Systems of linear Dyson–Schwinger equations

Henry Kißler

June 11, 2019

Abstract

Systems of Dyson–Schwinger equation represent the equations of motion in quantum field theory. In this paper, we follow the combinatorial approach and consider Dyson–Schwinger equations as fixed point equations that determine the perturbation series by usage of graph insertion operators. We discuss their properties under the renormalization flow, prove that fixed points are scheme independent, and construct solutions for coupled systems with linearized arguments of the insertion operators.

1 Introduction

In quantum field theory, the equations of motions are formulated terms of systems of coupled Dyson–Schwinger equations. Starting from a tree-level term, Dyson–Schwinger equations recursively determine the full perturbation series and hence yield all quantum corrections to the classical theory. The coefficients of this series are generally sums of Feynman graphs which translate into integrals that possibly diverge. This issue is solved by employing various methods commonly known as renormalization. Consequently, a perturbation series turns into a formal power series with finite coefficients which is called a Green's function. Green's functions can be thought of as the building blocks of quantum field theory—they contain the information of all observables. However, the process of renormalization requires the introduction of auxiliary scales which induce some freedom of choice. Ultimately, renormalized Green's function depend on the choice of a scale which is referred to as renormalization scheme dependence, whereas observables extracted from renormalized Green's functions are expected to be independent of any kind of choice, indicating a subtle underlying structure.

Whereas the original formulation of Dyson–Schwinger equations reaches back to the very early stages of the foundation of quantum field theory and

the related search for methods of renormalization, a recently proposed combinatorial description of Dyson–Schwinger equations provides fresh insights how the structure of divergences of the perturbation series determines the renormalization flow. In [1], Kreimer introduced a combinatorial version of Dyson–Schwinger equations that utilizes operators to insert subdivergences into skeleton graphs (Feynman graphs without subdivergences). Crucially, these insertion operators are known to be compatible to the Hopf algebra structure of Feynman graphs which was introduced in [2, 3] as mathematically rigorous description of the renormalization process. This allows for a combinatorial derivation of the Callan–Symanzik equations [4, 5], which can be understood as a first elucification on the subtle underlying structure.

Generally, a solution of a Dyson–Schwinger equation requires the evaluation of infinite many skeleton graphs and hence solutions can only be constructed after truncating the set of skeletons. So far, the above combinatorial methods have been successfully employed to solve particular truncations for various cases of a single Dyson–Schwinger equation. A particularly simple truncation was constructed by linearizing the argument of a single insertion operator for a single equation. This linear Dyson–Schwinger equation can be thought of some kind of rainbox approximation and was shown to be solved in terms of scaling solutions [6]. Other examples are the chord diagram expansions with a single skeleton [7] and an arbitrary number of skeleton graphs [8] which are, however, still restricted to a single equation. In this article, we generalize the single equation version of a linear Dyson–Schwinger equation to a coupled system of linear Dyson–Schwinger equations.

The article is organized as follows. Section 2 briefly reviews the terminology and essential results of Hopf algebras and combinatorial Dyson–Schwinger equations. Also, we provide a proof that fixed points are scheme independent in kinematic renormalization. Section 3 reproduces known results pertaining to the single equation case and establishes our conventions that naturally generalize to systems of linear Dyson–Schwinger equations which are discussed in section 4. We provide solutions and demonstrate that their properties critically depend on whether a matrix defined in terms of analytic expressions of the skeleton graphs is diagonalizable or degenerates. Finally, we discuss the physical relevance of the different cases.

2 Fixed points & Renormalization flow

This section provides a brief introduction to the combinatorial formulation of Dyson–Schwinger equations and the essential terminology of Hopf-algebraic

renormalization. After recalling the combinatorial analysis of the renormalization flow and deriving the celebrated Callan–Symanzik equation, we show that its fixed points are scheme independent within the class of kinematic renormalization schemes.

Recall that, in quantum field theory, a Lagrangian specifies a set of fields and interactions, which mediate forces between these fields. Each field defines a specific edge type and each interaction defines some type of vertex. The union of all these edge types and vertex types is commonly called the set of residues. Graphs build from these edges and vertices are called Feynman graphs and the challenge that is the subject of perturbation theory is to construct all Feynman graphs and evaluate their analytic expressions which are obtained by applying a set of rules which is derived from the Lagrangian and commonly called Feynman rules. The analytical structure of the Feynman rules associates a weight to each graph which yields a characterization of the divergent graphs. Renormalization provides methods to assign finite result to the divergent graphs and gives rise to a Hopf algebra $(H, m, \mathbb{1}, \Delta, \bar{\mathbb{1}}, S)$ [2, 3].

Here, H denotes the $\mathbb{Q}$-vector space generated by all divergent Feynman graphs those connected components are bridgeless. The product $m : H \otimes H \to H$ is given by the disjoint union and the unit $\mathbb{1}$ is the empty graph. The counit is denoted by $\bar{\mathbb{1}}$ and the coproduct $\Delta : H \to H \otimes H$ is the algebra homomorphism defined by

$$\Delta(\Gamma) = \sum_{\gamma \trianglelefteq \Gamma} \gamma \otimes \Gamma/\gamma, \tag{1}$$

where $\Gamma \in H$ is a Feynman graph, the sum goes over all subgraphs $\gamma \trianglelefteq \Gamma$ which connected components are divergent, and Γ/γ denotes the graph obtained by contracting each connected component of γ in Γ to a point. The antipode $S : H \to H$ is defined by

$$S(\Gamma) = -\Gamma - \sum S(\Gamma')\Gamma'', \tag{2}$$

where we used Sweedler's notation i.e. $\Delta(\Gamma) = \Gamma \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma + \sum \Gamma' \otimes \Gamma''$.

2.1 Combinatorial Dyson–Schwinger equations

In this article, the main objects of interested are the Green's functions, which are defined as formal series in the coupling parameter α with coefficients in H

$$X^r(\alpha) = \mathbb{1} \pm \sum_{\mathrm{res}(\Gamma)=r} \alpha^{|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)}, \tag{3}$$

where r refers to some residue (edge or vertex type), $|\Gamma|$ denotes the number of independent cycles of the graph Γ, which is commonly called loop number in the physics literature, and $\mathrm{sym}(\Gamma)$ is the symmetry factor of the graph Γ that is the order of the automorphism group of Γ. By convention, the plus sign applies if r is a vertex type, but minus applies if r is an edge type. It is worth remarking that, in this work, we will always restrict to a single coupling parameter α.

Recently, Kreimer proposed a combinatioral method to describe Green's functions by means of graph insertions [1]. As opposed to the traditional approaches, Kreimer's description reveals some algebraic properties underlying Green's functions and is manifestly compatible with the underlying Hopf-algebraic structure of Feynman graphs which allows for a detailed analysis of the renormalization process. Here, we recall the essential results of Kreimer's construction.

