Theory of Gravitation

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Abstract

In general relativity, the density of gravitational energy is undefined: it is said to be 'nonlocalizable'. In what follows, a new theory solves the problem of gravitational dynamics. It provides covariant expressions for the energy, momentum, stress, force and power. The theory predicts both longitudinal and transverse gravitational waves. It is time to launch a search for longitudinal waves, in the data at LIGO and Virgo.

1. Introduction

The precis of general relativity is that it provides covariant expressions (tensors) for all physical quantities and for the laws that relate those quantities. However, no such expressions exist for gravitational energy, momentum, stress, force and power. This, in itself, shows that general relativity cannot be correct. The four-dimensional tensor formalism is incapable of describing gravitational dynamics [1].

The theory presented here derives from the fact that special relativity was invented without the use of four-vectors. Only scalars and three-vectors were used by the founders. This is evident in the basic physical elements of the theory: (time, space), (energy, momentum), (charge, current), etc. It is by expressing these elements in terms of coordinates that a new theory of gravitation emerges. It yields a fully covariant treatment of gravitational dynamics.

2. Field equations

The theory of special relativity concerns the motion and orientation of orthonormal frames of reference. A displacement $d\mathbf{r}$ is projected onto an orthonormal 3-frame: $\mathbf{i} \cdot d\mathbf{r}$, $\mathbf{j} \cdot d\mathbf{r}$, $\mathbf{k} \cdot d\mathbf{r}$. These projections, together with the time interval dt, undergo a Lorentz transformation, which leaves the fundamental interval invariant

$$ds^2 = c^2 dt^2 - d\mathbf{r}^2 \tag{1}$$

The physical displacements may also be expressed in terms of a coordinate system $\{x^\mu\}$

$$c\,dt = e_0(x)dx^0 \qquad\qquad d\mathbf{r} = \mathbf{e}_i(x)dx^i \tag{2}$$

where $e_\mu = (e_0, \mathbf{e}_i)$ is a scalar, 3-vector basis. The interval (1) then takes the form

$$\begin{aligned} ds^2 &= (e_0 dx^0)^2 - \mathbf{e}_i \cdot \mathbf{e}_j \, dx^i dx^j \\ &= g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{3}$$

where

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} & \\ 0 & & & \end{pmatrix} \tag{4}$$

The theory of gravitation concerns the structure of this metrical coordinate system. A Lorentz transformation may take place at any point. It will not involve the coordinates $\{x^\mu\}$.

An observer is free to choose a new coordinate system $\{x^{\mu'}\}$. In order to retain the distinction between scalars and 3-vectors, the coordinate transformations are restricted to the form

$$x^{0'} = x^{0'}(x^0) \qquad\qquad x^{i'} = x^{i'}(x^j) \tag{5}$$

Displacements (2) will then be invariant, while the metric transforms as a tensor

$$g_{0'0'} = \frac{\partial x^0}{\partial x^{0'}} \frac{\partial x^0}{\partial x^{0'}}\, g_{00} \qquad\qquad g_{i'j'} = \frac{\partial x^m}{\partial x^{i'}} \frac{\partial x^n}{\partial x^{j'}}\, g_{mn} \tag{6}$$

The Christofel symbols

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}\, g^{\mu\rho} \left(\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda}\right) \tag{7}$$

yield the Ricci tensor

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} \tag{8}$$

The gravitational field equations

$$\frac{c^4}{8\pi G}\Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\Big) + T^{(m)}_{\mu\nu} = 0 \tag{9}$$

follow from the Einstein-Hilbert action

$$\delta \int \frac{c^4}{16\pi G}\, g^{\mu\nu} R_{\mu\nu} \sqrt{-g}\, d^4x + \delta \int L^{(m)} \sqrt{-g}\, d^4x = 0 \tag{10}$$

There are seven field equations, corresponding to the seven variations $\delta g^{\mu\nu} = (\delta g^{00}, \delta g^{ij})$. Components R_{0i} and $T^{(m)}_{0i}$ do not appear in (9).

