AdS_3-kerr, moving brane, Cardy-Verlinde formula

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A relation like Hubble equation is derived on a (1+1)-dimensional brane moving in the AdS_{1+2}-kerr space-time. The relation is used to obtain a Cardy-Verlinde formula on the brane.

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A. Introduction

Holography is a subject, in development, for long, almost silently, through works of 't hooft, Susskind, Bekenstein [1] and many others. It came to the limelight after Maldacena's conjecture [2–4] giving one concrete example of holography. The statement was that a CFT on boundary of AdS space-time completely represents gravity in the bulk. In cosmology, soon the search for holographic behaviour started through the work of Fischler-Susskind[5]. They tried to put a bound on the entropy of the closed Universe. But the entropy of the Universe was bounded from above by some undetermined constant. As it is, this was the first of the three major stages of development, holography in cosmology, has passed through uptill now. Related to the work of Fischler-Susskind, grew out a large number of works[6]. Major conceptual developments are the Hubble horizon bound and screen surface[7]. With those followed the second stage of development[8]. Consequences of Hubble bound/screen surface were explored. Parallely, was developing, vigorously, the subject of AdS/CFT. It included non-rotating AdS spaces[9] as well as rotating AdS-Kerr spaces [10]. The third stage[18] of development along the line was initiated by E. Verlinde [14]. He found a signature of $1+1$-dimensional conformal field theory in cosmology in $n+1$-dimensional spaces. The particular relationship in context, is Cardy's formula [15]. Soon thereafter, appeared his work with Savonije[16]. There, they found out that underlying FRW cosmology on moving brane of Randall-Sundrum kind [17] of dimension $n+1$, in the background of AdS_{n+2}-Schwarzscild spaces, is a formula that resembles Cardy's formula. Interestingly, entropy of the cosmology is completely known here. Considerable amount of works have been done in the recent past to get Cardy-Verlinde like formula in various cosmologies on bulk and on branes. But uptill now, in the literature, is missing a Verlinde type approach to obtain one formula for cosmology on brane, in the background of Kerr-AdS spaces. In this paper, we present a Verlinde type analysis in the simplest setup. We study a $1+1$-dimensional moving brane in AdS_{1+2}-Kerr black-hole space-time.

In the next section, we write down the orthonormal frame attached with the brane. Then we obtain the induced metric, the Hubble equation, on it in the fourth section. In between, in the third section, we discuss the metric of AdS_{2+1}- Kerr space-time. In the fifth one, we start with the energy density on the brane, obtain entropy, temperature, casimir energy, Cardy-Verlinde formula associated with the induced cosmology and draw some conclusions. We end up with a discussion in the last section.

B. Orthonormal frame

Let us describe, in this section, the orthonormal frame attached with the moving brane. This, we will see, will be required to obtain the induced metric on the brane. This orthonormal frame in conjuction with the induced metric, will determine the Hubble equation also.

The orthonormal frame, attached with the rotating as well as radially moving brane, in the AdS_{1+2}-Kerr space-time is

$$s_\mu = (-\sqrt{g_{\phi\phi}}w, 0, \sqrt{g_{\phi\phi}}) \tag{1}$$
$$v_\mu = (v_t, v_r, 0) \tag{2}$$
$$n_\mu = (n_t, n_r, 0) \tag{3}$$

$n^\mu \partial_\mu$ is a space-like unit vector normal to the brane. $v^\mu \partial_\mu$ is a time-like unit vector on the world-sheet. The unit vector along the brane is $s^\mu \partial_\mu$.

Here we note that

$$n_r = -\sqrt{-\frac{g^{tt}}{g^{rr}}}v_t \tag{4}$$
$$n_t = \sqrt{-\frac{g^{rr}}{g^{tt}}}v_r \tag{5}$$
$$\tag{6}$$

The negative sign in n_r corresponds to our choice of the normal pointing inward.

Again, the Israel Junction condition yields

$$n^r = -k_-[\partial_r \ln s^\phi]^{-1} \tag{7}$$

where,

$$s^\phi = \frac{1}{\sqrt{g_{\phi\phi}}} \tag{8}$$

As a result, the frame is fully determined by the embedding metric.
In the appendix, we will deal with the frame again.

