UNITARY QUANTUM GROUPS VS QUANTUM REFLECTION GROUPS

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ABSTRACT. We study the intermediate liberation problem for the real and complex unitary and reflection groups, namely O_N, U_N, H_N, K_N. For any of these groups G_N, the problem is that of understanding the structure of the intermediate quantum groups $G_N \subset G_N^\times \subset G_N^+$, in terms of the recently introduced notions of "soft" and "hard" liberation. We solve here some of these questions, our key ingredient being the generation formula $H_N^{[\infty]} =< H_N, T_N^+ >$, coming via crossed product methods. Also, we conjecture the existence of a "contravariant duality" between the liberations of H_N and of U_N, as a solution to the lack of a covariant duality between these liberations.

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INTRODUCTION

The quantum analogues of the compact Lie groups $G_N \subset U_N$, taken in an operator algebra sense, were introduced by Woronowicz in [33], [34]. In the case where the square of the antipode is the identity, $S^2 = id$, which is of particular interest, these quantum groups appear as closed subgroups $G_N \subset U_N^+$ of Wang's unitary quantum group [30].

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The liberation philosophy from [11] amounts in regarding each such subgroup $G_N\subset U_N^+$ as a liberation of its classical version, $G_N^c\subset U_N$. This philosophy, and its various technical modifications, cover a fairly large class of interesting closed subgroups $G_N\subset U_N^+$.

A number of compact Lie groups $G_N\subset U_N$ admit full liberations $G_N^+\subset U_N^+$, in a certain technical sense, and the classification and study of the various intermediate liberations $G_N\subset G_N^\times\subset G_N^+$ is a key open problem. This problem was solved in [12], [14] and subsequent papers for the orthogonal group O_N, the conclusions being as follows:

(1) There is only one "easy" intermediate liberation, namely the half-classical orthogonal group O_N^*, appearing via the relations $abc=cba$ between coordinates.
(2) This quantum group O_N^* and its subgroups can be studied by using a variety of techniques, and generally speaking, are quite well understood.

This solution is actually something quite unique, the situation for the other liberable compact Lie groups $G_N\subset U_N$ being considerably more complicated.

These questions are of particular interest for the unitary group U_N, as well as for the hyperoctahedral group $H_N=\mathbb Z_2\wr S_N$, and for its complex version $K_N=\mathbb T\wr S_N$. These groups, together with O_N, and with their liberations, are indeed as follows:

$$\begin{tikzcd}
& K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
& K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

As explained in a number of papers, including [4], this cubic diagram is of key importance, in order to have some 3D intuition on the subgroups $H_N\subset G_N\subset U_N^+$.

Summarizing, we have 4 main liberation problems. The solution for O_N was explained above. The classification problem for H_N was solved some time ago in [26], [27], [28], and the classification problem for U_N was recently solved in [23], [24], as follows:

(1) We have liberations $H_N\subset H_N^\Gamma\subset H_N^{[r]}\subset H_N^+$, consisting of a family indexed by groups $\mathbb Z_2^{*\infty}\to\Gamma\to\mathbb Z_2^\infty$, followed by a series indexed by integers $r\in\mathbb N$.
(2) We have liberations $U_N\subset U_N^{(r)}\subset U_N^C\subset U_N^+$, consisting of a series indexed by integers $r\in\mathbb N$, followed by a family indexed by cosemigroups $C\subset\mathbb N$.

As for the liberation problem for the complex reflection group K_N, this remains a missing piece of the puzzle, with no classification being available here yet.

The present paper was motivated by the fact that the above-mentioned classification results for H_N, U_N have some obvious similarity between them. We have indeed a family followed by a series, and a series followed by a family, and this suggests the existence of a "contravariant duality" between these quantum groups, as follows:

$$\begin{tikzcd}
U_N \arrow[r] \arrow[d, dash, dotted] & U_N^{(r)} \arrow[r] \arrow[d, dash, dotted] & U_N^C \arrow[r] \arrow[d, dash, dotted] & U_N^+ \arrow[d, dash, dotted] \\
H_N^+ & H_N^{[r]} \arrow[l] & H_N^\Gamma \arrow[l] & H_N \arrow[l]
\end{tikzcd}$$

At the first glance, this might sound a bit extravagant. Indeed, we have some natural and well-established correspondences $H_N \leftrightarrow U_N$ and $H_N^+ \leftrightarrow U_N^+$, obtained in one sense by taking the real reflection subgroup, $H = U \cap H_N^+$, and in the other sense by setting $U =< H, U_N >$. Thus, our proposal of duality "obviously" goes the wrong way.