3. Gravitational energy, momentum and stress

The rate of change of the basis system is defined in terms of connection coefficients $Q^{\lambda}_{\mu\nu}$

$$\nabla_\nu e_\mu = e_\lambda Q^{\lambda}_{\mu\nu} \tag{11}$$

This formula separates into scalar and 3-vector parts

$$\begin{aligned} \nabla_\nu e_0 &= e_0 Q^0_{0\nu} &(12)\\ \nabla_\nu \mathbf{e}_i &= \mathbf{e}_j Q^j_{i\nu} &(13) \end{aligned}$$

where $Q^j_{0\nu} = Q^0_{i\nu} \equiv 0$. In terms of the metrical functions (4),

$$\begin{aligned} \partial_\lambda g_{00} &= 2 g_{00} Q^0_{0\lambda} &(14)\\ \partial_0 g_{ij} &= g_{in} Q^n_{j0} + g_{jn} Q^n_{i0} &(15)\\ \partial_k g_{ij} &= g_{in} Q^n_{jk} + g_{jn} Q^n_{ik} &(16) \end{aligned}$$

If $Q^i_{jk} = Q^i_{kj}$ and if the two terms in (15) are assumed to be equal, then

$$\begin{aligned} Q^0_{0\lambda} &= \Gamma^0_{0\lambda} = \frac{1}{2} g^{00} \partial_\lambda g_{00} &(17)\\ Q^i_{j0} &= \Gamma^i_{j0} = \frac{1}{2} g^{in} \partial_0 g_{nj} &(18)\\ Q^i_{jk} &= \Gamma^i_{jk} = \frac{1}{2} g^{in} \left(\partial_k g_{jn} + \partial_j g_{nk} - \partial_n g_{jk}\right) &(19) \end{aligned}$$

Together, they comprise the formula

$$Q^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + g^{\mu\rho} g_{\lambda\eta} Q^\eta_{[\nu\rho]} \tag{20}$$

where

$$Q^\mu_{[\nu\lambda]} \equiv Q^\mu_{\nu\lambda} - Q^\mu_{\lambda\nu} \tag{21}$$

The non-zero components of $Q^\mu_{[\nu\lambda]}$ are

$$Q^0_{[0i]} = Q^0_{0i} = \frac{1}{2} g^{00} \partial_i g_{00} \qquad Q^i_{[j0]} = Q^i_{j0} = \frac{1}{2} g^{in} \partial_0 g_{nj} \tag{22}$$

They transform as tensor components

$$Q^{0'}_{[0'i']} = \frac{\partial x^n}{\partial x^{i'}}\, Q^0_{[0n]} \qquad\qquad Q^{i'}_{[j'0']} = \frac{\partial x^{i'}}{\partial x^m}\frac{\partial x^n}{\partial x^{j'}}\frac{\partial x^0}{\partial x^{0'}}\, Q^m_{[n0]} \tag{23}$$

This field strength tensor serves to define the gravitational energy tensor

$$T^{(g)}_{\mu\nu} = \frac{c^4}{8\pi G}\Big\{Q^\rho_{[\lambda\mu]}Q^\lambda_{[\rho\nu]} + Q_\mu Q_\nu - \frac{1}{2}g_{\mu\nu}g^{\eta\tau}(Q^\rho_{[\lambda\eta]}Q^\lambda_{[\rho\tau]} + Q_\eta Q_\tau)\Big\} \tag{24}$$

where $Q_\mu = Q^\rho_{[\rho\mu]}$. For a static Newtonian potential ψ

$$g_{00} = 1 + \frac{2}{c^2}\psi \tag{25}$$

so that $Q^\mu_{[\nu\lambda]}$ is given by

$$Q^0_{[0i]} = \frac{1}{c^2}\,\partial_i\psi \qquad\qquad Q^i_{[j0]} = 0 \tag{26}$$

It follows that

$$T^{(g)}_{00} = \frac{1}{8\pi G}(\nabla\psi)^2 \tag{27}$$

$$T^{(g)}_{0i} = 0 \tag{28}$$

$$T^{(g)}_{ij} = \frac{1}{4\pi G}\Big\{\partial_i\psi\,\partial_j\psi - \frac{1}{2}\delta_{ij}(\nabla\psi)^2\Big\} \tag{29}$$

which is the Newtonian stress-energy tensor.