As a main result of [1], the Green's function (3) can be understood as solution of the combinatorial fixed point equation

$$X^r = \mathbb{1} + B_+^r \left(X^r Q^k \right) \tag{4}$$

where we have introduced the insertion operators B_+^r, and the invariant charge $Q(\alpha)$. In this sense, $X^r(\alpha)$ is commonly referred to as combinatioral Green's function. The insertion operator above abbreviates a sum of linear operators

$$B_+^r = \sum_{k \geq 1} \alpha^k B_+^{k;r} \tag{5}$$

where each $B_+^{k;r}$ maps a product of Feynman graphs to a sum of Feynman graphs obtained by summing over all possibilities to insert the graph from its argument into a skeleton graph with loop order k and residue r. The domain of these insertion operators is naturally extended to formal power series by employing the operator on each coefficient of the series. The invariant charge Q is a quotient of formal series where the numerator is the product of all Green's functions of vertex type and each Green's function of edge type contributes some factors in the denominator. If we assume that there is only a single vertex type for simplicity, then the invariant charge reads

$$Q = \frac{(X^v)^2}{\prod\limits_{e \sim v} X^e} \tag{6}$$

where v denotes the vertex type and each half-edge e that is incident to v contributes a factor in the product. At this point, the reader should be aware that in gauge theories, there is a gauge parameter which is renormalized and hence must be understood as second coupling parameter. Under these circumstances, we expect the presence of another invariant charge. In general, every renormalized parameter is expected to introduce an invariant charge. Indeed, this has been explicitly verified for the renormalization quantum electrodynamics and quantum chromodynamics in [9, 10].

Another crucial result of [1] is that the insertion operators $B_+^{k;r}$ are Hochschild one-cocyles that is

$$\Delta \circ B_+^{k;r} = B_+^{k;r} \otimes \mathbb{1} + \left(\mathrm{id} \otimes B_+^{k;r}\right) \circ \Delta \tag{7}$$

where Δ denotes the coproduct of the Kreimer's Hopf algebra of Feynman graphs. This simple relations allows to commute the coproduct with the insertion operators and hence provides a technique for inductive derivations of coproduct formulas of the entire combinatorial Green's functions as opposed to derive the coproduct for each graph individually and sum up the results. This implies the identities [11, 5]

$$\Delta X^r = \sum_{k \geq 0} X^r Q^k \otimes X^r|_k \tag{8}$$

$$\Delta Q = \sum_{k \geq 0} Q^{k+1} \otimes Q|_k \tag{9}$$

where $X^r|_k$ denotes the k-th coefficient of the series X^r. These identities encode the combinatorial structure of the subdivergences of the Green's functions and form the basis for an analysis of their behavior under change of the renormalization scale.

2.2 Feynman rules

The Feynman rules $\mathbf{\Phi}$ map a Feynman graph to an analytic expression in some target algebra. It turns out that the integrals which are meant to represent these analytic expressions are commonly plagued by divergences. A common way to resolve this issue is to introduce an regulator and define a divergent integral by analytic continuation. Therefore the target algebra of the Feynman rules is defined to be the space of Laurent series in the regulator with complex coefficients. In this article, we will employ analytic regularization [12]. The explicit analytic shape of the set of Feynman rules

depends on the actual model under consideration. Here, we will restrict ourselves mainly to the case of a scalar field theory for the sake of simplicity. Also, we follow the conventions of [13] and denote the external kinematic data of a Feynman graph into a set of angles $\{\theta\}$ and the scale S. In this setting, there is a simple construction that assigns a finite expression to the aforementioned divergent expressions. This is accomplished by the renormalized Feynman rules

$$\mathbf{\Phi}^R_{(S,S_0,\Theta,\Theta_0)}(\Gamma) = \sum_{f\,\text{forest}} (-1)^{|f|} \mathbf{\Phi}_{(S_0,\Theta_0)}(f) \mathbf{\Phi}_{(S,\Theta)}(\Gamma/f), \tag{10}$$

where Γ is a Feynman graph, $\mathbf{\Phi}_{(S,\theta)}(\Gamma)$ denotes the Feynman rules employed to the graph Γ and evaluating its external momenta at the scale S and angles $\{\theta\}$, f is a forest of the graph Γ (that is a set of divergent subgraphs with any two of them being either nested or disjoint), $|f|$ denotes the number of elements of the forest f, and Γ/f is the Feynman graph obtained by contracting each of the elements of the forest f to a point in Γ. It is worth remarking that the external kinematic data of the forests $\mathbf{\Phi}_{(S_0,\theta_0)}(f)$ are evaluated at a reference scale S_0 and reference angles $\{\theta_0\}$ with is commonly referred to as renormalization point. This is the essence of a kinematic renormalization scheme: the subdivergences are evaluated at some renormalization point. Obviously, a change of the renormalization point will also change results and we will devote large parts of the next section to this subject.

Finally, we remark that the combinatorial structure of the renormalized Feynman rules in (10) allows for a concise expression using Hopf-algebraic terminology. More precisely, it holds [13]

$$\mathbf{\Phi}^R_{(S,S_0,\theta,\theta_0)} = (\mathbf{\Phi}_{(S_0,\theta_0)} \circ S) \star \mathbf{\Phi}_{(S,\theta)} := m \circ \big((\mathbf{\Phi}_{(S_0,\theta_0)} \circ S) \otimes \mathbf{\Phi}_{(S,\theta)}\big) \circ \Delta \tag{11}$$

where S is the antipode and Δ is the coproduct of the Hopf algebra of Feynman graph whereas m is the product in the target algebra of the Feynman rules. More generally, the right-hand side defines the convolution product $\star : \mathrm{Hom}(A,C) \times \mathrm{Hom}(A,C) \to \mathrm{Hom}(A,C)$ for some algebra A and some coalgebra C.

2.3 Callan–Symanzik equation and fixed points

It is well-known that scale dependence of a superficially divergent graph is a polynomial in the logarithm of the quotient of the scale S and the

renormalization point S_0 [13], hence we can define $L = \ln \frac{S}{S_0}$ and expand

$$\mathbf{\Phi}^R(X^r) = 1 + \sum_{k\geq 1} \gamma_k^r(\alpha) L^k, \tag{12}$$

$$\mathbf{\Phi}^R(Q) = 1 + \sum_{k\geq 1} \beta_k^r(\alpha) L^k, \tag{13}$$

where coefficient functions γ_k^r and β_k^r possibly depend on the angles $\{\theta\}$ and $\{\theta_0\}$. In the equation above and the following, we suppress this angle dependence for the sake of a concise notation.

Due to the underlying Hopf-algebraic structure [14, 15], a combinatorial analysis [4, 5] reveals that these coefficient functions are highly restricted and satisfy the following set of equations

$$k\gamma_k^r(\alpha) = \left[\gamma_1^r(\alpha) + \beta_1^r(\alpha)\alpha\partial_\alpha\right]\gamma_{k-1}^r(\alpha), \tag{14}$$

$$k\beta_k^r(\alpha) = \beta_1^r(\alpha)\left[1 + \alpha\partial_\alpha\right]\beta_{k-1}^r(\alpha) \quad \forall k \geq 2. \tag{15}$$

With this set of differential equations, the dynamics of the Green's function G^r is completely determined by the functions γ_1^r and β_1. Note that the derived equations imply the celebrated Callan–Symanzik equation

$$\left(\partial_L + \beta\alpha\partial_\alpha + \gamma^r\right) G^r = 0 \tag{16}$$

where we identified the first coefficient function in our expansion with the anomalous dimension $\gamma^r = -\gamma_1^r$ and the beta function $\beta = -\beta_1$ with appropriate signs. The Callan–Symanzik equation might be read as follows: a change in the reference scale (that is in L) can be compensated by a change of the coupling parameter α and the amplitude of G^r as described by the anomalous dimension γ^r.