C. Bulk space-time

For completeness, let us write down the AdS_{1+2}-kerr space-time metric below. This is the metric in the ref.[11, 13], written in the canonical form[19]. The metric in the ref[11, 13] can be obtained also from BTZ metric[20] by coordinate transformation.

$$ds^2_{1+2} = -N^2dt^2 + \frac{r^2dr^2}{\Delta_r} + g_{\phi\phi}(d\phi - wdt)^2 \tag{9}$$

where, the angular velocity, w, of the space-time is,

$$w = \frac{g^{\phi t}}{g^{tt}} = -\frac{g_{\phi t}}{g_{\phi\phi}} \tag{10}$$

Moreover, the metric components are

$$N^2 = \frac{r^2\Delta_r}{\Sigma^2} \tag{11}$$

$$w = \frac{a\Xi}{\Sigma^2}[r^2 + a^2 - \Delta_r] \tag{12}$$

$$g_{\phi\phi} = \frac{\Sigma^2}{r^2\Xi^2} \tag{13}$$

$$g_{\phi t} = \frac{a}{r^2\Xi}[\Delta_r - (r^2 + a^2)] \tag{14}$$

$$g_{tt} = -\frac{\Delta_r - a^2}{r^2} \tag{15}$$

$$g^{tt} = -\frac{1}{N^2} \tag{16}$$

with,

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta_r \tag{17}$$

$$\Delta_r = (r^2 + a^2)(1 + \frac{r^2}{L^2}) - 2G_3Mr^2 \tag{18}$$

$$\Xi = 1 - \frac{a^2}{L^2} \tag{19}$$

and

$$g_{tt}g_{\phi\phi} - g^2_{\phi t} = -\frac{\Delta_r}{\Xi^2}$$

Now, given all that, we observe,

$$n^2 - v^2 = -N^2dt^2 + \frac{r^2dr^2}{\Delta_r} \tag{20}$$

where, n and v are the one form fields, normal and tangential to the brane respectively. These are given as

$$n = n_\mu dx^\mu, v = v_\mu dx^\mu \tag{21}$$

This enables us to write the AdS_{1+2} metric as

$$ds^2_{1+2} = n^2 - v^2 + g_{\phi\phi}(d\phi - wdt)^2 \tag{22}$$

D. space-time on the brane

It turns out that,

$$v = -d\tau \tag{23}$$

Therefore, the induced metric on the brane is,

$$ds^2_{1+1} = -d\tau^2 + g_{\phi\phi}(d\phi - wdt)^2 \tag{24}$$

where, ϕ is the angular position of a point measured w.r.t the embedding space-time rest frame. w is the angular velocity of a point on the brane. At this point, let us intoduce a coordinate $\phi^{'}$ on the brane, such that, $d\phi' = d\phi - wdt$. Now, we recall that

$$\frac{d\phi'}{d\tau} = v^\phi - wv^t = 0 \tag{25}$$

Hence, $\phi^{'}$ is a comoving coordinate on the brane.
Consequently, the induced metric on the brane turns out as,

$$ds^2_{1+1} = -d\tau^2 + a^2_{sc}d(\phi')^2 \tag{26}$$

This is the metric of a $1+1$-dimensional homogeneous cosmology, with the scale-factor, a_{sc}, given by

$$a^2_{sc} = g_{\phi\phi}(r(\tau)) \tag{27}$$

$1+1$-dimensional cosmologies were studied previously in different context[21]. In those works, $1+1$-dimensional cosmologies provided a laboratory for toying with issues like duality, inflation, graceful exit etc. Here, we will be concerned with holographic aspect. Moreover, there is one interesting thing. We will soon see that our $1+1$-dimensional cosmology has a matter with equation of state $p = 3\rho$. ρ is also negative. This matter becomes important when the Universe is of very small size.

1. *Hubble equation*

Let us recall that $v_\mu v^\mu = -1$. This yields the following relation

$$\frac{dr}{d\tau} = -\frac{\Delta_r}{r^2} + (n^r)^2 \tag{28}$$

Now, the Hubble parameter is defined as follows,

$$H = \frac{d\ln\sqrt{g_{\phi\phi}}}{d\tau} \tag{29}$$

As a result, after setting k_- to $\frac{1}{L}$ to tune the cosmological constant on the brane to zero, we get from the above two equations,

$$H^2 = \frac{1}{L^2}[1 - \frac{\Delta_r}{r^2}\frac{1}{(n^r)^2}] \tag{30}$$

Let us notice that at the horizon, $H = \frac{1}{L}$, as in the AdS-Schwarzschild case[16].
Moreover,

$$n^r = \frac{\Xi}{Lr} a_{sc}^2 \tag{31}$$

Consequently, the Hubble parameter on the moving 1+1-dimensional brane, satisfies the equation,

$$H^2 = -\frac{1}{a_{sc}^2} + \frac{1}{a_{sc}^2}\frac{2G_3M}{\Xi^2}(2-\Xi) - \frac{1}{a_{sc}^4}(\frac{2G_3Ma}{\Xi^2})^2 \tag{32}$$