The conservation law for energy and momentum is found by summing the expression $e_\mu T^{\mu\nu}\,dV_\nu$ over a closed, infinitesimal region δR

$$\begin{aligned}\sum_{\delta R} e_\mu T^{\mu\nu}\,dV_\nu &= \Big\{e_\mu\,\partial_\nu(\sqrt{-g}\,T^{\mu\nu}) + (\nabla_\nu e_\mu)\sqrt{-g}\,T^{\mu\nu}\Big\}d^4x \\ &= e_\mu\,\Big\{\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\,T^{\mu\nu}) + Q^\mu_{\lambda\nu}T^{\lambda\nu}\Big\}\sqrt{-g}\,d^4x \end{aligned} \tag{30}$$

where $T^{\mu\nu} = T^{\mu\nu}_{(g)} + T^{\mu\nu}_{(m)}$ is the total energy tensor and

$$dV_\nu = \sqrt{-g}\,(dx^1dx^2dx^3,\; dx^0dx^2dx^3, \ldots) \tag{31}$$

Energy and momentum are conserved, if

$$\operatorname{div} T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\,T^{\mu\nu}) + Q^\mu_{\lambda\nu}T^{\lambda\nu} = 0 \tag{32}$$

Make use of (20) to find

$$\operatorname{div} T^{\mu\nu} = T^{\mu\nu}_{\ ;\nu} + g^{\mu\nu}Q^\beta_{[\alpha\nu]}T_\beta{}^\alpha \tag{33}$$

where $T^{\mu\nu}_{\ ;\nu}$ is the (contracted) covariant derivative. The divergence of the mixed tensor is

$$\operatorname{div} T_\mu{}^\nu = T_\mu{}^\nu{}_{;\nu} + Q^\beta_{[\alpha\mu]}T_\beta{}^\alpha \tag{34}$$

4. Gravitational force and power

The motion of a particle in a gravitational field is described by the Lagrangian

$$L = mc\sqrt{g_{\mu\nu}(x)u^\mu u^\nu} \tag{35}$$

and the resulting equation of motion

$$mc\left\{\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda\right\} = 0 \tag{36}$$

where $u^\mu = dx^\mu/ds$. The energy and momentum of the particle are given by

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}} \qquad\qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} = \frac{E}{c^2}\,\mathbf{v} \tag{37}$$

They yield the power formula

$$\frac{dE}{ds} = \mathbf{v}\cdot\frac{d\mathbf{p}}{ds} \tag{38}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the physical velocity. The gravitational power and force are found by expressing the energy and momentum in terms of coordinates

$$E = mc^2 e_0 u^0 \qquad\qquad \mathbf{p} = mc\ \mathbf{e}_i u^i \tag{39}$$

The rate of change of $e_\mu u^\mu$ is

$$
\begin{aligned}
\frac{d(e_\mu u^\mu)}{ds} &= e_\mu \frac{du^\mu}{ds} + \frac{de_\mu}{ds} u^\mu = e_\mu \Big\{ \frac{du^\mu}{ds} + Q^\mu_{\nu\lambda} u^\nu u^\lambda \Big\} \\
&= e_\mu \Big\{ \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \Big\} + e^\mu \, Q^\lambda_{[\nu\mu]} u^\nu u_\lambda
\end{aligned} \tag{40}
$$

where (20) has been used. Substitute the equation of motion (36) to find

$$
mc \, \frac{d(e_\mu u^\mu)}{ds} = e^\mu \, mc \, Q^\lambda_{[\nu\mu]} u^\nu u_\lambda \tag{41}
$$

Separate this formula into scalar and 3-vector parts, then substitute the components (22) to find that the energy and momentum change as follows:

$$
\frac{dE}{ds} = e^0 \, \frac{mc^2}{2} \Big\{ -\partial_n g_{00} \, u^n u^0 + \partial_0 g_{mn} \, u^m u^n \Big\} \tag{42}
$$

$$
\frac{d\mathbf{p}}{ds} = \mathbf{e}^i \, \frac{mc}{2} \Big\{ \partial_i g_{00} \, u^0 u^0 - \partial_0 g_{in} \, u^0 u^n \Big\} \tag{43}
$$