Another consequence of the differential equations (14) and (15) pertains to the scheme independence of so-called fixed points. Following the usual terminology, a fixed point α_* is defined to be a point of the zero locus of the beta function that is $\beta(\alpha_*) = 0$. As an immediate consequence of (15), α_* being a fixed point is equivalent to the condition

$$\mathrm{ev}_{\alpha_*}\, \mathbf{\Phi}^R_{(S,S_0,\theta,\theta_0)}(Q) = 1. \tag{17}$$

Here, ev_{α_*} denotes the evaluation of the coupling parameter α at α_*. Further, it is worth emphasizing that a priori this identity is only satisfied for a particular choice of the renormalization point $(S_0, \{\theta_0\})$, that is a particular renormalization scheme. Whereas the beta function generally does depend

on the choice of a renormalization scheme, there is a broad agreement that the fixed points are scheme independent in the physics literature. Here, we provide a proof of this statement for a special class of renormalization schemes.

Proposition 2.1. *Within the class of kinematic renormalization schemes, as defined by* (10)*, the fixed points do not depend on the choice of the renormalization point.*

In other words, if the fixed point α_* obeys (17) at some renormalization point $(S_0, \{\theta_0\})$, then α_* satisfies equation (17) at any renormalization point.

Proof. We assume α_* to satisfy (17) and show that α_* also evaluates to unity when the renormalization point is changed to $(S_1, \{\theta_1\})$.

First, we notice that in a kinematic scheme the renormalized Feynman rules (10) at the renormalization point $(S_1, \{\theta_1\})$ are related to the renormalization point $(S_0, \{\theta_0\})$ by

$$\mathbf{\Phi}^R_{(S,S_1,\theta,\theta_1)} = \left(\mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)}\right)^{-1} \star \mathbf{\Phi}^R_{(S,S_0,\theta,\theta_0)},$$

which can be easily derived from the convolution formula (11). Next recall that the target algebra is commutative; this implies that the convolution inverse is given by concatenation with the antipode (see e.g. proposition II.4.1 in [16])

$$\left(\mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)}\right)^{-1} = \mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)} \circ S. \tag{18}$$

Now, we use the coproduct formula of the invariant charge (9) to evaluate the convolution

$$\mathbf{\Phi}^R_{(S,S_1,\theta,\theta_1)}(Q) = \sum_{k\geq 0} \left(\mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)}\right)^{-1}(Q^{k+1})\ \mathbf{\Phi}^R_{(S,S_0,\theta,\theta_0)}(Q|_k). \tag{19}$$

As the renormalized Feynman rules satisfy the character property, i.e. $\mathbf{\Phi}^R(\Gamma_1\Gamma_2) = \mathbf{\Phi}^R(\Gamma_1)\mathbf{\Phi}^R(\Gamma_2)$, it is sufficient to show that $\mathrm{ev}_{\alpha_*}(\mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)})^{-1}(Q) = 1$.

This can be shown by the following derivation.

$$\begin{aligned}
\mathrm{ev}_{\alpha_*} \circ \left(\mathbf{\Phi}^R_{S_1,S_0,\theta_1,\theta_0}\right)^{-1}(Q) &= \mathrm{ev}_{\alpha_*} \circ \mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)} \circ S(Q) \\
&= \mathrm{ev}_{\alpha_*} \circ \mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)} \left(\sum_{k\geq 0} 1 \cdot S(Q|_k) \right) \\
&= \mathrm{ev}_{\alpha_*} \circ \mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)} \left(\sum_{k\geq 0} Q^{k+1} S(Q|_k) \right) \\
&= \mathrm{ev}_{\alpha_*} \circ \mathbf{\Phi}^R_{(S_1,S_0,\theta_1,\theta_0)} \circ (\mathrm{id} \star S)(Q) \\
&= 1
\end{aligned}$$

The first equality is due to (18). In the second equality, we expanded the invariant charge and multiplied a factor 1, which is replaced in the third equality due to $1 = \mathrm{ev}_{\alpha_*} \mathbf{\Phi}^R_{S,S_0,\theta,\theta_0}(Q^{n+1})$, which holds for all scales and angles $(S, \{\theta\})$, and the character property of the renormalized Feynman rules. The forth equality uses

$$(\mathrm{id} \star S)(Q) = \sum_{k\geq 0} Q^{k+1} S(Q|_k) \tag{20}$$

which follows from the coproduct formula for the invariant charge (9). Finally, make use of the fact that the antipode is the convolution inverse of the identity, hence $\mathrm{id} \star S$ the identity with respect to the convolution product which is well-known to vanish except for the subspace spanned by $\mathbb{1}$ on which it acts as identity (see e.g. proposition II.3.1 in [16]). Therefore, we have $(\mathrm{id} \star S)(Q) = \mathbb{1}$.

Finally, applying the last derivation shows that (19) evaluates to unity when the coupling is evaluated at the fixed point α_*; this finishes the proof. □

Thanks to the proposition 2.1, at a fixed point, the invariant change evaluates to unity in any kinematic renormalization scheme. Further, coefficients of the invariant charge generate a Hopf subalgebra, which can be seen from the coproduct formula for the invariant charge (9) and an expansion of (20). Therefore, the fixed point condition can algebraically implemented by considering the quotient Hopf algebra $H/\langle Q - \mathbb{1}\rangle$ where the quotient $\langle Q - \mathbb{1}\rangle$ denotes the algebra generated by the coefficients of the invariant charge Q except for $Q|_0 = \mathbb{1}$.

We conclude that at the fixed point, the coproduct of the Green's function $X^r(\alpha)$ becomes grouplike

$$\Delta X^r = X^r \otimes X^r \tag{21}$$

and, following the preceding discussion of the renormalization flow, the Callan–Symanzik equation simplifies to

$$(\partial_L + \gamma^r)\, G^r = 0. \tag{22}$$

This simple partial differential equation is solved by so-called scaling solutions

$$G^r(L, \alpha) = \exp\left(-L\gamma^r(\alpha)\right) = \left(S/S_0\right)^{-\gamma^r(\alpha)} \tag{23}$$

which we expect to describe the physical phenomena at a fixed point.

3 Linear Dyson–Schwinger equations

This paragraph gives a short account on how to solve a single linear Dyson–Schwinger equation. This discussion is useful to establish our conventions and to gain a first insight about the solution theory. Here, we basically recall the result of Kreimer [6]. This will be generalized to systems of linear Dyson–Schwinger equations in the next paragraph.

