On Evaluation of Certain Gaussian-type Integrals

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Abstract

In this paper we evaluate certain Gaussian-type integrals via contour integration in the complex plane along with the application of Cauchy-Goursat theorem.

1. Introduction

We consider $\lim_{R\to\infty} \int_0^R \exp(-(a+i\lambda)x^2)\,dx$ where a and λ are real numbers such that $a > 0$ and $\lambda \neq 0$.

2. Results

Let f be a complex-valued function in the complex plane defined by $f(z) = \exp(-\gamma z^2)$ such that $\gamma = \sqrt{a^2+\lambda^2}$.

Case 1. $a > 0, \lambda > 0$.

$\Gamma_1 : z = x;$ $\quad\Gamma_2 : z = R + \mu x + i\nu x;$ $\quad\Gamma_3 : z = x + \mu R + i\nu R;$ $\quad\Gamma_4 : z = \mu x + i\nu x;$ $\quad(0 \le x \le R)$

such that $\mu = \sqrt{\dfrac{\gamma+a}{2\gamma}},\ \nu = \sqrt{\dfrac{\gamma-a}{2\gamma}}$.

By Cauchy-Goursat theorem,

$$\int_{\Gamma_1} \exp(-\gamma z^2)\,dz + \int_{\Gamma_2} \exp(-\gamma z^2)\,dz - \int_{\Gamma_3} \exp(-\gamma z^2)\,dz - \int_{\Gamma_4} \exp(-\gamma z^2)\,dz = 0.$$

Hence

$$\int_0^R \exp(-(a+i\lambda)x^2)\,dx = (\mu-i\nu)\Big(\int_0^R \exp(-\gamma x^2)\,dx + \int_0^R \exp(-\gamma(R+\mu x+i\nu x)^2)(\mu+i\nu)\,dx - \int_0^R \exp(-\gamma(x+\mu R+i\nu R)^2)\,dx\Big).$$

$$\begin{aligned}\left|\int_0^R \exp(-\gamma(R+\mu x+i\nu x)^2)(\mu+i\nu)\,dx\right| &\le \int_0^R \left|\exp(-\gamma(R+\mu x+i\nu x)^2)(\mu+i\nu)\right| dx\\ &= \int_0^R \exp(-\gamma((R+\mu x)^2-\nu^2x^2))\,dx\\ &= \int_0^R \exp\Big(-\gamma\Big(R^2+2R\mu x+\frac{a}{\gamma}x^2\Big)\Big)\,dx\\ &\le \int_0^R \exp(-\gamma R^2)\,dx\\ &= \frac{R}{\exp(\gamma R^2)}.\end{aligned}$$

$$\begin{aligned}\left|\int_0^R \exp(-\gamma(x+\mu R+i\nu R)^2)\,dx\right| &\le \int_0^R \left|\exp(-\gamma(x+\mu R+i\nu R)^2)\right| dx\\ &= \int_0^R \exp(-\gamma(x^2+2x\mu R+\mu^2R^2-\nu^2R^2))\,dx\\ &\le \int_0^R \exp(-\gamma(\mu^2R^2-\nu^2R^2))\,dx\\ &= \int_0^R \exp(-aR^2)\,dx\\ &= \frac{R}{\exp(aR^2)}.\end{aligned}$$

Since

$$\lim_{R\to\infty}\frac{R}{\exp(\gamma R^2)}=0 \quad\text{and}\quad \lim_{R\to\infty}\frac{R}{\exp(aR^2)}=0,$$

by squeeze theorem

$$\lim_{R\to\infty}\left|\int_0^R \exp(-\gamma(R+\mu x+i\nu x)^2)(\mu+i\nu)\,dx\right|=0 \quad\text{and}\quad \lim_{R\to\infty}\left|\int_0^R \exp(-\gamma(x+\mu R+i\nu R)^2)\,dx\right|=0.$$

Thus

$$\lim_{R\to\infty}\int_0^R \exp(-\gamma(R+\mu x+i\nu x)^2)(\mu+i\nu)\,dx=0 \quad\text{and}\quad \lim_{R\to\infty}\int_0^R \exp(-\gamma(x+\mu R+i\nu R)^2)\,dx=0.$$

It follows that

$$\lim_{R\to\infty}\left((\mu-i\nu)\left(\int_0^R \exp(-\gamma x^2)\,dx+\int_0^R \exp(-\gamma(R+\mu x+i\nu x)^2)(\mu+i\nu)\,dx-\int_0^R \exp(-\gamma(x+\mu R+i\nu R)^2)\,dx\right)\right)=(\mu-i\nu)\sqrt{\frac{\pi}{4\gamma}}.$$

To conclude

$$\lim_{R\to\infty}\int_0^R \exp(-(a+i\lambda)x^2)\,dx=(\mu-i\nu)\sqrt{\frac{\pi}{4\gamma}}.$$

Case 2. $a>0, \lambda<0$.

$\Gamma_1: z=x;\quad \Gamma_2: z=R+\mu x-i\nu x;\quad \Gamma_3: z=x+\mu R-i\nu R;\quad \Gamma_4: z=\mu x-i\nu x;\quad (0\le x\le R)$

such that $\mu=\sqrt{\frac{\gamma+a}{2\gamma}},\ \nu=\sqrt{\frac{\gamma-a}{2\gamma}}$.

By the same argument as in case 1, it follows that

$$\lim_{R\to\infty}\int_0^R \exp(-(a+i\lambda)x^2)\,dx=(\mu+i\nu)\sqrt{\frac{\pi}{4\gamma}}.$$

3. References

[1] R. V. Churchill, and J. W. Brown, *Complex Variables and Applications*, 5th ed.,McGraw-Hill Book Company, Inc., New York, 1990.