These equations express the power and force which are exerted by the gravitational field. In the Newtonian limit (25), $u^0 = 1$ and $u^n = v^n/c$ so that

$$
\frac{dE}{dt} = -m \, \nabla\psi \cdot \mathbf{v} \qquad\qquad \frac{d\mathbf{p}}{dt} = -m \, \nabla\psi \tag{44}
$$

If other forces are present, they will appear in the equation of motion. For example, a charged particle in combined gravitational and electromagnetic fields is described by

$$
L = mc \sqrt{g_{\mu\nu}(x) u^\mu u^\nu} + \frac{q}{c} \, A_\mu(x) u^\mu \tag{45}
$$

and

$$
mc \Big\{ \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \Big\} = \frac{q}{c} \, F^\mu_{\ \nu} u^\nu \tag{46}
$$

It follows that (compare (41))

$$
mc \, \frac{d(e_\mu u^\mu)}{ds} = e^\mu \Big\{ mc \, Q^\lambda_{[\nu\mu]} u^\nu u_\lambda + \frac{q}{c} \, F_{\mu\nu} u^\nu \Big\} \tag{47}
$$

The scalar and 3-vector parts of the Lorentz force are then added to the right-hand side of (42) and (43).

5. The weak-field approximation

(In this section, $T^{\mu\nu}_{(m)} = T^{\mu\nu} = \rho c^2 u^\mu u^\nu$.)
If the coordinate system is nearly rectangular, then the metric tensor may be expanded

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad\qquad |h_{\mu\nu}| \ll 1 \tag{48}$$

Substitution into (8) yields

$$R_{\mu\nu} = \frac{1}{2}\left\{\eta^{\lambda\rho}\partial_\lambda\partial_\rho h_{\mu\nu} + \partial_\mu\partial_\nu h^\lambda_{\ \lambda} - \partial_\mu\partial_\lambda h^\lambda_{\ \nu} - \partial_\nu\partial_\lambda h^\lambda_{\ \mu}\right\} \tag{49}$$

The four conditions

$$\partial_\nu h^\lambda_{\ \lambda} = 2\,\partial_\lambda h^\lambda_{\ \nu} \tag{50}$$

leave three independent components $h_{\mu\nu}$, and they greatly simplify the Ricci tensor

$$R_{\mu\nu} = \frac{1}{2}\,\partial^\lambda\partial_\lambda h_{\mu\nu} \tag{51}$$

Rewrite the field equations in the form

$$R^\mu_{\ \nu} = -\frac{8\pi G}{c^4}\left(T^\mu_{\ \nu} - \frac{1}{2}\,\delta^\mu_{\ \nu}\,T\right) \tag{52}$$

in order to obtain

$$\partial^\lambda\partial_\lambda h^\mu_{\ \nu} = -\frac{16\pi G}{c^4}\left(T^\mu_{\ \nu} - \frac{1}{2}\,\delta^\mu_{\ \nu}\,T\right) \tag{53}$$

The retarded solution is

$$h^\mu_{\ \nu}(\mathbf{x},t) = -\frac{4G}{c^4}\int \frac{(T^\mu_{\ \nu} - \frac{1}{2}\delta^\mu_{\ \nu}\,T)}{|\mathbf{x}-\mathbf{x}'|}\,|_{\mathrm{ret}}\, d^3x' \tag{54}$$

If the source is at rest, then $T^0_{\ 0} = T = \rho c^2$ and $T^i_{\ j} = 0$ so that (54) yields

$$ds^2 = (1 + \frac{2}{c^2}\,\psi)(dx^0)^2 - (1 - \frac{2}{c^2}\,\psi)(dx^2 + dy^2 + dz^2) \tag{55}$$

where ψ is the Newtonian potential.

5.1 Wave generation

When motion of the source is significant, radiation is produced which propagates into distant regions. In those regions, the solution (54) simplifies to

$$h^{\mu}_{\ \nu}(\mathbf{x},t) = -\frac{4G}{c^4 r}\int (T^{\mu}_{\ \nu} - \frac{1}{2}\,\delta^{\mu}_{\ \nu}\,T)|_{t-r/c}\,d^3x' \tag{56}$$

