The Covariant Derivative Operator and Scalars

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Abstract

The writing delineates some peculiar aspects of the covariant operator. It appears that the metric coefficients have to disappear.

Introduction

The successive operations of two covariant derivative [1]operators on a tensor is usually non commutative. But there are many intricate issues involved in the issue. It seems that the metric coefficients have to vanish.

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Calculations

We consider the covariant derivative operator.For scalars in a torsion free field

$$\nabla_i \nabla_j f = \nabla_j \nabla_i f \quad (1)$$

Since $g^{\alpha\beta} P_\alpha Q_\beta$is a scalar we have

$$\nabla_i \nabla_j \left(g^{\alpha\beta} P_\alpha Q_\beta \right) = \nabla_j \nabla_i \left(g^{\alpha\beta} P_\alpha Q_\beta \right) \quad (2)$$

Since $\nabla_i g^{\alpha\beta} = 0$

$$g^{\alpha\beta} \nabla_i \nabla_j \left(P_\alpha Q_\beta \right) = g^{\alpha\beta} \nabla_j \nabla_i \left(P_\alpha Q_\beta \right)$$

$$\Rightarrow g^{\alpha\beta} \nabla_i \nabla_j \left(P_\alpha Q_\beta \right) - g^{\alpha\beta} \nabla_j \nabla_i \left(P_\alpha Q_\beta \right) = 0$$

$$\Rightarrow g^{\alpha\beta} \left[\nabla_i \nabla_j - \nabla_j \nabla_i \right] \left(P_\alpha Q_\beta \right) = 0 \; for\ arbitrary P_\alpha\ and\ Q_\beta \; (3)$$

We have,

$$\nabla_{\mathrm{i}} \nabla_j \left(P_\alpha Q_\beta \right) = \nabla_{\mathrm{i}} \left(P_\alpha \nabla_j Q_\beta + Q_\beta \nabla_j P_\alpha \right)$$

$$\nabla_{\mathrm{i}} \nabla_j \left(P_\alpha Q_\beta \right) = P_\alpha \nabla_{\mathrm{i}} \nabla_j Q_\beta + \left(\nabla_{\mathrm{i}} P_\alpha \right) \left(\nabla_j Q_\beta \right) + \left(\nabla_i Q_\beta \right) \left(\nabla_{\mathrm{j}} P_\alpha \right) + Q_\beta \nabla_{\mathrm{i}} \nabla_j P_\alpha \; (4)$$

$$\nabla_{\mathrm{j}}\nabla_i(P_\alpha Q_\beta) = \nabla_{\mathrm{j}}(P_\alpha \nabla_i Q_\beta + Q_\beta \nabla_i P_\alpha)$$

$$\nabla_{\mathrm{j}}\nabla_i(P_\alpha Q_\beta) = P_\alpha \nabla_{\mathrm{j}}\nabla_i Q_\beta + (\nabla_{\mathrm{j}} P_\alpha)(\nabla_i Q_\beta) + (\nabla_j Q_\beta)(\nabla_{\mathrm{i}} P_\alpha) + Q_\beta \nabla_{\mathrm{j}}\nabla_i P_\alpha \quad (5)$$

From (4) and (5)

$$[\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i](P_\alpha Q_\beta) = P_\alpha(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i)Q_\beta + Q_\beta(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i)P_\alpha (6)$$

Using (3)

$$g^{\alpha\beta}[\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i](P_\alpha Q_\beta) = g^{\alpha\beta} P_\alpha(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i)Q_\beta + g^{\alpha\beta} Q_\beta(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i)P_\alpha = 0 \quad (7)$$

Next we apply the following formula[2] on (7)

$$[\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i]A_p = R^n{}_{pij} A_n (8)$$

$$g^{\alpha\beta} P_\alpha R^n{}_{\beta ij} Q_n + g^{\alpha\beta} Q_\beta R^n{}_{\alpha ij} P_n = 0 \; for\ arbitray P_\alpha, Q_\beta \; (9)$$

$$g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + [g^{\alpha\beta} P_\alpha R^k{}_{\beta ij} Q_k]_{k=1,2,3} + g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n + [g^{\alpha k} Q_k R^n{}_{\alpha ij} P_n]_{k=1,2,3} = 0 \quad (10)$$