Let (Γ, e) be the ordered pair of a skeleton graph Γ and a distinguished internal edge of this skeleton $e \in I_\Gamma$ that matches the residue of Γ, i.e. $e = \mathrm{res}(\Gamma)$. The distinguished edge is meant to serve as an insertion place for graphs that have a matching residue. As the skeleton Γ itself satisfies this requirement, it can be inserted into itself. Further, such a chain of skeletons can again be inserted into the skeleton graph and so on. This iterative procedure of chaining the skeleton is formalized by defining the shift operator

$$B_+ : H \to H \tag{24}$$

which takes a graph from the Hopf algebra H as argument and inserts it into the distinguished edge of the skeleton Γ provided that the argument has the residue $\mathrm{res}(\Gamma)$, otherwise the shift operator is defined to map the argument to zero. Now, a set of coefficients in the Hopf algebra of graphs H is defined by setting $c_0 = \mathbb{1}$ and using the shift operator $c_n = B_+ \circ \cdots \circ B_+(c_0)$ with n times B_+. In other words, c_n is the chain that is built from n skeletons

Γ. It was noticed (see [6] and references therein) that the shifting operator acts on the set of these coefficients as a Hochschild one-cocycle

$$\Delta \circ B_+ = (B_+ \otimes \mathbb{1}) + (\mathrm{id} + B_+) \circ \Delta. \tag{25}$$

This easily implies a closed coproduct formula for the coefficients

$$\Delta c_i = \sum_{j=0}^{i} c_j \otimes c_{i-j} \tag{26}$$

and proves that the coefficients are spanning a Sub-Hopf algebra of the connected Hopf algebra H. From the physical perspective, this condition ensures that the set of graphs given by means of the coefficients c_n is renormalizable (in terms of Z-factors) – therefore, the formal series generated by these coefficients is a renormalizable truncation of the full perturbation series and hence a reasonable object to study. So let us define

$$X(\alpha) = \sum_{i \geq 0} c_i \alpha^i \tag{27}$$

where α denotes the coupling parameter. Note that the coproduct formula (26) from above resurrects in the form

$$\Delta X = X \otimes X. \tag{28}$$

Following the physical interpretation as discussed in [10], this formula allows us to anticipate the outcome of our succeeding discussion: the truncated perturbation series X does not require a renormalization of the coupling α; only its scaling dimension is required to be adjusted. Consequently, the series X is expected to yield scaling solutions after employing Feynman rules. At the end of this paragraph, we will explicitly derive the perturbation series for one specific example and confirm the anticipated result. For now, let us continue the discussion and note that the perturbation series satisfies the following equation

$$X = \mathbb{1} + \alpha B_+(X). \tag{29}$$

This type of equation is the central object of our discussion; it is a combinatorial Dyson–Schwinger equation that can be considered to iteratively define the perturbation series X. However, in contrast to the Dyson–Schwinger equation that describes the full perturbation series, the argument of the

B_+ operator is a linear expression in the series X, therefore, it is called a linear Dyson–Schwinger equation.

In the following, the Feynman rules of the skeleton is abbreviated by the integral

$$\mathbf{\Phi}(\Gamma) = \int \Omega_\Gamma \tag{30}$$

over the differential form Ω_Γ associated to the Feynman graph Γ. In this way, we find a simple law for the commutation of the Feynman rules and the shift operator $\mathbf{\Phi} \circ B_+(\cdot) = \int(\Omega_\Gamma \mathbf{\Phi}(\cdot))$, where the output of the Feynman rules is also integrated over in the right-hand side of the equation. Now, the Green's function associated to the truncated perturbation series is defined by $G(\alpha) = \mathbf{\Phi}(X(\alpha))$. Applying the above commutation law, the linear Dyson–Schwinger equation is turned into the linear integral equation

$$G(\alpha) = \mathbf{\Phi}(X(\alpha)) = 1 + \alpha \int \Omega_\Gamma G(\alpha) \tag{31}$$

with the Green's function $G(\alpha)$ as the unknown. This is the equation we are interested in solving. Of course, an exact solution to this type of problem requires us to determine the truncation in use und hence we need to provide a concrete description of the skeleton and its Feynman rules.

As an example, we consider a one-loop Euclidean one-scale scalar Feynman graph at four dimensional spacetime – in other words the considered graph consists of two vertices that are connected by two edges. As the associated integral is logarithmically divergent, a regularization schemes is required. Recall that in this paper, we will use analytic regularization at the distinguished edge which turns out to be well compatible with the consequent integration in the equation. These conventions for the skeleton Γ result in the following expression

$$\mathbf{\Phi}_{q^2}(\Gamma) = \int \frac{d^4k}{\pi^2} \frac{1}{(k^2)^{1+\rho}(k+q)^2} = \frac{1}{\rho(1-\rho)}(q^2)^{-\rho}, \tag{32}$$

where ρ denotes the analytic regulator. The physical limit is given by $\rho \to 0$ and the pole in the final expression exhibits the logarithmic divergence of the integral. For the following discussion, it is useful to separate the kinematic scaling factor $(q^2)^{-\rho}$ and define the Laurent series

$$f(\rho) = \frac{1}{\rho(1-\rho)} = (q^2)^\rho \mathbf{\Phi}_{q^2}(\Gamma), \tag{33}$$

which determines the analytic expression of the skeleton up to its trivial kinematic behaviour.

Now, we can start to construct solutions of the linear integral equation (31). However, as the considered skeleton gives rise to an unrenormalized expression, a solution of this integral equation also is an unrenormalized Green's function. To construct a renormalized Green's function, we need to treat the divergence that arises from the skeleton integrand. A fairly easy approach to this issue is to improve the convergence behaviour by subtracting a suitable term from the integrand. Here, such as subtrahend is constructed by evaluating the external kinematic data of the integrand at the renormalization point μ. In this way, the renormalized integral vanishes when the external momentum is evaluated at the renormalization point $q^2 = \mu^2$ – hence a physical boundary condition is implemented by performing kinematic renormalization. This results in the following integral equation

$$G(\alpha, q, \mu) = 1 + \alpha \left\{ \int \frac{d^4k}{\pi^2} \frac{G(\alpha, k, \mu)}{(k^2)^{1+\rho}(k+q)^2} \right\}_{q^2=\mu^2}^{q^2}. \tag{34}$$

To proceed with the construction of a solution of this renormalized linear integral equation, scaling solutions of the type

$$G(\alpha; q, \mu) = \left({}^{q^2}/_{\mu^2}\right)^{-\gamma(\alpha)} \tag{35}$$

are considered; firstly because this type of ansatz satisfies the boundary condition $G(\alpha, \mu, \mu) = 1$ and secondly as scaling solutions have been anticipated from our considerations concerning the subdivergence structure as in equation (28). Note that identifications of the scale $S = q^2$ and the renormalization point $S_0 = \mu^2$, reproduces the scaling solutions as predicted by (23). The unknown $\gamma(\alpha)$ is usually called anomalous (scaling) dimension and needs to be determined. Insertion of the ansatz into the integral equation yields that the ansatz is a solution iff the anomalous dimension satisfies the condition

$$1 = \alpha f(\gamma(\alpha)). \tag{36}$$

In our concrete example for the skeleton Γ, this derives to a quadratic equation in the anomalous dimensional, which is easily solved by the following expression

$$\gamma(\alpha) = \frac{1}{2}\left(1 - \sqrt{1 - 4\alpha}\right) \tag{37}$$

where the negative root has to be chosen due to the condition $G(0, q, \mu) = 1$. Finally, the anomalous dimension $\gamma(\alpha)$ in combination with the scaling ansatz (35) is a solution of the renormalized linear integral equation and hence the sought truncated Green's function.

