Polar Hypercomplex Integers

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Abstract. We introduce a special class of complex numbers, wherein their absolute values and arguments given in a polar coordinate system are integers, which when considered within the complex plane, constitute Unicentered Radial Lattice and similarly for quaternions.

Keywords: complex plane; integer lattice; polar coordinate system; quaternion

1. Introduction

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{ a + b\mathbf{i} \mid a, b \in \mathbf{Z} \}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2-dimensional integer lattice. The Gaussian integers form unique factorization domain: it is irreducible if and only if it is a prime(Gaussian primes). The field of Gaussian rationals consists of the complex numbers whose real and imaginary part are both rational(see, e.g., [3]).

The norm of a Gaussian integer is its product with its conjugate:

$N(a + b\mathbf{i}) = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2$.

The norm is multiplicative, that is, one has:

$N(zw) = N(z)N(w), \; z, w \in \mathbf{Z}[\mathbf{i}]$.

The following is unsolved problem regarding Gaussian Integers: if you are allowed only steps of bounded size, is it possible to walk to ∞ stepping only on Gaussian primes?

Another well-known integral subclass of complex numbers are Eisenshtein integers: complex numbers of the form: $z = a + b\omega$, where a and b are integers and $\omega^2 + \omega + 1 = 0$. The Eisenshtein integers form a triangular lattice in the complex plane, in contrast with Gaussian integers, which form a square lattice in the complex plane. The Eisenstein integers form a commutative ring as well and similar to Gaussian integers form a Euclidean domain, which supposes unique factorization of Eisenshtein integers into Eisenshtein primes.

Similar integral subclasses can be defined for quaternions: Lipschitz and Hurwitz Integers(quaternions).

Quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers(see, e.g., [1], [2]).

The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \, \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$.

This norm makes it possible to define the distance $d(p, q) = \|p - q\|$, which makes **H** into a metric space.

Lipschitz Integer(quaternion) is defined as:

$$L := \{ q:\ q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a \in \mathbf{Z},\ b \in \mathbf{Z}, c \in \mathbf{Z},\ d \in \mathbf{Z} \}.$$

Lipschitz Integer(quaternion) is a quaternion, whose components are all integers.

Hurwitz Integer(quaternion) is defined as:

$$H := \{ q:\ q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{Z} + 1/2 \}.$$

Thus, Hurwitz Integer(quaternion) is a quaternion, whose components are either all integers or all half-integers.

2. Polar Complex Integers

Let us introduce a new subclass of complex numbers and a new approach for their definition accordingly: Polar Complex Integers.

Its well-known for a complex number $z = \mathrm{Re}(z) + \mathrm{Im}(z)\mathbf{i} = a + \mathbf{i}b$, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $\mathbf{i}^2 = -1$, to use an alternative option for coordinates in the complex plane: polar coordinate system that uses the distant of the point z from the origin and the angle, subtended between the positive real axis and the line segment in a counterclockwise sense(see, e.g., [4], [5]).

The absolute value of the complex number: $r = |z|$ is the distance to the origin of the point, representing the complex number z in the complex plane.

The argument of z: φ, is the angle of the radius with the positive real axis. Note that there are two notations of angle φ: in degree and in radian.

Together, r and φ gives another way of representing complex numbers, the polar form. Recovering the original rectangular co-ordinates from the polar form is done by the formula called trigonometric form:

$$z = r(\cos \varphi + \mathbf{i} \sin \varphi).$$

Recall that addition of two complex numbers can be done geometrically by constructing the corresponding parallelogram.

Given two complex numbers:

$z_1 = r_1 (\cos \varphi_1 + \mathbf{i} \sin \varphi_1)$ and $z_2 = r_2 (\cos \varphi_2 + \mathbf{i} \sin \varphi_2)$, multiplication of z_1 and z_2 in polar form is given by:

$$z_1 z_2 = r_1 r_2 (\cos (\varphi_1 + \varphi_2) + \mathbf{i} \sin (\varphi_1 + \varphi_2)).$$

Similarly, division is given by:

$$z_1 / z_2 = = r_1 / r_2 (\cos (\varphi_1 - \varphi_2) + \mathbf{i} \sin (\varphi_1 - \varphi_2)).$$

Using polar form, let us introduce the following new subclass of complex numbers, Polar Complex Integers:

$$\mathbf{P} := \{z:\ z = r(\cos\varphi + \mathbf{i}\sin\varphi) \mid z \in \mathbf{C},\ r \in \mathbf{Z}, \varphi \in \mathbf{Z},$$
$$-180^\circ < \varphi \le 180^\circ\}.$$

Theorem 1. *Polar Complex Integers are closed under multiplication.*

Proof. It follows from the formula:

$$z_1 z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + \mathbf{i}\sin(\varphi_1 + \varphi_2)). \qquad \square$$

Theorem 2. *Polar Complex Integers are not closed under addition*.

Proof. Let us consider $z_1 = 0 + 1\mathbf{i}$ and $z_2 = 1 + 0\mathbf{i}$.