Since P_α and Q_β are arbitrary we make Q_0 five times its previous value keeping the other components unchanged. With a four vector like the velocity four vector it is not possible to change one component keeping the others constant, the norm being not only an invariant but also a constant. We have to consider such four vectors for which the norm is not a constant but just an invariant.

$$5g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + [g^{\alpha\beta} P_\alpha R^k{}_{\beta ij} Q_k]_{k=1,2,3} + 5g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n + [g^{\alpha k} Q_k R^n{}_{\alpha ij} P_n]_{k=1,2,3} = 0 \quad (11)$$

Subtracting (10) from (11)

$$g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n = 0$$

$$g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} + g^{\alpha 0} R^n{}_{\alpha ij} P_n = 0 \quad (12)$$

We expand (12) to write

$$g^{0\beta} P_0 R^0{}_{\beta ij} + g^{k\beta} P_\alpha R^0{}_{kij} + g^{\alpha 0} R^0{}_{\alpha ij} P_0 + g^{\alpha 0} R^k{}_{\alpha ij} P_k = 0 \; (13)$$

Setting

$$P_0 \to 500 P_0$$

$$500 g^{0\beta} P_0 R^0{}_{\beta ij} + g^{k\beta} P_\alpha R^0{}_{kij} + 500 g^{\alpha 0} R^0{}_{\alpha ij} P_0 + g^{\alpha 0} R^k{}_{\alpha ij} P_k = 0 (14)$$

Taking the difference between (14) and (13) we obtain

$$499 g^{0\beta} P_0 R^0{}_{\beta ij} + 499 g^{\alpha 0} R^0{}_{\alpha ij} P_0 = 0$$

$$g^{0\beta} R^0{}_{\beta ij} + g^{\alpha 0} R^0{}_{\alpha ij} = 0 \ (15)$$

We could make similar type of adjustments with other components sometimes changing all of them simultaneously in different proportions.

The only solution would be to have$g^{\alpha\beta} = 0$

A closer Look at a Formula

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta} A^{\alpha\beta}\right) = A_{\alpha\beta} \nabla_\gamma B^{\alpha\beta} + B^{\alpha\beta} \nabla_\gamma A_{\alpha\beta} \ \ (16)$$

Proof:

We consider the following relations

$$\nabla_\gamma A^{\alpha\beta} = A^{\alpha\beta}{}_{;\gamma} = \frac{\partial A^{\alpha\beta}}{\partial x^\gamma} + \Gamma_{\gamma s}{}^{\alpha} A^{s\beta} + \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}$$

$$\nabla_\gamma B_{\alpha\beta} = B_{\alpha\beta;\gamma} = \frac{\partial B_{\alpha\beta}}{\partial x^\gamma} + \Gamma^s{}_{\gamma\alpha} B_{s\beta} + \Gamma^s{}_{\gamma\beta} B_{\alpha s}$$

[The above relations do not assume $A^{\alpha\beta}$ and $B_{\alpha\beta}$ as symmetric tensors]

We obtain,

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta} A^{\alpha\beta}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha} A^{s\beta} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^s{}_{\gamma\alpha} B_{s\beta} + \Gamma^s{}_{\gamma\beta} B_{\alpha s}\right)$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta} A^{\alpha\beta}\right) = B_{\alpha\beta}\left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}\right) + A^{\alpha\beta}\left(\Gamma^s{}_{\gamma\alpha} B_{s\beta} + \Gamma^s{}_{\gamma\beta} B_{\alpha s}\right) + A^{\alpha\beta} \nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta} \nabla_\gamma A_{\alpha\beta}$$

$$= -\Gamma_{\gamma s}{}^{\alpha} g^{s\beta} B_{\alpha\beta} - \Gamma_{\gamma s}{}^{\beta} g^{\alpha s} B_{\alpha\beta} + \Gamma^s{}_{\gamma\alpha} A^{\alpha\beta} T_{s\beta} + \Gamma^s{}_{\gamma\beta} A^{\alpha\beta} B_{s\alpha} + A^{\alpha\beta} \nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta} \nabla_\gamma A_{\alpha\beta}$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta} A^{\alpha\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} B_{\alpha\beta} + \Gamma^s{}_{\gamma\alpha} A^{\alpha\beta} B_{s\beta}\right) + \left(\Gamma^s{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s} B_{\alpha\beta}\right) + A^{\alpha\beta} \nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta} \nabla_\gamma A_{\alpha\beta} \ \ (17)$$

[In the above α, s, β are dummy indices]

We work out the two parentheses separately.