The $\dot{H}$ equation is

$$\dot{H} = \frac{1}{a_{sc}^2} - \frac{1}{a_{sc}^2}\frac{2G_3M}{\Xi^2}(2-\Xi) + \frac{2}{a_{sc}^4}(\frac{2G_3Ma}{\Xi^2})^2 \tag{33}$$

$\dot{H}$ is related to the decceleration parameter. Again, it is interesting to find that in the presence of angular momentum, the Universe can contract upto a minimum scale factor

$$a_{sc} = \frac{2G_3Ma}{\Xi^2}\frac{1}{[\frac{2G_3M}{\Xi^2}(2-\Xi)-1]^{\frac{1}{2}}} \tag{34}$$

This corresponds to a minimum radial distance, r_{min}, within the black-hole, the brane can enter

$$r_{min}^2 = a^2\frac{(\frac{2G_3M}{\Xi^2})^2\Xi(\Xi-1) - 2\frac{2G_3M}{\Xi^2}+1}{\frac{2G_3M}{\Xi^2}(2-\Xi)-1} \tag{35}$$

E. Cardy-Verlinde formula

Let us now study the Cardy-Verlinde formula for the full range of Ξ. The range extends from zero to one.

1. *energy density*

Let us note that H^2 is of the form

$$H^2 = -\frac{1}{a_{sc}^2} + \frac{16\pi G_3}{L}\rho \tag{36}$$

where,

$$\rho = \frac{L}{16\pi G_3}[\frac{\frac{2G_3M}{\Xi^2}(2-\Xi)}{a_{sc}^2} - \frac{(\frac{2G_3Ma}{\Xi^2})^2}{a_{sc}^4}] \tag{37}$$

As a result,

$$p = \frac{L}{16\pi G_3}[\frac{\frac{2G_3M}{\Xi^2}(2-\Xi)}{a_{sc}^2} - 3\frac{(\frac{2G_3Ma}{\Xi^2})^2}{a_{sc}^4}] \tag{38}$$

Again in coming from equation (37) to equation (38) we have used the following,

$$\rho = \rho_1 + \rho_2 \tag{39}$$

with the equation of states obeyed respectively as,

$$p_1 = \rho_1 \tag{40}$$
$$p_2 = 3\rho_2 \tag{41}$$

We recall that in 1+1-dimension, the stiff-matter has the same equation of state, as that of the conformal matter. Before going to the next subsection, let us note some pertinent points:
(i) RS prescription of using relation[16] between G_{1+2} and G_{1+1} does not work here. So we keep $\frac{16\pi G_3}{L}$ in H^2 equation, in the spirit of higher dimensional case. There, $\frac{16\pi G_{n+2}}{nL}$ does give G_{n+1} on the brane. Then ρ turns out to be $n+1$-dimensional energy density. In the same way, we consider our H^2 equation as giving the Hubble equation on the $(1+1)$-dimensional brane. ρ is the energy density on the brane.
(ii) ρ is non-zero due to the brane's presence in AdS_{1+2}.
(iii) Ala Verlinde, if we do analysis, keeping $\frac{16\pi G_{1+2}}{L}$ for G_{1+1}, the Cardy-Verlinde formula does come for a (1+1)-dimensional brane in the AdS_{1+2}-Schwarzschild black-hole.
(iv) Our analysis has the same status as that mentioned in (iii), but in the AdS_{1+2}-kerr black-hole background.
(v)H^2-equation, in the $a \to 0$ limit, goes over to the H^2-equation in the AdS-Schwarzschild background.
(vi)Now, let us notice the second equation of state (41). Individually, it violates positive energy condition and leads to velocity of sound more than the velocity of light. But here the previous equation (39) tells that it is part of the total energy density. Moreover, for $a_{sc} > a_{sc}^{min}$, $\rho > p$. This ensures that velocity of sound in the medium, in the full phase of life of the brane-universe, is less than the velocity of light[22]. There is no violation of energy condition[22] too. Note that we have assumed as in ref.[22] that the Ricci scalar on the brane, R_{1+1}, satisfies the ansatz

$$R_{1+1} = 8\pi G_{1+1}T_a^a \tag{42}$$

We are yet to get to a concrete justification for the ansatz in the moving brane context.

