Sums of powers of the terms of Lucas sequences with indices in arithmetic progression

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Abstract

We evaluate the sums $\sum_{j=0}^k u_{rj+s}^{2n}\, z^j$, $\sum_{j=0}^k u_{rj+s}^{2n-1}\, z^j$ and $\sum_{j=0}^k v_{rj+s}^n\, z^j$, where r, s and k are any integers, n is any nonnegative integer, z is arbitrary and (u_n) and (v_n) are the Lucas sequences of the first kind and of the second kind, respectively. As natural consequences we obtain explicit forms of the generating functions for the powers of the terms of Lucas sequences with indices in arithmetic progression. This paper therefore extends the results of P. Staňică who evaluated $\sum_{j=0}^k u_j^{2n}\, z^j$ and $\sum_{j=0}^k u_j^{2n-1}\, z^j$; and those of B. S. Popov who obtained generating functions for the powers of these sequences.

1 Introduction

The Lucas sequences of the first kind, $(u_n(p,q))$, and of the second kind, $(v_n(p,q))$, with complex parameters p and q, are defined by

$$u_0 = 0,\; u_1 = 1;\; u_n = pu_{n-1} - qu_{n-2}, (n \ge 2); \tag{1.1}$$

and

$$v_0 = 2,\; v_1 = p;\; v_n = pv_{n-1} - qv_{n-2}, (n \ge 2). \tag{1.2}$$

The most well-known Lucas sequences are the Fibonacci sequence, $(f_n) = (u_n(1,-1))$ and the sequence of Lucas numbers, $(l_n) = (v_n(1,-1)$. Thus, the Fibonacci numbers, f_n, and the Lucas numbers, l_n are defined by:

$$f_0 = 0,\;\; f_1 = 1,\;\; f_n = f_{n-1} + f_{n-2}\; (n \ge 2) \tag{1.3}$$

and

$$l_0 = 2,\;\; l_1 = 1,\;\; l_n = l_{n-1} + l_{n-2}\; (n \ge 2)\,. \tag{1.4}$$

Denote by α and β, the zeros of the characteristic polynomial $x^2 - px + q$ of the Lucas sequences. Then

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}\,, \tag{1.5}$$

$$\alpha+\beta=p,\quad \alpha-\beta=\sqrt{p^2-4q}\quad \text{and } \alpha\beta=q\,. \tag{1.6}$$

The difference equations (1.1) and (1.2) are solved by the Binet-like formulas

$$u_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\quad v_n=\alpha^n+\beta^n\,. \tag{1.7}$$

It follows that

$$u_{2n}=\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}=\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)(\alpha^n+\beta^n)=u_nv_n\,. \tag{1.8}$$

Properties of Lucas sequences can be found in the book by Ribenboim [4, Chapter 1]. The Mathworld [7] and Wikipedia [8] articles are also good sources of information on the subject, with many references to useful materials. The books by Koshy [2] and by Vajda [6] are excellent reference materials on Fibonacci numbers and Lucas numbers.

Extension of the definitions of u_n and v_n to negative subscripts is provided by writing the recurrence relations as $u_{-n}=(pu_{-n+1}-u_{-n+2})/q$ and $v_{-n}=(pv_{-n+1}-v_{-n+2})/q$. Using the Binet-like formulas in (1.7) and the identities

$$\frac{x^{-n}-y^{-n}}{x-y}=-(xy)^{-n}\frac{x^n-y^n}{x-y} \tag{1.9}$$

and

$$x^{-n}+y^{-n}=(xy)^{-n}(x^n+y^n)\,, \tag{1.10}$$

with $x=\alpha$ and $y=\beta$, we see that

$$u_{-n}=-u_n/q^n\,,\quad v_{-n}=v_n/q^n\,. \tag{1.11}$$

As advertised in the abstract, our aim is to evaluate the sums $\sum_{j=0}^k u_{rj+s}^{2n}\,z^j$, $\sum_{j=0}^k u_{rj+s}^{2n-1}\,z^j$ and $\sum_{j=0}^k v_{rj+s}^{n}\,z^j$ and, naturally, the generating functions $\sum_{j=0}^\infty u_{rj+s}^{2n}\,z^j$, $\sum_{j=0}^\infty u_{rj+s}^{2n-1}\,z^j$ and $\sum_{j=0}^\infty v_{rj+s}^{n}\,z^j$. To this end we shall make use of the following identities:

Lemma 1. *The following identities hold for nonnegative integer n, integers r, s and k and arbitrary x, y and z:*

$$2\sum_{j=0}^k(x^{rj+s}-y^{rj+s})^{2n}z^j=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\sum_{j=0}^k(xy)^{(rj+s)i}(x^{2(n-i)(rj+s)}+y^{2(n-i)(rj+s)})z^j\,, \tag{1.12}$$

$$0=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\sum_{j=0}^k(xy)^{(rj+s)i}(x^{2(n-i)(rj+s)}-y^{2(n-i)(rj+s)})z^j\,, \tag{1.13}$$

$$2\sum_{j=0}^k(x^{rj+s}-y^{rj+s})^{2n-1}z^j=\sum_{i=0}^{2n-1}(-1)^i\binom{2n-1}{i}\sum_{j=0}^k(xy)^{(rj+s)i}(x^{(2n-1-2i)(rj+s)}-y^{(2n-1-2i)(rj+s)})z^j\,, \tag{1.14}$$

$$0=\sum_{i=0}^{2n-1}(-1)^i\binom{2n-1}{i}\sum_{j=0}^k(xy)^{(rj+s)i}(x^{(2n-1-2i)(rj+s)}+y^{(2n-1-2i)(rj+s)})z^j\,, \tag{1.15}$$

$$2\sum_{j=0}^k(x^{rj+s}+y^{rj+s})^{n}z^j=\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^k(xy)^{(rj+s)i}(x^{(n-2i)(rj+s)}+y^{(n-2i)(rj+s)})z^j\,, \tag{1.16}$$

$$0=\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{k}(xy)^{(rj+s)i}(x^{(n-2i)(rj+s)}-y^{(n-2i)(rj+s)})z^j\,. \tag{1.17}$$

Proof. By the binomial theorem and a change of the order of summation, we have

$$\sum_{j=0}^{k}(x^{rj+s}-y^{rj+s})^{2n}z^j=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\sum_{j=0}^{k}(xy)^{(rj+s)i}y^{(2n-i)(rj+s)}z^j\,. \tag{1.18}$$

Interchanging x and y in (1.18) gives

$$\sum_{j=0}^{k}(x^{rj+s}-y^{rj+s})^{2n}z^j=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\sum_{j=0}^{k}(xy)^{(rj+s)i}x^{(2n-i)(rj+s)}z^j\,. \tag{1.19}$$

Addition of (1.18) and (1.19) gives identity (1.12), while their subtraction gives identity (1.13). The proof of (1.14) — (1.17) is similar. □

We also require the sum of the terms of Lucas sequences with indices in arithmetic progression:

Lemma 2 ([1, Theorem 1])**.** *The following identities hold for integers r, k and s and arbitrary z:*

$$\sum_{j=0}^{k}u_{rj+s}z^j=\frac{q^r u_{rk+s}z^{k+2}-u_{rk+r+s}z^{k+1}+q^s u_{r-s}z+u_s}{q^r z^2-v_r z+1}\,, \tag{1.20}$$

$$\sum_{j=0}^{k}v_{rj+s}z^j=\frac{q^r v_{rk+s}z^{k+2}-v_{rk+r+s}z^{k+1}-q^s v_{r-s}z+v_s}{q^r z^2-v_r z+1}\,. \tag{1.21}$$

2 The main results

Theorem 1 (**Sums of powers of the terms of Lucas sequences with indices in arithmetic progression**)**.** *The following identities hold for integers r, s and k, nonnegative integer n and arbitrary z:*

$$\begin{aligned}2(p^2-4q)^n\sum_{j=0}^{k}u_{rj+s}^{2n}z^j&=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}q^{si}\frac{q^{r(2n+ki)}v_{(rk+s)(2n-2i)}z^{k+2}-q^{ri(k+1)}v_{(rk+r+s)(2n-2i)}z^{k+1}}{q^{2rn}z^2-q^{ri}v_{r(2n-2i)}z+1}\\&\quad-\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}q^{si}\frac{q^{s(2n-2i)+ri}v_{(r-s)(2n-2i)}z-v_{s(2n-2i)}}{q^{2rn}z^2-q^{ri}v_{r(2n-2i)}z+1}\,,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&2(p^2-4q)^{n-1}\sum_{j=0}^{k}u_{rj+s}^{2n-1}z^j\\&\quad=\sum_{i=0}^{2n-1}(-1)^i\binom{2n-1}{i}q^{si}\frac{q^{r(2n-1+ki)}u_{(rk+s)(2n-1-2i)}z^{k+2}-q^{ri(k+1)}u_{(rk+r+s)(2n-1-2i)}z^{k+1}}{q^{(2n-1)r}z^2-q^{ri}v_{r(2n-1-2i)}z+1}\\&\qquad+\sum_{i=0}^{2n-1}(-1)^i\binom{2n-1}{i}q^{si}\frac{q^{s(2n-1-2i)+ri}u_{(r-s)(2n-1-2i)}z+u_{s(2n-1-2i)}}{q^{(2n-1)r}z^2-q^{ri}v_{r(2n-1-2i)}z+1}\,,\end{aligned} \tag{2.2}$$

$$
\begin{aligned}
2\sum_{j=0}^{k} v_{rj+s}^{n} z^{j} &= \sum_{i=0}^{n} \binom{n}{i} q^{si} \frac{q^{r(n+ki)} v_{(rk+s)(n-2i)} z^{k+2} - q^{ri(k+1)} v_{(rk+r+s)(n-2i)} z^{k+1}}{q^{rn} z^2 - q^{ri} v_{r(n-2i)} z + 1} \\
&\quad - \sum_{i=0}^{n} \binom{n}{i} q^{si} \frac{q^{s(n-2i)+ri} v_{(r-s)(n-2i)} z - v_{s(n-2i)}}{q^{rn} z^2 - q^{ri} v_{r(n-2i)} z + 1}\,.
\end{aligned}
\tag{2.3}
$$