The fact that we have been able to derive this closed expression for the anomalous dimension crucially depends on the specific shape of the Laurent series (33). However, the scaling ansatz is general enough to solve the renormalized integral equation in a much more general setting.

Proposition 3.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function with a pole of first order at $\rho = 0$, denote its Laurent series as*

$$f(\rho) = \sum_{j \geq -1} f_j \rho^j \tag{38}$$

and define an anomalous dimension as the formal power series with complex coefficients

$$\gamma(\alpha) = \sum_{j \geq 1} \gamma_j \alpha^j \tag{39}$$

which obeys $\gamma(0) = 0$ and $\alpha f(\gamma(\alpha)) = 1$. Then, the first coefficient of the anomalous dimension is determined by the residuum of the meromorphic function, that is $\gamma_1 = f_{-1}$, and the consecutive coefficients are recursively determined by

$$\gamma_{n+1} = \sum_{1 \leq j \leq n} f_{j-1} \sum_{n_1 + \cdots + n_j = n} \gamma_{n_1} \cdots \gamma_{n_j} \quad \text{for} \quad n \geq 1. \tag{40}$$

Proof. This is a ready consequence of Faà di Bruno's formula and proposition 4 of [6]. □

To conclude, we have argued that the renormalized Dyson–Schwinger equation is solved by scaling solutions, which are characterized by their anomalous dimension $\gamma(\alpha)$. This anomalous dimension is completely determined by equation (36) once the Laurent series $f(z)$ of the analytically regularized skeleton is known.

4 Systems of linear Dyson–Schwinger equations

This section generalizes the above discussion of a single linear Dyson–Schwinger equation to systems.

As we have seen in the preceding paragraph, a linear Dyson–Schwinger equation is solved by a scaling solution. The respective anomalous (scaling) dimension is determined by the Laurent series of the analytically regularized skeleton graph, which is used to define the truncated perturbation series.

The motivation to study systems of linear Dyson–Schwinger equation is to lose the restriction to a single skeleton, a single insertion place, and a single residue. So let $\Gamma_1, \ldots, \Gamma_N$ be skeleton graphs of different residues and use the convention $\mathrm{res}(\Gamma_i) = i$. Further, each skeleton Γ_i is paired with a list of internal edges $(e_1, \ldots, e_N)$ where $e_j \in I_{\Gamma_i}$ and $\mathrm{res}(e_i) = i$. As in the case of a single skeleton, the internal edges are considered as insertion places. Hence given a skeleton $(\Gamma_i, (e_1, \ldots, e_N))$, another skeleton Γ_j can be inserted into Γ_i at the edge e_j; this operation defines a set of shifting operators

$$B_{ij} : H \to H \tag{41}$$

which, again, vanishes whenever its argument has a residue that is not compatible (i.e. of type j) in straight analogy to the single skeleton case.

For the rest of this article, the following system of linear Dyson–Schwinger equations is examined

$$X_j(\alpha) = \mathbb{1} + \alpha \sum_{1 \leq k \leq N} B_{jk}(X_k(\alpha)) \quad \text{for} \quad j = 1, \ldots N. \tag{42}$$

where we restrict ourselves to a single perturbative order parameter α. As in the single skeleton case, such a system recursively defines an unique set of formal power series $X_1(\alpha), \ldots, X_N(\alpha)$, which we call the solution of the system and consider it as a truncation of the physical perturbation series.

The consecutive steps are again: first apply the Feynman rules $\mathbf{\Phi}$ to the system in order to convert it into a set of linear integral equations, second renormalize by subtracting each integrand of the skeletons at a common renormalization point μ, third solving this set of renormalized linear integral equations

$$G_j(\alpha, q, \mu) = 1 + \alpha \sum_{1 \leq k \leq N} \left[\int \Omega_{\Gamma_j} G_k(\alpha, p_k, \mu) \right]. \tag{43}$$

Here, G_j is the renormalized Green's function, p_j is the momentum assigned to the edge $e_k \in I_{\Gamma_j}$, and the brackets denote the kinematic subtraction at $q^2 = \mu^2$, for some $\mu \in \mathbb{R}$ which is called renormalization point. As in the case of a single equation, we expect the solution to be determined by analytic

expressions of skeletons. Therefore, we define the following set of Laurent series

$$\mathbf{\Phi}_{q^2} \circ B_{ij}(\mathbb{1}) = (q^2)^{-\rho} f_{ij}(\rho), \tag{44}$$

where the index j expresses that the edge e_j has been chosen to be analytically regularized and to introduce the ρ dependence in the analytic expression. This data gives rise to a $N \times N$ Matrix $(f_{jk}(\rho))$ which we will refer to as the skeleton matrix. More generally, matrices will be represented by capitalized characters in following. In this convention, the entries of the skeleton matrix read $F_{jk} = f_{jk}$. Now, in the construction of solutions, two cases have to be distinguished: first the canonical case, that is the skeleton matrix is diagonalizable, and second the degenerated case, that is the skeleton matrix is defective.

4.1 The canonical case

Firstly let us assume that the $N \times N$ matrix of the analytic expressions of the analytically regularized skeletons is diagonalizable

$$F(\rho) = A(\rho) \begin{bmatrix} \lambda_1(\rho) & & \\ & \ddots & \\ & & \lambda_N(\rho) \end{bmatrix} A^{-1}(\rho) \tag{45}$$

with eigenvalues $\lambda_i(\rho)$, $i = 1, \ldots, N$ and an invertible matrix $A(\rho)$. By assumption, every entry of $F(\rho)$ is a Laurent series with a pole of at most order one at $\rho = 0$. As implicit functions of the characteristic polynomial with analytic coefficients, the eigenvalues λ_i are analytic functions at some disk around $\rho = 0$. At $\rho = 0$, their have a pole of first order at most which can be seen as follows. First notice that the modulus of the eigenvalue

$$|\lambda(\rho)| = \frac{\|\lambda(\rho)\, v(\rho)\|}{\|v(\rho)\|} = \frac{\|F(\rho)\, v(\rho)\|}{\|v(\rho)\|} \leq \|F(\rho)\| \tag{46}$$

is bounded by the sum of the modulus of each entry of the matrix F. Therefore,

$$\rho^2 |\lambda(\rho)| \leq \|\rho^2 F(\rho)\| \to 0 \quad \text{for} \quad \rho \to 0. \tag{47}$$

In other words, the eigenvalues do not have poles of order two or higher.

Consequently, a diagonalization of the system (45) does not alter the pole order of the Laurent series and, after diagonalization, the system can

be considered as N independent linear Dyson–Schwinger equations, which are solved by scaling solutions as demonstrated above. The consecutive task is now to construct an appropriate superposition of scaling solutions.