2. *entropy on the brane*

Now, the entropy[11, 13] of AdS_{1+2}-kerr black-hole space-time is

$$S = \frac{A_+}{4G_3} = \frac{2\pi a_{sc}^+}{4G_3} \tag{43}$$

Following GPKW prescription[3] for the AdS/CFT correspondence, the total entropy on the (1+1)-dimensional brane is also S. Again, the perimeter of the brane is

$$P = \int_0^{2\pi}\sqrt{g_{\phi\phi}}d\phi' \tag{44}$$
$$= 2\pi\sqrt{g_{\phi\phi}} \tag{45}$$

Hence, the entropy per unit length, s, of the brane is

$$s = \frac{S}{P} \quad (46)$$

$$= \frac{1}{4G_3}\frac{a_{sc}^{+}}{a_{sc}} \quad (47)$$

where, a_{sc}^{+} is the scale factor at the horizon.

3. temperature on the brane

Hawking temperature, T_H, of AdS_{1+2}-kerr black-hole space-time [11, 13] is

$$T_H = \frac{1}{2\pi}\frac{\frac{r_+^4}{L^2} - a^2}{r_+(r_+^2 + a^2)} \quad (48)$$

This is the temperature of the brane too. But the Hawking temperature is measured w.r.t the bulk metric.
Hence we consider now the scaling[3] following[16]. Here we note that,

$$Lim_{g_{\phi\phi}\to\infty}\frac{L^2}{g_{\phi\phi}}ds_{1+2}^2 = -dt^2 + L^2(d\phi - w_\infty dt)^2 \quad (49)$$

But, we are considering the thermodynamics in $(1+1)$-dimensional brane metric

$$ds^2 = -d\tau^2 + g_{\phi\phi}(d\phi - wdt)^2$$

We also note,

$$w - w_\infty = \frac{2G_3Ma}{\Xi^2}\frac{1}{a_{sc}^2} \quad (50)$$

As a result,

$$w_H - w_\infty = \frac{a}{L^2} \quad (51)$$

for astrophysical black-hole. Since, we are considering classical black-hole, L is also very large. Consequently, we can put in the equation (49), $w_\infty = w$ reliably. Then, we see that the brane time, τ, is related to the bulk time, t, by,

$$\tau = \frac{a_{sc}}{L}t \quad (52)$$

Again, GPKW[3] prescribes that the temperature on the brane, T_b, is related to the Hawking temperature, T_H, by conformal scaling.
As the temperature is inverse in dimension to time,

$$T_b = T_H\frac{L}{a_{sc}} \quad (53)$$

$$= \frac{L}{2\pi a_{sc}^{+}}\frac{\frac{r_+^4}{L^2} - a^2}{\Xi r_+^2}\frac{1}{a_{sc}} \quad (54)$$

where, a_{sc}^{+} is the scale factor at the horizon. So, the thermal energy density on the brane is,

$$T_b s = \frac{L}{8\pi G_3}\frac{1}{\Xi(r_+)^2}(\frac{(r_+)^4}{L^2} - a^2)\frac{1}{a_{sc}^2} \quad (55)$$

4. angular energy on the brane

Angular velocity of the space-time, w, [11, 13]is

$$w = a\Xi\frac{(r^2+a^2) - \Delta_r}{(r^2+a^2)^2 - a^2\Delta_r} \quad (56)$$

whereas, quantity which enters in thermodynamics is $w - w_\infty$, multiplied by angular momentum, J, after scaling incorporated.
Now, angular momentum, at $\frac{r_+}{L} >> 1$, of the brane is $J = \frac{1}{2}\frac{Ma}{\Xi^2}$ if we assume, AdS/CFT correpondence to work as in Kerr-AdS_5[10]. Then the angular energy density, W_J, on the brane, as seen from the asymptotic, comes as,

$$W_J = \frac{J(w - w_\infty)}{2\pi a_{sc}}\frac{L}{a_{sc}} \quad (57)$$

$$= \frac{L}{16\pi G_3}2(\frac{2G_3Ma}{\Xi^2})^2\frac{1}{a_{sc}^4} \quad (58)$$

Moreover,

$$W_J = -2\rho_2 \quad (59)$$

5. towards Casimir energy

We define a quantity, γ_J, on the rotating brane as

$$\gamma_J = \frac{1}{2}(\rho + p - T_b s + 2W_J)a_{sc}^2 \quad (60)$$

$$= \frac{L}{16\pi G_3}[\frac{a^2}{L^2} + 2\frac{a^2}{r_+^2} + 1 + 2G_3M\frac{\frac{2a^2}{L^2}}{\Xi}] \quad (61)$$