Proof. Set $(x, y) = (\alpha, \beta)$ in (1.12), (1.14) and (1.16) and make use of Lemma 2. $\square$

In particular, we have

$$
\begin{aligned}
2(p^2-4q)^n \sum_{j=0}^{k} u_{j}^{2n} z^{j} &= \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} \frac{q^{2n+ki} v_{k(2n-2i)} z^{k+2} - q^{i(k+1)} v_{(k+1)(2n-2i)} z^{k+1}}{q^{2n} z^2 - q^{i} v_{2n-2i} z + 1} \\
&\quad - \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} \frac{q^{i} v_{2n-2i} z - 2}{q^{2n} z^2 - q^{i} v_{2n-2i} z + 1}\,,
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
&2(p^2-4q)^{n-1} \sum_{j=0}^{k} u_{j}^{2n-1} z^{j} \\
&\qquad = \sum_{i=0}^{2n-1} (-1)^i \binom{2n-1}{i} \frac{q^{2n-1+ki} u_{k(2n-1-2i)} z^{k+2} - q^{i(k+1)} u_{(k+1)(2n-1-2i)} z^{k+1}}{q^{2n-1} z^2 - q^{i} v_{2n-1-2i} z + 1} \\
&\qquad\quad + \sum_{i=0}^{2n-1} (-1)^i \binom{2n-1}{i} \frac{q^{i} u_{2n-1-2i} z}{q^{2n-1} z^2 - q^{i} v_{2n-1-2i} z + 1}\,,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
2\sum_{j=0}^{k} v_{j}^{n} z^{j} &= \sum_{i=0}^{n} \binom{n}{i} \frac{q^{n+ki} v_{k(n-2i)} z^{k+2} - q^{i(k+1)} v_{(k+1)(n-2i)} z^{k+1}}{q^{n} z^2 - q^{i} v_{n-2i} z + 1} \\
&\quad - \sum_{i=0}^{n} \binom{n}{i} \frac{q^{i} v_{n-2i} z - 2}{q^{n} z^2 - q^{i} v_{n-2i} z + 1}\,.
\end{aligned}
\tag{2.6}
$$

Stănică [5] obtained results which may be considered equivalent to (2.4) and (2.5). We remark however that his expressions still contain the irrational numbers α and β.

Dropping terms proportional to z^k, in the limit as k approaches infinity, we obtain the generating fuctions given in Corollary 2.

Corollary 2 (Generating functions for powers of the terms of Lucas sequences with indices in arithmetic progression). *The following identities hold for integers r and s and nonnegative integer n:*

$$
2(p^2-4q)^n \sum_{j=0}^{\infty} u_{rj+s}^{2n} z^{j} = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} q^{si} \frac{-q^{s(2n-2i)+ri} v_{(r-s)(2n-2i)} z + v_{s(2n-2i)}}{q^{2rn} z^2 - q^{ri} v_{r(2n-2i)} z + 1}\,,
\tag{2.7}
$$

$$
2(p^2-4q)^{n-1} \sum_{j=0}^{\infty} u_{rj+s}^{2n-1} z^{j} = \sum_{i=0}^{2n-1} (-1)^i \binom{2n-1}{i} q^{si} \frac{q^{s(2n-1-2i)+ri} u_{(r-s)(2n-1-2i)} z + u_{s(2n-1-2i)}}{q^{(2n-1)r} z^2 - q^{ri} v_{r(2n-1-2i)} z + 1}\,,
\tag{2.8}
$$

$$
2\sum_{j=0}^{\infty} v_{rj+s}^{n} z^{j} = \sum_{i=0}^{n} \binom{n}{i} q^{si} \frac{-q^{s(n-2i)+ri} v_{(r-s)(n-2i)} z + v_{s(n-2i)}}{q^{rn} z^2 - q^{ri} v_{r(n-2i)} z + 1}\,.
\tag{2.9}
$$

We have, in particular,

$$2(p^2-4q)^n\sum_{j=0}^{\infty}u_j^{2n}z^j=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\frac{-q^iv_{2n-2i}z+2}{q^{2n}z^2-q^iv_{2n-2i}z+1}\,, \tag{2.10}$$

$$2(p^2-4q)^{n-1}\sum_{j=0}^{\infty}u_j^{2n-1}z^j=\sum_{i=0}^{2n-1}(-1)^i\binom{2n-1}{i}\frac{q^iu_{2n-1-2i}z}{q^{(2n-1)}z^2-q^iv_{(2n-1-2i)}z+1}\,, \tag{2.11}$$

$$2\sum_{j=0}^{\infty}v_j^nz^j=\sum_{i=0}^{n}\binom{n}{i}\frac{-q^iv_{n-2i}z+2}{q^nz^2-q^iv_{(n-2i)}z+1}\,. \tag{2.12}$$

Popov [3] obtained results equivalent to (2.10) — (2.12).

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