Proposition 4.1. *For $i = 1, \dots, N$, define the anomalous dimension γ_i to satisfy $\alpha\lambda_i(\gamma_i(\alpha)) = 1$ and $\gamma_i(0) = 0$, then the integral equation* (43) *is solved by the following set of scaling solutions*

$$G_j(\alpha, q, \mu) = \sum_{k=1}^{N} \widetilde{A}_{jk} \left(\frac{q^2}{\mu^2}\right)^{-\gamma_k(\alpha)} \tag{48}$$

where the coefficients are determined by means of the transformation matrix (45) *and the anomalous dimensions: $\widetilde{A}_{jk} = \sum_{l=1}^{N} A_{jk}(\gamma_k(\alpha)) A^{-1}_{kl}(\gamma_k(\alpha))$.*

Before providing the proof of this proposition, we would like to remark that the existence of the anomalous dimensions γ_i, as mentioned in the assumption, can be deduced from the case of a single linear Dyson–Schwinger equation. In proposition 3.1, Equation (40) gives a recursive definition where the coefficients of the Laurent series of the skeleton graph is simply replaced by the Laurent series of an eigenvalue λ_i from the matrix (45) of the linear system.

Proof. First, note that the coefficients are normalized in the following sense

$$\sum_{1 \le k \le N} \widetilde{A}_{jk} = \sum_{1 \le k,l \le N} A_{jk} A^{-1}_{kl} = \sum_{1 \le l \le N} \delta_{jl} = 1. \tag{49}$$

Further, recall that the anomalous dimensions satisfy $\gamma(0) = 0$ by definition. Therefore, the ansatz obeys the normalization

$$G_j(0, q, \mu) = 1. \tag{50}$$

That is the ansatz (48) fulfills the linear integral equation (43) at the order α^0. In order to show that that the integral equation is satisfied to all orders,

we first prove the following useful equality.

$$\begin{aligned}\alpha\sum_{k=1}^{N}F_{jk}(\gamma_l)\widetilde{A}_{kl} &= \alpha\sum_{k,m=1}^{N}F_{jk}(\gamma_l)A_{kl}(\gamma_l)A^{-1}_{lm}(\gamma_l)\\ &= \alpha\sum_{k,m,n=1}^{N}A_{jn}(\gamma_l)\lambda_n(\gamma_l)A_{nk}(\gamma_l)A_{kl}(\gamma_l)A^{-1}_{lm}(\gamma_l)\\ &= \sum_{m=1}^{N}A_{jl}(\gamma_l)\underbrace{\alpha\lambda_l(\gamma_l)}_{=1}A^{-1}_{lm}(\gamma_l) = \widetilde{A}_{jl}\end{aligned} \tag{51}$$

In the last line, we made use of the defining property of the anomalous dimensions $\alpha\lambda_l(\gamma_l(\alpha)) = 1$ (see proposition 3.1). Now, using this identity, the rest of the proof consists of verifying that the ansatz (48) satisfies the integral equation (43).

$$\begin{aligned}\alpha\sum_{k=1}^{N}\left\{\int\Omega_{\gamma_j}G_k(\alpha,p_k,\mu)\right\} &= \alpha\sum_{k,l=1}^{N}\widetilde{A}_{kl}\left\{\int\Omega_{\gamma_j}\left(\frac{p_k^2}{\mu^2}\right)^{-\gamma_l(\alpha)}\right\}\\ &= \alpha\sum_{k,l=1}^{N}\widetilde{A}_{kl}F_{jk}(\gamma_l(\alpha))\left[\left(\frac{q^2}{\mu^2}\right)^{-\gamma_l(\alpha)}-1\right]\\ &= \sum_{l=1}^{N}\widetilde{A}_{jl}\left[\left(\frac{q^2}{\mu^2}\right)^{-\gamma_l(\alpha)}-1\right] = G_j(\alpha,q,\mu)-1\end{aligned}$$

Here, the p_k is again the momentum assigned to the edge $e_k \in I_{\gamma_j}$ and the braces in the first line are meant to indicate the subtraction of the integrand at the renormalization point $q^2 = \mu^2$. For the last step, we have used the identity (51). We conclude that the scaling ansatz (48) solves the linear integral equation.

□

4.2 The degenerated case

Now it remains to discuss systems with a skeleton matrix which is not diagonalizable. A general degenerated system decouples in separate systems of Jordan blocks. Here, we provide a solution for the Jordan block of size M and afterwards discuss how the general degenerated case relates to these

solutions. Now assume the skeleton matrix to be the $M \times M$ Jordan block

$$F(z) = \begin{bmatrix} \lambda(z) & & & & \\ 1 & \lambda(z) & & & \\ & 1 & \lambda(z) & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda(z) \end{bmatrix}. \tag{52}$$

In this set up the first integral equation decouples and we immediately recognize a solution of the first equation from the previous discussion

$$G_1(\alpha, q, \mu) = (s/\mu)^{-\gamma(\alpha)} \tag{53}$$

where the anomalous dimension $\gamma(\alpha)$ obeys $\alpha\lambda(\gamma(\alpha)) = 1$ and is hence determined by proposition 3.1.

Prior to solving the other equations of the degenerated system, we prove a brief lemma that will be useful for the construction of solutions. First, let us define some convenient notation. Given a Laurent series $F \in \mathbb{C}((\alpha))$, the commutator of the k-times derivative with respect to α and the F is defined by its natural action on some Laurent series

$$\left[\partial_\alpha^k, F\right] G = \partial_\alpha^k (FG) - F\left(\partial_\alpha^k G\right) \quad \forall G \in \mathbb{C}((\alpha)). \tag{54}$$

For the sake of a concise notation, the Laurent series G will be suppressed in the following.

Lemma 4.2. *Let $(p_j)_{j\geq 1}$ be the sequence of polynomials*

$$p_j(X) = \sum_{k=0}^{j} c_{jk} X^k \quad \textit{where} \quad c_{jk} \in \mathbb{C}[\alpha], \tag{55}$$

that obey the following properties

1. $\deg(p_j) = j - 1 \quad \forall j \geq 1$
2. $p_j(0) = 1 \quad \forall j \geq 1$
3. $\left[p_{j+1}(\partial_\alpha), \frac{1}{\alpha}\right] = -p_j(\partial_\alpha)$

then, the coefficients $c_{jj} = 0$ and $c_{j0} = 1$ for all $j \geq 1$ and the remaining coefficients are determined by the recursion

$$c_{j+1k} = \alpha\left(c_{jk} + \frac{\alpha}{k} c_{jk-1}\right) \quad \forall j \geq 1, k \geq 1. \tag{56}$$

As this sequence is utilized in the construction of solutions, we like to list the first three polynomials of the sequence for the convenience of the reader.

$$p_1(X) = 1 \tag{57}$$

$$p_2(X) = 1 + \alpha^2 X \tag{58}$$

$$p_3(X) = 1 + \alpha^2(1+\alpha)X + \frac{\alpha^4}{2}X^2 \tag{59}$$

Also note that the identity $c_{jj} = 0$ and the recursion (56) determines the highest order coefficients by induction to be

$$c_{j+1j} = \frac{\alpha^{2j}}{j!}. \tag{60}$$

It follows the proof of the lemma.