With the second term in the first parenthesis to the right we interchange as follows

$$\alpha \leftrightarrow s$$

$$\left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} B_{\alpha\beta} + \Gamma^s{}_{\gamma\alpha} A^{\alpha\beta} B_{s\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} T_{\alpha\beta} + \Gamma^{\alpha}{}_{\gamma s} A^{s\beta} B_{\alpha\beta}\right) = 0$$

We do not have to worry about reflections on the left side of (5)because alpha and beta on the left side also disappear on contraction.

Indeed recalling (17) and using the relation:$B_{\alpha\beta}A^{\alpha\beta} = B_{\mu\nu}A^{\mu\nu}$ we may rewrite it [equation (2)] in the following form :

$$\frac{\partial}{\partial x^\gamma}\left(B_{\mu\nu}A^{\mu\nu}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha}A^{s\beta} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha}B_{s\beta} + \Gamma^{s}{}_{\gamma\beta}B_{\alpha s}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

There is no α, β on the left side of the above.

With the second term in the second parenthesis

$$\beta \leftrightarrow s$$

$$\left(\Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}B_{\alpha\beta}\right) = \left(\Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{\alpha s} - \Gamma_{\gamma\beta}{}^{s}A^{\alpha\beta}B_{\alpha s}\right) = 0$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

Formula (16) is often used implicitly in mainstream literature[3]

Considering (3.1.20) from[3]

$$t^a\nabla_a\left(g_{bc}u^b w^c\right) = 0$$

Analyzing the above relation

$$\nabla_a\left(g_{bc}u^b w^c\right) \equiv \frac{\partial}{\partial x^a}\left(g_{bc}u^b w^c\right)$$

We have using (16)

$$\nabla_a\left(g_{bc}u^b w^c\right) = g_{bc}\nabla_a\left(u^b w^c\right) + u^b w^c\nabla_a g_{bc}$$

etc. etc.

We may prove from separate premises: $\nabla_a g_{bc} = 0$ and then set out to show that dot product is preserved for parallel transport that is we may prove

$$\frac{\partial}{\partial \mathrm{x}^{\mathrm{a}}}\left(g_{bc}u^b w^c\right) = \nabla_a\left(g_{bc}u^b w^c\right) = 0$$

In such an endeavor we have to use relation (16)

Scalars and Arbitrary Transformations

We consider a scalar χ across various manifolds corresponding to all possible transformations(non singular transformations). On a given manifold labeled manifold one we consider two points P and Q with distinct values of the scalar:$\chi(P) \neq \chi(Q)$. We consider a non linear transformation both P and Q are mapped to point S on manifold two.

Due to invariance $\chi(P) = \chi'(S); \chi(Q) = \chi'(S) \Rightarrow \chi(P) = \chi(Q)$. But this stands in contradiction to what we had assumed earlier: $\chi(P) \neq \chi(Q)$. To avoid the contradiction we have to disallow such non linear transformations as are many to one. To accommodate non linear transformations in a consistent manner we have to consider scalars that are constant on the same manifold so that we may not have instances like $\chi(P) \neq \chi(Q)$ on the same manifold to work out the contradiction shown.

For many to one or one to many transformations the line element is not preserved. Only one tone transformations have to be considered. Linear transformations are one to one. Non linear transformations like Cartesian to spherical or Cartesian to cylindrical are one tone. But these one to one non linear transformations do have some associated problems.

Let's consider Cartesian to spherical transformations. Determinant of the Jacobian is given by

$$|J| = r^2 Sin^2\theta$$

On the z axis[poles],$\theta = 0 \; or \; \pi \Rightarrow Sin\theta = 0 \Rightarrow |J| = 0$

Therefore the transformation is of a singular nature for all points on the x-axis. It is to be taken note of that for such points though r and θ are uniquely defined ϕhas no unique value.