For degree notation, where $z_1 = 1(\cos 90^\circ + \mathbf{i}\sin 90^\circ)$ and $z_2 = 1(\cos 0^\circ + \mathbf{i}\sin 0^\circ)$, absolute value of $z_1 + z_2$ is an irrational number. $\square$

Theorem 3. *Polar Complex Integers are not closed under division.*

Proof. It follows from the formula:

$$z_1 / z_2 = = r_1 / r_2 (\cos(\varphi_1 - \varphi_2) + \mathbf{i}\sin(\varphi_1 - \varphi_2)). \qquad \square$$

Corollary 1. *Polar Complex Integers are mutually primes if and only if their absolute values are mutually primes*.

Theorem 4. *Polar Complex Integers form countable infinite set.*

Proof. It follows from the definition. $\square$

Similarly to aforementioned Hurwitz integers, let us introduce Polar Complex Hurwitz-like Integers:

$$\mathbf{PH} := \{z:\ z = r(\cos\varphi + \mathbf{i}\sin\varphi) \mid z \in \mathbf{C},\ r \in \mathbf{Z} + 1/2,\ \varphi \in \mathbf{Z} + 1/2,$$
$$-180^\circ < \varphi \le 180^\circ\},$$

and similarly to aforementioned Gaussian Rationals, the corresponding set of Polar Complex Rationals can be introduced as well.

Theorem 5. *Polar Complex Hurwitz-like Integers form countable infinite set.*

Proof. It follows from the definition. □

3. Unicentered Radial Lattices of Polar Complex Integers and Polar Complex Hurwitz-like Integers

As we mentioned above, when considered within the complex plane, the Gaussian integers constitute the 2-dimensional integer lattice and the Eisenshtein integers form a triangular lattice in the complex plane, in contrast with Gaussian integers, which form a square lattice in the complex plane.

As it follows from the definition:

$$\mathbf{P} := \{z:\ z = r(\cos\varphi + \mathbf{i}\sin\varphi) \mid z \in \mathbf{C},\ r \in \mathbf{Z}, \varphi \in \mathbf{Z},$$

$$-180^{\circ} < \varphi \leq 180^{\circ}\},$$

by fixing the integer radius $r \in \mathbf{Z}$, Polar Complex Integers, when considered within the complex plane, constitute Unicentered Radial Lattice.

Accordingly, for the Polar Complex Hurwitz-like Integers, as it follows from the definition :

$$\mathbf{PH} := \{z:\ z = r(\cos\varphi + \mathbf{i}\sin\varphi) \mid z \in \mathbf{C},\ r \in \mathbf{Z} + 1/2\ , \varphi \in \mathbf{Z} + 1/2,$$

$$-180^{\circ} < \varphi \leq 180^{\circ}\},$$

by fixing the integer radius $r \in \mathbf{Z}$, Polar Complex Hurwitz-like Integers, when considered within the complex plane, constitute Unicentered Radial Lattice as well.

4. Polar Quaternionic Integers

Similarly, we can introduce Polar Quaternionic Integers.

Indeed, its well known to represent quaternions as pairs of complex numbers: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \leftrightarrow (a + b\mathbf{i},\ c + d\mathbf{i})$ (Cayley-Dickson construction).

Correspondingly, considering each of two parts in polar form:

$$a + b\mathbf{i} = r(\cos \varphi + \mathbf{i} \sin \varphi), \quad c + d\mathbf{i} = \rho(\cos \phi + \mathbf{i} \sin \phi),$$

let us introduce Polar Quaternionic Integers:

$$\begin{aligned}\mathbf{PQ} := \{\, q:\ & q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \leftrightarrow (a + b\mathbf{i},\ c + d\mathbf{i}),\\ & a + b\mathbf{i} = r(\cos \varphi + \mathbf{i} \sin \varphi),\ c + d\mathbf{i} = \rho(\cos \phi + \mathbf{i} \sin \phi) \mid\\ & q \in \mathbf{H},\ r \in \mathbf{Z}, \varphi \in \mathbf{Z},\ \rho \in \mathbf{Z},\ \phi \in \mathbf{Z},\\ & -180^\circ < \varphi \le 180^\circ,\ -180^\circ < \phi \le 180 \,\},\end{aligned}$$

and Polar Quaternionic Hurwitz-like Integers:

$$\begin{aligned}\mathbf{PQH} := \{\, q:\ & q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \leftrightarrow (a + b\mathbf{i},\ c + d\mathbf{i}),\\ & a + b\mathbf{i} = r(\cos \varphi + \mathbf{i} \sin \varphi),\ c + d\mathbf{i} = \rho(\cos \phi + \mathbf{i} \sin \phi) \mid\\ & q \in \mathbf{H},\ r \in \mathbf{Z} + 1/2, \varphi \in \mathbf{Z} + 1/2,\ \rho \in \mathbf{Z} + 1/2,\ \phi \in \mathbf{Z} + 1/2,\\ & -180^\circ < \varphi \le 180^\circ,\ -180^\circ < \phi \le 180 \,\},\end{aligned}$$

and similarly to aforementioned Gaussian Rationals, the corresponding set of Polar Quaternion Rationals can be introduced as well.

5. Conclusions

We unveiled a special class of complex numbers, wherein their absolute values and arguments, given in a polar coordinate system are integers, which when considered within the complex plane, constitute Unicentered Radial Lattice and similarly for quaternions.

References

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