Proof. First, the commutator of k times the derivative with respect to α and $1/\alpha$ reads

$$\left[\partial_\alpha^k, \frac{1}{\alpha}\right] = \sum_{l=0}^{k-1} \binom{k}{l} \left(\partial_\alpha^{k-l}\frac{1}{\alpha}\right) \partial_\alpha^l = \sum_{l=0}^{k-1} a_{kl}\partial_\alpha^l, \tag{61}$$

where we have introduced coefficients $a_{kl} := \frac{k!\alpha^l}{l!\alpha^{1+k}}$. Next, the commutator from the left-hand side of property (3) is derived

$$\left[p_{j+1}(\partial_\alpha), \frac{1}{\alpha}\right] = \sum_{k=1}^{j} c_{j+1k}\left[\partial_\alpha^k, \frac{1}{\alpha}\right] = \sum_{1\le k\le j}\ \sum_{0\le l\le k-1} c_{j+1k}a_{kl}\partial_\alpha^l. \tag{62}$$

Comparison with the right-hand side of property (3), that is with the coefficients of $-p_j(\partial_\alpha)$ yields the following restriction for the coefficients

$$-c_{jl} = \sum_{l+1\le k\le j} c_{j+1k}a_{kl}\partial_\alpha^l. \tag{63}$$

Now, changing the set of variables by defining $\widetilde{c}_{jk} := c_{jk}l!/(-\alpha)^l$, this equation reads

$$-\alpha\widetilde{c}_{jl} = \sum_{k=l+1}^{j} \widetilde{c}_{j+1k} \quad \forall j\ \forall l = 0, \ldots, j-1. \tag{64}$$

It is useful to visualize the equation in a matrix notation.

$$-\alpha \begin{bmatrix} \widetilde{c}_{jj-1} \\ \widetilde{c}_{jj-2} \\ \vdots \\ \widetilde{c}_{j0} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \widetilde{c}_{j+1j} \\ \widetilde{c}_{j+1j-1} \\ \vdots \\ \widetilde{c}_{j+11} \end{bmatrix} \tag{65}$$

Now, by inverting the matrix on the right-hand side, a recursive description of the coefficients is readily derived.

$$\begin{bmatrix} \widetilde{c}_{j+1j} \\ \widetilde{c}_{j+1j-1} \\ \vdots \\ \widetilde{c}_{j+11} \end{bmatrix} = -\alpha \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{c}_{jj-1} \\ \widetilde{c}_{jj-2} \\ \vdots \\ \widetilde{c}_{j0} \end{bmatrix}. \tag{66}$$

Finally, it is easy to see that this matrix equation is equivalent to the recursion (56).

□

Proposition 4.3. *A solution for the Dyson–Schwinger system* (43) *associated to the Jordan block* (52) *is solved by*

$$G_j(\alpha, q, \mu) = p_j(\partial_\alpha) \left(\frac{q^2}{\mu^2}\right)^{-\gamma(\alpha)} \qquad j = 1, \cdots, M \tag{67}$$

where $p_j(X)$ is the jth polynomial of the sequence defined in lemma 4.2 and the anomalous dimension γ is determined by $\alpha\lambda(\gamma(\alpha)) = 1$ and $\gamma(0) = 0$ (see proposition 3.1).

Proof. The proof consists of a simple evaluation of all the integrals of the Dyson–Schwinger system. Note that there is no summation over repeated indices indented if not explicitly denoted otherwise.

$$\begin{aligned}
\alpha \left\{ \int \Omega_{\Gamma_j} G_j(\alpha, p_j, \mu) \right\} &= \alpha p_j(\partial_\alpha) \underbrace{F_{jj}(\Gamma(\alpha))}_{=1/\alpha} \left[\left(\frac{q^2}{\mu^2}\right)^{-\gamma(\alpha)} - 1 \right] \\
&= p_j(\partial_\alpha) \left[\left(\frac{q^2}{\mu^2}\right)^{-\gamma(\alpha)} - 1 \right] + \alpha \underbrace{\left[p_j(\partial_\alpha), \frac{1}{\alpha} \right]}_{=-p_{j-1}(\partial_\alpha)} \left[\left(\frac{q^2}{\mu^2}\right)^{-\gamma(\alpha)} - 1 \right] \\
&= G_j(\alpha, q, \mu) - 1 - \alpha \left[G_{j-1}(\alpha, q, \mu) - 1 \right]
\end{aligned}$$

Here, we used the defining property of the anomalous dimension in the first line and exploited the third property of the polynomial sequence from lemma 4.2. The next non-trivial integral evaluates as follows

$$\begin{aligned}\alpha\left\{\int\Omega_{\Gamma_j}G_{j-1}(\alpha,p_j,\mu)\right\} &= \alpha p_{j-1}(\partial_\alpha)\underbrace{F_{jj-1}(\gamma(\alpha))}_{=1}\left[\left(\frac{q^2}{\mu^2}\right)^{-\gamma(\alpha)}-1\right]\\ &= \alpha\left[G_{j-1}(\alpha,q,\mu)-1\right].\end{aligned}$$

The remaining integrals vanish thanks to the fact that the Jordan block (52) is a sparse matrix

$$\alpha\left\{\int\Omega_{\Gamma_j}G_k(\alpha,p_k,\mu)\right\}=0\quad\text{for}\quad k\neq j,j-1.$$

Adding these expression together, we conclude that the ansatz (67) solves the Dyson–Schwinger integral system

$$\sum_{k=1}^{N}\alpha\left\{\int\Omega_{\Gamma_j}G_k(\alpha,p_k,\mu)\right\}=G_j(\alpha,q,\mu)-1.$$

□

Remark. *Now, if we consider a skeleton matrix that is similar to a Jordan block*

$$F(z)=S\widetilde{F}(z)S^{-1}\tag{68}$$

where $\widetilde{F}$ *is a Jordan block of the form* (52)*, then a solution of the system* (43) *is given by*

$$G(\alpha,q,\mu)=S\widetilde{G}(\alpha,q,\mu).\tag{69}$$

Here, $\widetilde{G}_j=\widetilde{p}_j(\partial_\alpha)(q^2/\mu^2)^{-\gamma(\alpha)}$ *and* $(\widetilde{p}_j)_{j\geq 0}$ *is a sequence of polynomials constructed with the recursion* (56) *following lemma 4.2 except for a replacement of the second condition by* $\widetilde{p}_j(0)=\sum_{k=1}^{M}S^{-1}_{jk}$*.*

Remark. *The solutions of the Jordan block of size two read*

$$G_1(L,\alpha)=\exp\left(-L\gamma(\alpha)\right),\quad G_2(L,\alpha)=\left(1-\alpha^2\gamma'(\alpha)L\right)G_1(L,\alpha).\tag{70}$$

The additional factor in G_2 *is due to the polynomial* p_2 *which has been worked out in* (58)*. However, an examination of these Green's functions*

shows that G_2 does not satisfies the Callan–Symanzik equation on or off the fixed point

$$(\partial_L + \gamma(\alpha))\, G_1(L,\alpha) = 0, \tag{71}$$

$$(\partial_L + \gamma(\alpha))\, G_2(L,\alpha) = -\alpha^2\gamma'(\alpha) G_1(L,\alpha) \neq 0. \tag{72}$$

From this we deduce that a degenerated skeleton matrix does not describe the physical behavior of the renormalization flow and will be excluded in discussion of physical scenarios in the conclusion.