We also define another quantity, ρ_J, as

$$\rho_J = \rho - \frac{\gamma_J}{a_{sc}^2} + \frac{W_J}{2} \quad (62)$$

$$= \frac{L}{16\pi G_3}\frac{1}{\Xi(r_+)^2}(\frac{(r_+)^4}{L^2} - a^2)\frac{1}{a_{sc}^2} \quad (63)$$

Therefore,

$$\rho_J + \frac{\gamma_J}{a_{sc}^2} = \rho + \frac{W_J}{2} > \rho \quad (64)$$

We also notice, that as $a^2 \to 0$, $\gamma_J \to \frac{L}{16\pi G_3}$ which is the casimir energy on the brane in the AdS-Schwarzschild case[16]. Whereas, for a^2 not sufficiently close to zero, there are two extremes. One is the limit of astrophysical black-holes. Then,

$$G_3M >> 1 \quad (65)$$

Consequently, γ_J takes the form,

$$\gamma_J = \frac{L}{16\pi G_3}\frac{2(1-\Xi)}{\Xi}2G_3M \quad (66)$$

6. around Cardy-Verlinde formula

As a result, it turns out that the entropy and energy densities obey the following relation,

$$s^2 = (4\pi)^2 f^2 \gamma_J \rho_J \tag{67}$$

where,

$$f^{-2} = \frac{\frac{r_+^4}{L^4} - \frac{a^2}{L^2}}{(\frac{r_+^2}{L^2} + \frac{a^2}{L^2})^2}[\frac{a^2}{L^2} + 2\frac{a^2}{r_+^2} + 1 + 2G_3 M \frac{\frac{2a^2}{L^2}}{\Xi}] \tag{68}$$

For $a^2 = 0$, the above relation reduces to the Cardy-Verlinde formula in the AdS-Schwarzschild case[16],

$$s^2 = (4\pi)^2 \gamma(\rho - \frac{\gamma}{r^2}) \tag{69}$$

Again, a^2 varies over a huge range from zero to L^2. Let us consider that a^2 is not sufficiently close to zero. Then, the relation (67) reduces to, for astrophysical black-holes,

$$s^2 = (4\pi)^2 f_{as}^2 \gamma_J (\rho - \frac{\gamma}{a_{sc}^2} + \frac{W_J}{2}) \tag{70}$$

where,

$$f_{as}^2 = \frac{\Xi}{2(1-\Xi)} \frac{L^2}{r_+^2} \tag{71}$$

Notice, $f^2 << 1$.

7. large distance limit

When the scale factor is large, ρ_2 and W_J becomes negligible. Consequently, definition of γ_J reduces to the familiar[16] form of Casimir energy

$$\frac{\gamma}{a_{sc}^2} = \frac{1}{2}(\rho + p - T_b s) \tag{72}$$

Simultaneously, the relation (67) between the entropy and energy densities reduce to the familiar relationship [16] going by the name of Cardy-Verlinde formula

$$s^2 = (4\pi)^2 f^2 \gamma(\rho - \frac{\gamma}{a_{sc}^2}) \tag{73}$$

Note the presence of the prefactor f^2 in the relation (67) as well as in the formula (73).

8. correspondence to Cardy's formula

Equation (73) is analogous to the Cardy's formula [15] for entropy in $1+1$-dimensional CFTs,

$$S^2 = (4\pi)^2 \frac{c}{24}(L_0 - \frac{c}{24}) \tag{74}$$

where,

$$\frac{c}{24} \leftrightarrow f\gamma \tag{75}$$

$$L_0 \leftrightarrow f\rho \tag{76}$$

9. more conclusions

(i) When a^2 is not sufficiently close to zero, there is another extreme for black-hole's mass. This is the limit when black-hole's mass is such that

$$2G_3 M = (1 + \frac{a}{L})^2 \tag{77}$$

This leads to

$$\frac{r_+{}^2}{L} = \frac{a}{L} \tag{78}$$

$$T_H = 0 \tag{79}$$

In this case, our analysis will go through provided we are at very large a_{sc}. Then from equation (73), it follows that Cardy-Verlinde type relationship does not exist in the bps limit.
(ii) It is also obvious, that in the limit when $\frac{a^2}{L^2}$ goes to one, Cardy-Verlinde relationship breaks down. In that limit, the brane rotates with the velocity of light.