To further analyze the situation we recall the geodesic equations:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

$$\frac{d^2\phi}{d\tau^2} + \Gamma^\phi{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

Now for Schwarzschild geometry as well as for flat space time[spherical coordinates]

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = Cot\theta$$

For $\theta = 0$orπ Cot $Cot\theta$ blows up: hence $\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta}$blow up. Therefore the geodesic equation fails for $\theta = 0$orπ.on the z-axis cannot lie on a geodesic. That implies that the north or the south poles cannot lie on geodesics that for an ordinary sphere great circles cannot pass through the poles. This is not true. Therefore the poles[z axis] has to be excluded from the transformation in the spherical system. Line elements with one end on the z axis arte not meaningful the spherical system.

$$ds^2 = ds'^2$$

is not a meaningful equation if the z axis happens to be the end point of a line element

Dot Product Preserving Transport

In parallel transport[4] the two vectors the transported parallel to themselves. In dot product preserving transport product the dot is preserved but the two vectors individually are not transported parallel to themselves but the components of each change in a continuous and as differentiable manner as we move along the curve: $t^i \nabla_i u^\alpha = 0; t^i \nabla_i v^\beta = 0$ are not followed as we move along the curve

We have due to the preservation of dot product,

$$t^i \nabla_i \left(g_{\alpha\beta} u^\alpha v^\beta\right) = 0 \ (17)$$

Since each vector is not transported parallel to itself we have

$$t^i \nabla_i u^\alpha \neq 0; t^i \nabla_i v^\beta \neq 0 \ \ (18)$$

.

We transform to a frame of reference where t^i has only one non zero component. We try to achieve this at some solitary point on the curve

$t^{k\prime} \nabla_{k\prime} \left(g_{\alpha\beta}{}' u^{\alpha}{}' v^{\beta\prime}\right) = 0$[no summation on k': prime denotes the new frame of reference and not differentiation]

$$\nabla_k' \left(g_{\alpha\beta}{}' u^{\alpha}{}' v^{\beta}{}'\right) = 0 (19)$$

$$u^{\alpha}{}' v^{\beta}{}' \nabla_{i\prime}\left(g_{\alpha\beta}{}'\right) + g_{\alpha\beta}{}' \nabla_{i\prime}\left(u^{\alpha\prime} v^{\beta}{}'\right) = 0$$

Since $\nabla_i \left(g_{\alpha\beta}\right) = 0$,we have,

$$g_{\alpha\beta}{}' \nabla_i \left(u^{\alpha\prime} v^{\beta\prime}\right) = 0 \ \ (20)$$

The vectors $u^{\alpha\prime}$and $v^{\beta}{}'$ and consequently their individual components are arbitrary. Therefore

$$g_{\alpha\beta}' = 0 \Rightarrow g_{\alpha\beta} = 0 \ \ (21)$$

At the specific chosen point $g_{\alpha\beta} = 0$ that is the metric tensor is the null tensor for the appropriate manifold. But the null tensor remains null in all frames of reference.

Next we choose different points and for each point we look for some corresponding manifold by transformation manifold for which t^i has only one non zero component

[the null tensor remains null in all frames of reference]That implies that the Riemann tensor , Ricci tensor and the Ricci scalar are all zero valued objects.

Equation (19) may be worked out[without any transformation] as

$$t^i[u^\alpha v^\beta \nabla_i g_{\alpha\beta} + g_{\alpha\beta}\nabla_i[u^\alpha v^\beta]] = 0$$

$$t^i g_{\alpha\beta}\nabla_i[u^\alpha v^\beta] = 0$$

$$t^i g_{\alpha\beta}[u^\alpha \nabla_i v^\beta + v^\beta \nabla_i u^\alpha] = 0$$

$$g_{\alpha\beta}[u^\alpha t^i \nabla_i v^\beta + v^\beta t^i \nabla_i u^\alpha] = 0$$

Since $u^\alpha$and v^β are arbitrary the only option would be $g_{\alpha\beta} = 0$

Alternatively we do the following: The four vectors u and v are moved along a continuous curve such that