5 Conclusion

The interplay of the Hopf algebra structure of Feynman graphs and combinatorial Dyson–Schwinger equations allows for remarkable insights into the behavior of Green's functions under the renormalization flow. In proposition 2.1, we showed that fixed points are scheme independent in kinematic renormalization schemes.

Further, we constructed solutions for systems of linear Dyson–Schwinger equations. As seen in previous work [6], a single linear Dyson–Schwinger equation is solved by a scaling solution (35). A superposition of these scaling solution solves a system only if the associated skeleton matrix is diagonalizable—see proposition 4.1. Solutions for a degenerated case have been constructed in proposition 4.3. Whereas a superposition of scaling solutions satisfies the Callan–Symanzik equation with a vanishing betafunction, the degenerated solutions does not allow for an universal beta function is hence not compatible with the Callan–Symanzik equations. Therefore, degenerated systems are exclude for further investigations of physical theories.

Conclusively, we expect systems of linear Dyson–Schwinger equations with a non-degenerate skeleton matrix to describe physical theories that feature scale invariance which is not broken by the renormalization process. Prominent examples of field theories that fit the concept of scale invariance and feature a vanishing beta function are supersymmetric models such as $N = 4$ super Yang–Mills theory or the quenched approximation of quantum electrodynamics.

However, in our discussion of the scheme invariance of fixed points in section 2, it transpired that, at a fixed point α_*, the invariant charge trivializes $Q(\alpha_*) = \mathbb{1}$. Therefore, the ideal generated by the non-trivial coefficients of the invariant charge can be divided out which implements the fixed point condition combinatorially into the Hopf algebra of Feynman graphs. In this

quotient space, Green's function are evaluated at the fixed point and obey the scale invariant version of the Callan–Symanzik equation (22). Hence, we expect systems of linear Dyson–Schwinger equations also to apply in physical scenarios without scale invariance when considered at a fixed point. This observation might be of central interest due to the recent results for critical exponents at the Wilson–Fisher fixed point [17, 18, 19, 20] where it was observed that universality establishes relations between various towers of field theories at different dimensions. Here, we expect the Hopf ideals of these varying theories together with linearized Dyson–Schwinger systems to shed light on the universality of the critical exponents.

6 Acknowledgments

The author likes to thank Dirk Kreimer, Karen Yeats, Eric Panzer, and Marco Berghoff for many valuable discussions pertaining to this work. It is acknowledged that parts of the results of this work were discussed in the master's thesis by the author of this paper[21]. This research was supported by Deutsche Forschungsgemeinschaft (DFG) through the grant KR 1401/5-1.

References

[1] D. Kreimer, Anatomy of a gauge theory, Annals Phys. 321 (2006) 2757–2781. arXiv:hep-th/0509135, doi:10.1016/j.aop.2006.01.004.

[2] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, Adv. Theor. Math. Phys. 2 (1998) 303–334. arXiv:q-alg/9707029.

[3] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. 1. The Hopf algebra structure of graphs and the main theorem, Commun. Math. Phys. 210 (2000) 249–273. arXiv:hep-th/9912092, doi:10.1007/s002200050779.

[4] D. Kreimer, K. Yeats, An Etude in non-linear Dyson-Schwinger Equations, Nucl. Phys. Proc. Suppl. 160 (2006) 116–121. arXiv:hep-th/0605096, doi:10.1016/j.nuclphysbps.2006.09.036.

[5] D. Kreimer, K. Yeats, Recursion and growth estimates in renormalizable quantum field theory, Commun. Math. Phys. 279 (2008) 401–427. arXiv:hep-th/0612179, doi:10.1007/s00220-008-0431-7.

[6] D. Kreimer, Etude for linear dyson-schwinger equations, Tech. Rep. IHES-P-2006-23, Inst. Hautes Etud. Sci., Bures-sur-Yvette (2006).

[7] N. Marie, K. Yeats, A chord diagram expansion coming from some Dyson-Schwinger equations, Commun. Num. Theor. Phys. 07 (2013) 251–291. arXiv:1210.5457, doi:10.4310/CNTP.2013.v7.n2.a2.

[8] M. Hihn, K. Yeats, Generalized chord diagram expansions of Dyson-Schwinger equationsarXiv:1602.02550.

[9] H. Kißler, Hopf-algebraic Renormalization of QED in the linear covariant Gauge, Annals Phys. 372 (2016) 159–174. arXiv:1602.07003, doi:10.1016/j.aop.2016.05.008.

[10] H. Kißler, *Computational and Diagrammatic Techniques for Perturbative Quantum Electrodynamics*, Ph.D. thesis, Humboldt-Universität zu Berlin (2017).

[11] C. Bergbauer, D. Kreimer, Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology, IRMA Lect. Math. Theor. Phys. 10 (2006) 133–164. arXiv:hep-th/0506190, doi:10.4171/028-1/4.

[12] J. C. Collins, Renormalization, Vol. 26 of Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1986.

[13] F. Brown, D. Kreimer, Angles, Scales and Parametric Renormalization, Lett. Math. Phys. 103 (2013) 933–1007. arXiv:1112.1180, doi:10.1007/s11005-013-0625-6.

[14] D. J. Broadhurst, D. Kreimer, Combinatoric explosion of renormalization tamed by Hopf algebra: Thirty loop Pade-Borel resummation, Phys. Lett. B475 (2000) 63–70. arXiv:hep-th/9912093, doi:10.1016/S0370-2693(00)00051-4.

[15] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. 2. The beta function, diffeomorphisms and the renormalization group, Commun. Math. Phys. 216 (2001) 215–241. arXiv:hep-th/0003188, doi:10.1007/PL00005547.

[16] D. Manchon, Hopf algebras, from basics to applications to renormalization, in: 5th Mathematical Meeting of Glanon: Algebra, Geometry and Applications to Physics Glanon, Burgundy, France, July 2-6, 2001, 2001. arXiv:math/0408405.

[17] J. A. Gracey, Six dimensional QCD at two loops, Phys. Rev. D93 (2) (2016) 025025. arXiv:1512.04443, doi:10.1103/PhysRevD.93.025025.

[18] J. A. Gracey, R. M. Simms, Higher dimensional higher derivative ϕ^4 theory, Phys. Rev. D96 (2) (2017) 025022. arXiv:1705.06983, doi:10.1103/PhysRevD.96.025022.

[19] J. A. Gracey, Renormalization of scalar field theories in rational space-time dimensions. arXiv:1703.09685.

[20] J. A. Gracey, R. M. Simms, Six dimensional Landau-Ginzburg-Wilson theory, Phys. Rev. D95 (2) (2017) 025029. arXiv:1701.03618, doi:10.1103/PhysRevD.95.025029.

[21] H. Kißler, On linear systems of Dyson–Schwinger equations, master's thesis, Humboldt-Universität zu Berlin (2012).