# Weights at the Gym and the Irrationality of $\zeta(2)$

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## A Gym Story

Free weights at the gym are circular and designate real numbers. They are stackable in descending order making nice pyramidal shapes. This makes them ideal for studying whether the sum of fractions can be expressed as a multiple of one of the terms, one of the fractions used. Coins can be made the same way and are more familiar. So, for example, can five dimes (1/10 of a dollar) and a quarter (1/4 of a dollar) be expressed with dimes? No. With quarters? Yes. Weights are all of the same thickness and their weights, more than their value, is the key. Once again we can ask if the sum of say a 25, a 20, a 15, and a 10 pound weight is expressible as a multiple of one of the weights used? Yes $7 \times 10 = 70$, the weight of the stack. We can regard these weights as fractions of 100: 1/4, 1/5, 3/20, and 1/10, respectively. With weights we can imagine a scale being used to determine whether one set equals another.

We will manufacturer weights with certain properties and pose a question. We will make weights that are circular and of proportional radii; that is all weights will have the same thickness, and have fractional values of a unit of the form $1/k^2$, where $k$ is a natural number and $k \geq 2$. We will assume that it is known that sums of the form

$$s_2 = \sum_{j=2}^{n} \frac{1}{j^2}$$

are not expressible as multiples of any of their terms [1]. Is the infinite sum

$$\zeta(2) - 1 = z_2 = \sum_{j=2}^{\infty} \frac{1}{j^2}$$

also not expressible as a multiple of any of its terms? If this is true then $z_2$ is, as we shall see, an irrational number.

To make the image more concrete, the reader should have the picture of stacked descending weights (on the left) on a balance with a counter stack (on the right) consisting of a multiple of one of the weights used. They don't balance. But we could make them balance by having too much and taking a wedge out of say the top weight on the right. We could also drill holes in the center of it it to make it keep its ability to be the foundation of a stack. Let's settle on two radii drawn on the weight: one is at 0 degrees and the other at some other degree greater than 0 and less than 360; the two lines define a sector with the desired weight; we will refer to the non-zero radius as the radial. To be clear call the top weight the one with the radial on it and all the ones below it call them the bottom weights. The bottom weights are clearly balanced by themselves; even in combination they will sum to a multiple of some single weight, so the top weights are key.

We stack up more and more weights, that is we add more and more terms and we also accumulate more and more top weights: portions of 1/4, 1/9, 1/16, etc.. that give the remainder weights for each partial. These partial weights get smaller and smaller, both because they have a wedge taken out of each and per our assembling process – we are adding smaller and smaller weights. As each weight is added, we can discard bottom weights and update the radials on top weights for all previous larger weights. We always have a stack of top weights for all the terms in our partial. The bottom stack for each eventually doesn't require updating – otherwise we would have weights greater than 1 and $z_2 < 1$. Does the radial line on the top weights converge to a rational weight? The answer to that is the question?

# The Math

Rudin's *Principles of Mathematical Analysis* has the following problem, problem 21, on page 82 [2]: If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space $X$, if $E_n \supset E_{n+1}$, and if

$$\lim_{n \to \infty} \text{diam } E_n = 0,$$

then

$$\bigcap_1^\infty E_n$$

consists of exactly one point.

Our intervals are $E_n = [\frac{x}{n^2}, \frac{x+1}{n^2}]$, a closed interval, where $x$ is the number of bottom weights and the top weight yields a value inside this interval. We know such an $x$ exists because rational numbers with denominators $k^2$ divide the unit into a finite number of sectors. The radial always defines an area (or weight) between the end points. Also, as the weights are getting lighter, $E_{n+1} \subset E_n$; and the diameters, literally with our weights, tend to 0. Applying the problem in Rudin, there is a convergence weight. Think of a grand weight, say 100 pounds, that has a radial on it as the convergence radial. A convergence point in the intersection will have to be between all $\frac{x}{n^2}$ and $\frac{x+1}{n^2}$. Say the convergence point is the rational $\frac{p}{q}$, then, as

$$\frac{pq}{q^2} = \frac{p}{q}$$

it is either an endpoint of some $E_n$ or residing in a bottom stack. In either case, $\frac{p}{q}$ is impossible.

## Why weights?

Intervals on the real line can overlap. It is intuitively a struggle to see this doesn't change the argument. An illustration: have $\frac{1}{4}$ lines and others one below the other. The claim is you can slide these back and forth to keep a line going through all intervals always between all endpoints. This is a puzzle to comprehend with intervals, but with concrete weights, it is easier to comprehend. When you weigh the weights, regardless of their position, they will always be between such and such and such and such weight – not at the endpoints. You can also jockey the weights around so that they reside one inside the other: 45's on top of 33's aligned at the centers. It is intuitively clearer that positions don't matter.

## References


[1] T.W. Jones, A Simple Proof that $\zeta(n)$ is Irrational (2017), available at `http://vixra.org/abs/1801.0140`.

[2] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976.