Here, the long wavelength limit is assumed, so that ret simply means time $(t - r/c)$. Formula (50) for $\nu = 0$ imposes a constraint on this solution, viz., $\partial_0 h^0_{\ 0} = \partial_0 h^n_{\ n}$. Explicitly,

$$\begin{aligned} h^0_{\ 0}(x) &= -\frac{2G}{c^4 r}\int (T^0_{\ 0} - T^n_{\ n})\,d^3x' && (57)\\ h^n_{\ n}(x) &= -\frac{2G}{c^4 r}\int (-3T^0_{\ 0} - T^n_{\ n})\,d^3x' && (58)\end{aligned}$$

so that the constraint becomes

$$\frac{d}{dt}\int T^0_{\ 0}\,d^3x' = 0 \tag{59}$$

The material energy changes very little in the weak-field approximation, i.e., little gravitational energy is produced. Thus, the $T^0_{\ 0}$ term will not contribute to the radiation, which leaves

$$\begin{aligned} h^0_{\ 0}(x) &= \frac{2G}{c^4 r}\int T^n_{\ n}\,d^3x' && (60)\\ h^i_{\ j}(x) &= -\frac{4G}{c^4 r}\int (T^i_{\ j} - \frac{1}{2}\,\delta^i_{\ j}\,T^n_{\ n})\,d^3x' && (61)\end{aligned}$$

These integrals may be transformed by means of the identity [2-5]

$$\int T^{ij}\,d^3x = \frac{1}{2}\int x^i x^j\,\partial_k\partial_l T^{kl}\,d^3x \tag{62}$$

The conservation law $\partial_\mu T^{\mu\nu} = 0$ (again, ignoring the gravitational part) gives $\partial_k\partial_l T^{kl} = \partial_0\partial_0 T^{00}$, and it follows that

$$\int T^{ij}\,d^3x = \frac{1}{2}\,\frac{d^2}{dt^2}\int \rho\,x^i x^j\ d^3x = \frac{1}{2}\,\frac{d^2 I^{ij}}{dt^2} \tag{63}$$

Finally,

$$h^0{}_0(x) = \frac{G}{c^4 r}\frac{d^2 I^n_n}{dt^2} = -\frac{G}{c^4 r}\frac{d^2 I}{dt^2} \tag{64}$$

$$h^i{}_j(x) = -\frac{2G}{c^4 r}\left(\frac{d^2 I^i{}_j}{dt^2} + \frac{1}{2}\delta^i{}_j\frac{d^2 I}{dt^2}\right) \tag{65}$$

The above expressions give the radiation field in terms of the motion of the source. The $h_{\mu\nu}$ also satisfy the field equations (53) for matter-free space, $\partial^\lambda\partial_\lambda h_{\mu\nu} = 0$, which admit plane wave solutions such as

$$h^\mu_\nu = A^\mu{}_\nu \cos(-K_\lambda x^\lambda) \qquad (K^0 = K) \tag{66}$$

Conditions (50) now take the form

$$A^0{}_0 = A^n{}_n \qquad K_i A^n{}_n = K_n A^n{}_i \tag{67}$$

If a particular direction is chosen (say K^3) then the following components remain

$$A^0{}_0 = A^3{}_3 \quad A^1{}_1 = -A^2{}_2 \quad A^1{}_2 = A^2{}_1 \tag{68}$$

while $A^2{}_3 = A^3{}_1 = 0$. The transverse part of A_{ij} is traceless

$$A_{\mu\nu} = \begin{pmatrix} A_{00} & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & -A_{00} \end{pmatrix} \tag{69}$$

5.2 Radiative energy flow

The flow of gravitational energy is determined by (24)

$$\begin{aligned} T^{(g)}_{0i} &= \frac{c^4}{8\pi G}\Big\{Q^0_{[0n]}Q^n_{[i0]} + Q^n_{[n0]}Q^0_{[0i]}\Big\} \\ &= \frac{c^4}{32\pi G}\Big\{\partial_k h^0{}_0\,\partial_0 h^k{}_i + \partial_i h^0{}_0\,\partial_0 h^k{}_k\Big\} \end{aligned} \tag{70}$$

The retarded fields satisfy $\partial_i h^\mu{}_\nu = -\partial_0 h^\mu{}_\nu\, n^i$ where n^i is the unit vector in the radial direction. It follows that

$$T^{(g)}_{0i} = -\frac{c^4}{32\pi G}\left\{\partial_0 h^0_{\ 0}\,\partial_0 h^k_{\ i}\, n^k + \partial_0 h^0_{\ 0}\,\partial_0 h^k_{\ k}\, n^i\right\} \tag{71}$$

