The coordinates therefore include those for the fourth section, such as $s_1 + s_2 + s_3 + s_4 = (1,–1, 1,–1)$. The first four arms’ coordinates are therefore (and the other *N* arms’ have again the negative values):

[(1, 0, 0, 0), (1,–1, 0, 0), (1,–1, 1, 0), (1,–1, 1,–1)]

[(0,–1, 0, 0), (0,–1, 1, 0), (0,–1, 1,–1), (–1,–1, 1,–1)]

[(0, 0, 1, 0), (0, 0, 1,–1), (–1, 0, 1,–1), (–1, 1, 1,–1)]

[(0, 0, 0,–1), (–1, 0, 0,–1), (–1, 1, 0,–1), (–1, 1,–1,–1)]

The 3D cube has six faces that are all squares (2-cubes). The *N*-hypercube has 2*N* “faces” that are all (*N*–1)-cubes. Therefore, the required enveloping 4D hypercube has an outer “skin” made from eight cubes. The cube with all x_4 coordinates being equal to –1 is, similar to the large square in Fig 5a, the “front face”, meaning the largest cube in Fig. 5b. The “back face” ($x_4 = 1$) is the smaller cube inside. The arms’ last sections always lead from a corner of the enveloping *N*-cube along an edge to the middle of that edge. Drawing the arms by hand according to the coordinates is therefore far easier when starting with each arm’s last section rather than with the first sections.

6 Concluding Remarks

Another possible generalization obtains the swastika starting from the enveloping shape, covering all its “faces” half, as is illustrated in Fig. 6a for a 2-cube envelope and in Fig. 6b for a 2-diamond. The possible shapes are now a rich field. Not only the last sections of the “arms” may be (*N* – 1) surfaces, but all sections can be also, or (*N* – 2) surfaces for instance. However, the obtainable results confirm the already obtained. For example, the 3-diamond (octahedron), although many beautiful 3D shapes can be developed in this way, does not offer an obvious solution for covering all its faces half (they are triangles). Generally speaking, considering surfaces does not help finding obvious hyper shapes from *N*-diamond envelopes. The cube envelope however has symmetrically added half-surfaces (blue) meeting each other at half-edges (shown in green) which coincide precisely with the 3D swastika’s arms’ last sections!

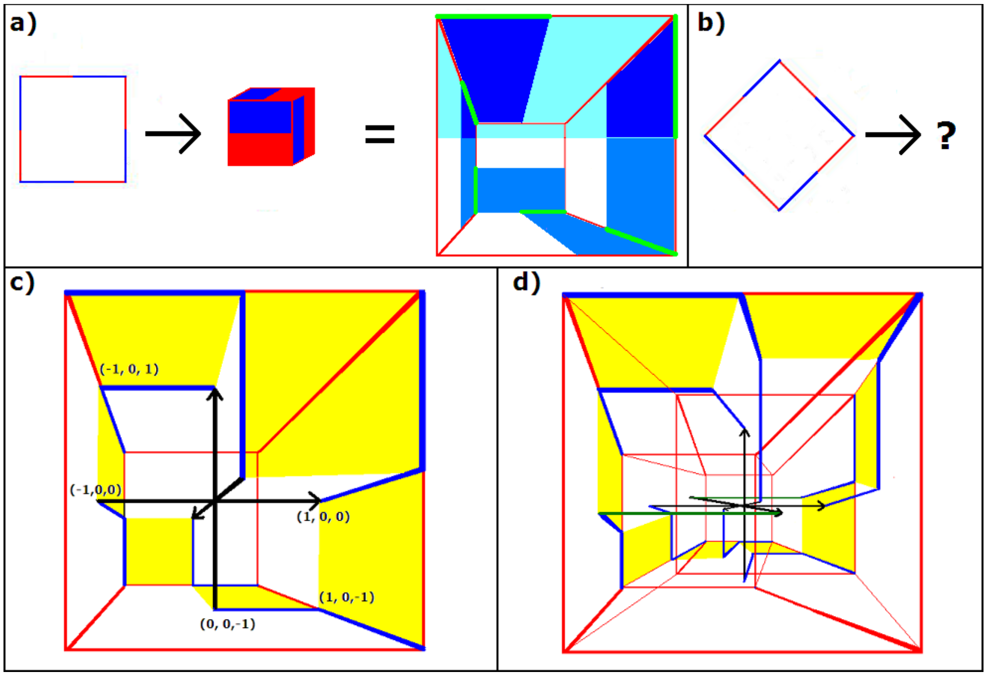

Fig. 6: The swastika as a general shape that starts with half-filling (blue) the faces of the envelope (red); a) the cube obtains the already presented geometry again; the lines where the added surfaces meet (green) reproduce the arms' last sections. b) The *N*-diamond does again not offer a unique solution. c,d) Added surfaces help understanding the 3D and 4D hyper geometry, for example providing a closed path around the very outside, although it looks as if partially being inside, especially in d).

Artists interested in "reclaiming symbols" or supporting didactics, developing visually intuitive representations of 4D hyper geometries, best take the *N*-cube-envelope approach; the considered surfaces still provide a very rich source of inspiration. For instance, Fig. 6c shows the halves of the added surfaces that are in between the arms' last sections and another arms second to last section. They line up in a closed path around the shape. They are also in the 4D shape (Fig. 6d), and of course two more than six. This helps understanding the shape, for example the closed path is still around the very outside, or where the eight half-filled cube-"faces" of that shape should be, which are 3D volumes of course, not just the yellow planes shown.

7 References

[1] Image Source: Public domain (downloaded Mai 2019)

[2] Sascha Vongehr, Science20 Science Column (now censored) (2016)