1. Dot product is preserved: $t^i \nabla_i(g_{\alpha\beta}u^\alpha v^\beta) = 0$
2. $t^i \nabla_i u^\alpha = 0$
3. $t^i \nabla_i v^\beta \neq 0$. This vector is not transported parallel to itself along the curve but it changes its magnitude and orientation embo0died in its components such that we have: $t^i \nabla_i v^\beta \neq 0$ and $t^i \nabla_i(g_{\alpha\beta}u^\alpha v^\beta) = 0$. In the classical three dimensional space one vectors is transported parallel to itself the other does not move parallel to itself; its magnitude and orientation change in such a manner that $\vec{u}.\vec{v}$ is preserved.

We have considered a sequence of vectors u and v along a curve such that the above conditions are fulfilled.

$$t^i \nabla_i(g_{\alpha\beta}u^\alpha v^\beta) = 0$$

$$t^i[u^\alpha v^\beta \nabla_i g_{\alpha\beta} + g_{\alpha\beta}\nabla_i[u^\alpha v^\beta]] = 0$$

$$t^i g_{\alpha\beta}\nabla_i[u^\alpha v^\beta] = 0$$

$$t^i g_{\alpha\beta}[u^\alpha \nabla_i v^\beta + v^\beta \nabla_i u^\alpha] = 0$$

$$g_{\alpha\beta}u^\alpha t^i \nabla_i v^\beta + g_{\alpha\beta}v^\beta t^i \nabla_i u^\alpha = 0$$

$$g_{\alpha\beta}u^\alpha t^i \nabla_i v^\beta = 0$$

Since $u^\alpha$and v^β are arbitrary the only option would be $g_{\alpha\beta} = 0$

Also

$$g_{\alpha\beta}T^{\beta} = 0$$

where $t^{i}\nabla_{i}v^{\beta} = T^{\beta}(t,x,y,z)$

We consider v^{β} getting transported on different manifolds where $g_{\alpha\beta}$ are different at ach point.We have

$$T^{\beta} = 0 \Longrightarrow t^{i}\nabla_{i}v^{\beta}$$

The Line Element and the Symmetric Nature of the Metric Coefficients

So long as we are on the same manifold, the line element is preserved. This is not true for distinct manifolds

Example: A room with a flat floor and a hemispherical roof is considered. A small arc is drawn on the roof and its projection is taken on the floor. With this transformation

$$ds'^{2} \neq ds^{2}$$

Only if

$$ds'^{2} = ds^{2}$$

then $g_{\mu\nu}$behaves as a rank two tensor. Indeed

$$ds'^{2} = ds^{2}$$

$$\Rightarrow \bar{g}_{\mu\nu}d\bar{x}^{\mu}d\bar{x}^{\nu} = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$$

$$= g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}d\bar{x}^{\mu}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}d\bar{x}^{\nu}$$

$$= g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}d\bar{x}^{\mu}d\bar{x}^{\nu}$$

$$\Rightarrow \bar{g}_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}$$

We revisit the idea[7] that the metric tensor is a symmetric tensor. Indeed

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu}$$

By interchanging the dummy indices μ and ν we have,

$$ds^2 = g_{\nu\mu} dx^\nu dx^\mu$$

$$\Rightarrow ds^2 = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) dx^\mu dx^\nu$$

$g_{\mu\nu} + g_{\nu\mu}$ is asymmetric quantity. The relation

$$g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) dx^\mu dx^\nu$$

is true for arbitrary $dx^\mu$and dx^ν.Therefore

$$g_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})$$

$$\Rightarrow g_{\mu\nu} = g_{\nu\mu}$$

For an *arbitrary* non singular transformation a symmetric tensor has to be produced. That is impossible unless the metric tensor is the null tensor. This corroborates our inference from the "Dot Product Preserving Transport.