The radial flux is given by the product

$$T^{(g)}_{0i} n^i = -\frac{c^4}{32\pi G}\,\partial_0 h^0_{\ 0}\left\{\partial_0 h^k_{\ i}\, n^k n^i + \partial_0 h^0_{\ 0}\right\} \tag{72}$$

Substitute (64) and (65) to find the energy flow per unit solid angle

$$\frac{d^2E}{dt\, d\Omega} = -cr^2 T^{(g)}_{0i} n^i = \frac{G}{16\pi c^5}\,\frac{d^3 I}{dt^3}\left(\frac{d^3 I^k_i}{dt^3}\, n^k n^i + \frac{d^3 I}{dt^3}\right) \tag{73}$$

Integration yields an average value for the angular term $\int n^k n^i d\Omega = \delta^i_k 4\pi/3$ and the power formula

$$\frac{dE}{dt} = \int c\, T^{(g)i}_0 n^i r^2 d\Omega = \frac{G}{6c^5}\left(\frac{d^3 I}{dt^3}\right)^2 \tag{74}$$

In a binary system, [6]

$$\frac{d^3 I}{dt^3} = -\frac{2m_1 m_2}{a(1-e^2)}\, e\, \sin\theta\, \dot{\theta} \tag{75}$$

Substitution into (74) gives

$$\frac{dE}{dt} = \frac{2Gm_1^2 m_2^2}{3c^5 a^2 (1-e^2)^2}\, e^2 \sin^2\theta\, \dot{\theta}^2 \tag{76}$$

while the average power over one period is

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{Gm_1^2 m_2^2 (m_1 + m_2)}{3c^5 a^5}\, e^2 (1 + \frac{e^2}{4})(1-e^2)^{-7/2} \tag{77}$$

This formula exhibits a strong dependence upon eccentricity. In particular, circular orbits will not radiate energy ($e = 0$), suggesting that they are more stable than eccentric orbits. Nevertheless, they do emit transverse waves in accordance with (65)

$$h^i_{\ j}(x) = -\frac{2G}{c^4 r}\,\frac{d^2 I^i_{\ j}}{dt^2} \tag{78}$$

5.3 Wave detection

The force (43) exerted on a detector initially at rest is due to $\partial_i g_{00}$ alone

$$\frac{d\mathbf{p}}{dt} = \frac{mc^2}{2}\,\mathbf{e}^i\,\partial_i g_{00} = \frac{mc^2}{2}\,\partial_0 h_{00}\,\mathbf{e}_i n^i \tag{79}$$

which is along the direction of propagation. Substitute (64) to find

$$\frac{d\mathbf{p}}{dt} = -\frac{Gm}{2c^3 r}\frac{d^3 I}{dt^3}\,\mathbf{e}_i n^i \tag{80}$$

Apart from the acceleration of the mirrors (referring to LIGO), the gravitational field directly changes the wavelength of light in the storage chambers, much as in the static redshift. In this regard, it is important that each mode possesses energy:

$$\begin{aligned}
T^{(g)}_{00} &= \frac{c^4}{16\pi G}\left\{Q^m_{n0}Q^n_{m0} + Q^m_{m0}Q^n_{n0} - 2\eta_{00}\eta^{mn}Q^0_{0m}Q^0_{0n}\right\} \\
&= \frac{c^4}{64\pi G}\left\{\partial_0 h^m_{\ n}\partial_0 h^n_{\ m} + \partial_0 h^m_{\ m}\partial_0 h^n_{\ n} + 2(\partial_0 h^0_{\ 0})^2\right\} \\
&= \frac{c^4}{32\pi G}\left\{(\partial_0 h^1_{\ 1})^2 + (\partial_0 h^1_{\ 2})^2 + 2(\partial_0 h^0_{\ 0})^2\right\}
\end{aligned} \tag{81}$$

where the final line pertains to plane waves along K^3 (see (69)). The effect on light can then be viewed as an exchange of energy.