F. Discussions

(i)We have retrieved standard form of Cardy-Verlinde formula, only when the Universe reaches the "steady state"[22].
(ii)In the AdS_{n+2}-Schwarzschild black-holes[16], gamma depends on r_+ and r both, at generic position, when n is not equal to one. For $n = 1$, gamma is constant, does not depend on either r_+ or, r. But, in this case of AdS_{1+2}-Kerr black-holes, gamma depends on r_+ for all r. Hence, it has no 'universal' limit where, like at the horizon in the Schwarzschild case[16], it goes to a constant.
(iii)There are two characteristics of the cosmology, one is the Casimir energy and another analogue of central charge. Both are large. First one is present, even if $a^2 = 0$, the second one appaears only at a^2 not sufficiently equal to zero. In other words, this is due to the presence of the cosmology in the background of Kerr-AdS space-time. The bulk space-time has two conserved quantities, which can be defined only globally.
(iv)It will be interesting to study the cosmology on the brane from the cosmological perspective.
(v)The immediate generalisation of our analysis to the AdS_{n+2}-kerr space-time, where $n > 1$, is equally interesting to be done.

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H. Appendix

The relevent part of the minimal action[12, 13, 16] is

$$I_b = \frac{1}{8\pi G_{1+2}}[\int \sqrt{h} K d^{1+1}x + k_b \int \sqrt{h} d^{1+1}x] \tag{80}$$

The Israel Junction condition[25], for a brane, is

$$[K_{MN} - K h_{MN}]^{+}_{-} = 8\pi G_{1+2} t_{MN} \tag{81}$$

(+) corresponds to the side of the brane towards the essential singularity of the black-hole.
Stress-energy tensor on the brane t_{MN} is as follows,

$$t_{MN} = -\frac{2}{\sqrt{h}}\frac{\delta I_b}{\delta h^{MN}} \tag{82}$$

$$= \frac{k_b}{8\pi G_{1+2}} h_{MN} \tag{83}$$

Again if the tension of the brane k_b is small, it does not disturb the space-time on both sides much [?]. In that case, the Israel Junction condition reduces to

$$K_{MN} = -k_- h_{MN} \tag{84}$$

where,

$$k_- = \frac{k_b}{2} \tag{85}$$

On the otherhand, if the brane is the end of the black-hole space-time or, the socalled cut-off brane, then the Israel Junction condition involves only the (+) side. As a result, the final relation is [26]

$$K_{MN} = -k_- h_{MN} \tag{86}$$

with

$$k_- = k_b \tag{87}$$

1. extrinsic curvature

Let us assume that e^M is one of the nonholonomic orthonormal vectors on the brane. $\vec{n}$ is an unit nonholonomic vector orthonormal to the brane. The extrinsic curvature [24] of the brane, is defined through

$$\mathcal{L}_{e^M}\vec{n} = -K^M_N e^N \tag{88}$$

Again, by definition, [?]

$$\mathcal{L}_{e^M}\vec{n} = -[\vec{n}, e^M] \tag{89}$$

As a result,

$$[\vec{n}, e^M] = K^M_N e^N \tag{90}$$

So, the diagonal components are given by

$$K^M_M =< e^M, [\vec{n}, e^M] > \tag{91}$$

Hence,

$$K^s_s =< \vec{s}, [\vec{n}, \vec{s}] > \tag{92}$$

This combined with the Israel Jn condition yields n^r equation (7) in the first section.

2. Time-like vector of a point on the brane

A point, x^μ, on the brane follows a non-geodesic path in the embedding space-time as it's under the brane tension also, beside gravity pull of the black-hole. $v^\mu = \frac{dx^\mu}{d\tau}$ [23], is tangent to the path. It satisfies,

$$v^\mu \nabla_\mu v^\lambda \neq 0 \tag{93}$$

The orthonormal frame attached with it also, in general, does not propagate parallely. In this case,

$$v^\mu \nabla_\mu n^\lambda \neq 0 \tag{94}$$

$$v^\mu \nabla_\mu s^\lambda = 0 \tag{95}$$

In the cosmology on the brane, the same trajectory of the same point on the brane is a geodesic path

$$v^a \nabla_a v^b = 0 \tag{96}$$

Azimuthal vector, s^a, is also parallely propagated along the time-like trajectory,

$$v^a \nabla_a s^b = 0 \tag{97}$$

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