Fine Analysis

From the proof that $g_{\mu\nu}$is a rank two tensor we may recall the following

$$\bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\mu d\bar{x}^\nu$$

Since $d\bar{x}^\mu$and $d\bar{x}^\nu$are arbitrary we have the neat relation $\bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$. But $d\bar{x}^\mu$and $d\bar{x}^\nu$ are arbitrary within the constraint that they have to be very small or sufficiently small. Is there any sort of fogginess in that sufficiently small qualification? let us delve into that:

We start with the relation

$$\bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

$$\Rightarrow \bar{g}_{\mu\nu}(d\bar{x}^\mu + h^\mu)(d\bar{x}^\nu + k^\nu) = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}(d\bar{x}^\mu + h^\mu)(d\bar{x}^\nu + k^\nu)$$

[h^μ and k^ν might be finite quantities]

$$\begin{aligned}\Rightarrow \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu + \bar{g}_{\mu\nu} d\bar{x}^\mu k^\nu + \bar{g}_{\mu\nu} h^\mu d\bar{x}^\nu + \bar{g}_{\mu\nu} h^\mu k^\nu \\ = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\mu d\bar{x}^\nu + g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\mu \bar{k}^\nu + g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} h^\mu d\bar{x}^\nu \\ + g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} h^\mu k^\nu\end{aligned}$$

The following relation has to hold for arbitrary finite[or infinitesimal] h^{μ} and k^{ν}

Ignoring this trouble $g_{\alpha\beta}$ is a tensor. Ignoring similar trouble it is a symmetric tensor

$$\bar{g}_{\mu\nu}d\bar{x}^{\mu}k^{\nu}+\bar{g}_{\mu\nu}h^{\mu}d\bar{x}^{\nu}+\bar{g}_{\mu\nu}h^{\mu}k^{\nu}$$
$$=g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}d\bar{x}^{\mu}\bar{k}^{\nu}+g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}h^{\mu}d\bar{x}^{\nu}+g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}h^{\mu}k^{\nu}$$

$$\bar{g}_{\mu\nu}h^{\mu}d\bar{x}^{\nu}+\bar{g}_{\mu\nu}h^{\mu}k^{\nu}=g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}h^{\mu}d\bar{x}^{\nu}+g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}h^{\mu}k^{\nu}$$

Riemannian Curvature Tensor

If we analyze in terms of the general coordinate systems[orthogonal or non orthogonal]we shall use $\mathrm{R}_{\alpha\alpha\gamma\delta}=\mathrm{R}_{\alpha\beta\gamma\gamma}=0$ for same components in all reference frames [and not $\mathrm{R}_{\alpha\beta\gamma\delta}=\mathrm{R}_{\alpha\gamma\beta\delta}=0$]Possibly $\mathrm{R}_{\alpha\beta\alpha\delta}$, $\mathrm{R}_{\alpha\beta\gamma\alpha}, \mathrm{R}_{\alpha\beta\gamma\delta}, \mathrm{R}_{\alpha\gamma\beta\delta}$and $\mathrm{R}_{\alpha\beta\beta\alpha}$ are non zero

The zeros will occur [components] every time we transform to some other arbitrary reference frame. All the components have to mix in order to produce the zeros in the same positions. The transformation elements will also change as we select different frames of reference.

Zeros occurring in all reference is impossible unless $\mathrm{R}_{\alpha\beta\gamma}$ =0 for each α, β, γ and δ.

The Riemannian tensor being zero, the Ricci tensor is also a null tensor and the Ricci scalar stands zero.

That the Ricci tensor is zero has been proved by an alternative method towards the beginning of the section.

As a simpler case we consider a rank one [contravariant] tensor in four dimensions which has as zero one of its components[any one] in all frames of reference

$$\bar{A}^{\mu}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}A^{\alpha}$$

Let

$$A^{1}=0, A^{k}\neq 0; k\neq 1$$

We can always arrange for transformations $\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}$[with non singular determinant of the Jacobian matrix]so that all $\bar{A}^{\mu}$ are zero. We do arrive at a contradiction unless A^{α} is the null tensor. This argument may be extended to higher rank tensors [including the covariant tensors]like the Riemann curvature tensor which has several zeros in each and every frame of reference.

Tensors are usually defined by relations like

$$\bar{A}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} A^{\alpha}$$

The only restriction on the transformation matrix is that $Det[Jacobian] \neq 0$

Conclusion

As mentioned earlier that there is some peculiarity about the covariant operators. Their behavior indicates that the metric coefficients have to disappear

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