The gravitational stress is given by

$$T^{(g)}_{ij} = \frac{c^4}{8\pi G}\left\{2Q^0_{0i}Q^0_{0j} - \frac{1}{2}\eta_{ij}\left[\eta^{00}(Q^m_{n0}Q^n_{m0} + Q^m_{m0}Q^n_{n0}) + 2\eta^{mn}Q^0_{0m}Q^0_{0n}\right]\right\} \tag{82}$$

which is diagonal for the field (69)

$$T^{(g)}_{11} = T^{(g)}_{22} = \frac{c^4}{32\pi G}\left\{(\partial_0 h^1_{\ 1})^2 + (\partial_0 h^1_{\ 2})^2\right\} \tag{83}$$

$$T^{(g)}_{33} = \frac{c^4}{32\pi G}\left\{(\partial_0 h^1_{\ 1})^2 + (\partial_0 h^1_{\ 2})^2 + 2(\partial_3 h^0_{\ 0})^2\right\} \tag{84}$$

The transverse modes exert an equal pressure in all directions, while the longitudinal mode exerts pressure along K^3.

5.4 LIGO and Virgo

The equation of motion for a light ray (36) may be written in terms of the energy and momentum components $p^\mu = \hbar k^\mu$

$$\frac{dk^\mu}{ds} + \Gamma^\mu_{\nu\lambda} k^\nu \frac{dx^\lambda}{ds} = 0 \tag{85}$$

By proceeding as in section 4, this yields the expression

$$d(e_\mu k^\mu) = e^\mu\, Q^\lambda_{[\nu\mu]} k^\nu dx_\lambda \tag{86}$$

Separate the scalar and 3-vector parts to find

$$\begin{aligned} d(e_0 k^0) &= e^0\, \frac{1}{2}\left\{-\partial_n h_{00}\, k^n dx^0 + \partial_0 h_{mn}\, k^m dx^n\right\} & (87)\\ d(\mathbf{e}_i k^i) &= \mathbf{e}^i\, \frac{1}{2}\left\{\partial_i h_{00}\, k^0 dx^0 - \partial_0 h_{in}\, k^0 dx^n\right\} & (88)\end{aligned}$$

These formulas give the change of frequency and wave vector, as the light ray passes through a gravitational field.

For the plane waves (66),

$$\partial_\rho h_{\mu\nu} = K_\rho A_{\mu\nu} \sin(-K_\lambda x^\lambda) \tag{89}$$

and the above formulas become

$$\begin{aligned} d(e_0 k^0) &= e^0\, \frac{1}{2}\left\{-K_n k^n A_{00}\, dx^0 + K_0 k^m A_{mn}\, dx^n\right\} \sin(-K_\lambda x^\lambda) & (90)\\ d(\mathbf{e}_i k^i) &= \mathbf{e}^i\, \frac{1}{2}\left\{K_i k^0 A_{00}\, dx^0 - K_0 k^0 A_{in}\, dx^n\right\} \sin(-K_\lambda x^\lambda) & (91)\end{aligned}$$

At LIGO and Virgo, the light rays are confined to the horizontal arms, which are taken to be along $\mathbf{e}_1$ and $\mathbf{e}_2$. The gravitational waves can arrive from any direction. However, only two special cases will be considered here: arrival along the vertical and arrival along one specific horizontal direction.

(a) Arrival along $\mathbf{e}_3$, $K^0 = K^3; A_{00} = -A_{33}, A_{11} = -A_{22}, A_{12} = A_{21}$
Effect on light ray $k^\mu = (k^0, k^1, 0, 0)$:

$$d(e_0 k^0) = e_0 \frac{1}{2} K^0 k^1 A_{11} dx^1 \sin(-K_\lambda x^\lambda) \tag{92}$$

$$d(\mathbf{e}_1 k^1) = \mathbf{e}_1 \frac{1}{2} K^0 k^0 A_{11} dx^1 \sin(-K_\lambda x^\lambda) \tag{93}$$

Effect on light ray $k^\mu = (k^0, 0, k^2, 0)$:

$$d(e_0 k^0) = e_0 \frac{1}{2} K^0 k^2 A_{22} dx^2 \sin(-K_\lambda x^\lambda) \tag{94}$$

$$d(\mathbf{e}_2 k^2) = \mathbf{e}_2 \frac{1}{2} K^0 k^0 A_{22} dx^2 \sin(-K_\lambda x^\lambda) \tag{95}$$

These changes are due to the transverse gravitational waves.

(b) Arrival along $\mathbf{e}_1$, $K^0 = K^1; A_{00} = -A_{11}, A_{22} = -A_{33}, A_{23} = A_{32}$
Effect on light ray $k^\mu = (k^0, k^1, 0, 0)$:

$$d(e_0 k^0) = e_0 \frac{1}{2} k^1 A_{00} (K^1 dx^0 - K^0 dx^1) \sin(-K_\lambda x^\lambda) \tag{96}$$

$$d(\mathbf{e}_1 k^1) = \mathbf{e}_1 \frac{1}{2} k^0 A_{00} (K^1 dx^0 - K^0 dx^1) \sin(-K_\lambda x^\lambda) \tag{97}$$

These changes are due to the longitudinal waves alone. They are nonzero if the light ray and the gravitational wave vector are antiparallel.
Effect on light ray $k^\mu = (k^0, 0, k^2, 0)$:

$$d(e_0 k^0) = e_0 \frac{1}{2} K^0 k^2 A_{22} dx^2 \sin(-K_\lambda x^\lambda) \tag{98}$$

$$d(\mathbf{e}_2 k^2) = \mathbf{e}_2 \frac{1}{2} K^0 k^0 A_{22} dx^2 \sin(-K_\lambda x^\lambda) \tag{99}$$

These changes are due to the transverse modes. In each case, only the terms along the light ray have been retained. They will produce interference.

5.5 The redshift and bending of light

For the time-independent field (55), formulas (87) and (88) simplify

$$d(e_0 k^0) = -e^0 \frac{1}{c^2} \partial_n \psi \, k^n dx^0 \qquad d(\mathbf{e}_i k^i) = \mathbf{e}^i \frac{1}{c^2} \partial_i \psi \, k^0 dx^0 \tag{100}$$

The redshift of a light beam which moves radially is found by setting $k^0 = k$, $dx^0 = dr$ and $e_0 k^0 = \omega/c = 2\pi/\lambda$ in the scalar equation

$$\frac{\Delta\omega}{\omega} = -\frac{\Delta\lambda}{\lambda} = -\frac{1}{c^2}\Delta\psi \tag{101}$$

The bending of a light ray that passes near mass M involves the comparison of wave vectors $\mathbf{k}(A)$ and $\mathbf{k}(B)$, which are located at points A and B far from mass M. The angle between them is found by referring the components $k^i(A)$ and $k^i(B)$ to the same basis $\mathbf{e}_i(A)$. The components are found from (85)

$$dk^i = -\Gamma^i_{00} k^0 dx^0 - \Gamma^i_{mn} k^m dx^n \tag{102}$$

For a light ray moving parallel to the z-axis ($k^0 dx^0 = k^3 dx^3$), the change in the x-direction is

$$dk^1 = -\frac{2}{c^2}\,\partial_1\psi\, k^3 dx^3 \tag{103}$$

Integration yields the angle

$$\frac{\Delta k^1}{k} = -\frac{4GM}{c^2 R} \tag{104}$$

where R is the radius of closest approach to mass M.

6. Concluding remarks

The search for longitudinal waves could begin with the experiments at LIGO and Virgo. At any given installation, if a gravitational wave arrives vertically, then its transverse components alter the wavelength of light in the storage chambers. For other directions of arrival, both transverse and longitudinal components contribute. A new analysis of the data could reveal the existence of longitudinal gravitational waves.

The tensor (24) has been used in [6] to calculate the gravitational energy of the Robertson-Walker metric. The conservation law (32) and the Friedmann equations then combined to show that gravity accounts for two-thirds of the energy in the Universe—it is the "dark energy." The current conditions in the Universe yield a positive acceleration for the expansion.

In [7], the covariant spinor derivative couples the gravitational tetrad to the spinor, in a gauge invariant manner. The coupling is such that the spinor and tetrad propagate together with the same velocity. It is similar to the coupling between electric and magnetic fields. The tetrad satisfies the gravitational field equations (9), while the spinor satisfies the Dirac equation for a free, massive fermion. The tensor (24) was applied to show that the energy and momentum are conserved.

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