On the other hand, obvious as well is the fact that these correspondences $H_N \leftrightarrow U_N$ and $H_N^+ \leftrightarrow U_N^+$ cannot be extended as to map the series to the series, and the family to the family, because the series/families would have to be "inverted", in order to do so.

Summarizing, our idea of a contravariant duality makes sense. In practice, however, working out such a result looks very technical, requiring an excellent knowledge of the papers [26], [27], [28] on one hand, of the papers [23], [24] on the other hand, and finally of newly developed soft and hard liberation theory from [4], [6], [7], [8] as well.

We were unable to reach to such a level of knowledge, and construct the duality. Instead, we will simply continuate here our soft and hard liberation work from [4], [6], [7], [8], by using input from [26], [27], [28] and from [23], [24]. We will reach to a number of interesting conclusions, that we intend to further use and refine in a number of future papers, with the idea in mind of building of fully functional soft and hard liberation theory.

Back to the duality itself, this seems to be something quite deep, lying one step above our soft and hard liberation program. It is our hope of course that such a duality can be established one day, and that the present considerations can help.

The paper is organized as follows: 1-2 contain preliminaries on the soft and hard liberation operations, in 3-4 we discuss the quantum reflection groups, in 5-6 we discuss the unitary quantum groups, and in 7-8 we comment on the duality question.

1. Liberation theory

We use Woronowicz's compact matrix quantum group formalism in [33], [34], under the extra assumption that the square of the antipode is the identity, $S^2 = id$.

We are interested in the liberation problem. Let us start with:

Definition 1.1. *A liberation of a compact Lie group $G_N \subset U_N$ is a quantum group*

$$G_N \subset G_N^\times \subset U_N^+$$

whose classical version is G_N itself.

Here the classical version if by definition obtained by dividing the algebra $C(G_N^\times)$ by its commutator ideal, and by [33] we obtain in this way a compact Lie group. Observe that G_N being classical, it is automatically contained in the classical version of $G_N^\times$.

There are many examples of such liberations. We first have the "full" liberations $G_N \to G_N^+$ in the sense of [11], that we will explain in a moment. We also have "half-liberations", denoted $G_N \to G_N^*$, that we will explain as well in a moment. Finally, we have many interesting examples of "genuine" intermediate liberations, either series usually denoted $G_N \to G_N^r$ with $r \in \mathbb{N}$, or uncountable families, usually denoted $G_N \to G_N^X$ with X being a discrete group-type object, coming from the work in [23], [24], [28].

At the level of generality of Definition 1.1, however, nothing much can be said, or at least we don't know how to do it, so far. Here are a few remarks on the subject:

Proposition 1.2. *The collection of liberations of a given compact Lie group $G_N \subset U_N$ has the following properties:*

1. *It is stable under the intersection operation $\cap$.*
2. *It is not necessarily stable under the generation operation $<\,,>$.*

Proof. Here (1) is something trivial, and (2) fails indeed, for instance for the hyperoctahedral group $H_N = \mathbb{Z}_2 \wr S_N$. Indeed, H_N has at least two liberations, namely the twisted orthogonal group O_N^{-1}, which appears as quantum symmetry group of the hypercube in $\mathbb{R}^N$, and the quantum group $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$, which is the quantum symmetry group of the coordinate axes of $\mathbb{R}^N$. And the point is that, according to [5], we have:

$$< O_N^{-1}, H_N^+ >= U_N^+$$

Thus, (2) fails, and in a particularly bad way, for H_N. See [5]. ☐

At a more constructive level now, one idea, which has emerged in recent times, is that the liberations of a compact Lie group $G_N \subset U_N$ should appear via operations of type $G_N^\times =< G_N, I_N^\times >$, with $I_N^\times \subset U_N^+$ being a "basic" quantum group. All this is quite conjectural for the moment, still requiring a lot of work. Formulating some improved conjectures on the subject, and partly solving them for some basic examples of compact Lie groups $G_N \subset U_N$, will be our main purpose in this paper.

Let us first discuss the case of the "well-established" full liberations, from [11]. These can be understood via a formula $G_N^\times =< G_N, I_N^\times >$ as above, as follows:

Definition 1.3. *The soft liberation of a compact Lie group $G_N \subset U_N$ is given by*

$$G_N^+ =< G_N, S_N^+ >$$

where $S_N^+ \subset U_N^+$ is Wang's quantum permutation group [31].

The terminology here comes from the fact that, as we will see soon, there is as well a second liberation operation, the "hard" one. For more on this, see [8].

As already mentioned, all this is inspired from [11]. To be more precise, we have the following result, which makes the link with the notion of liberation from there:

Proposition 1.4. *The soft liberation operation has the following properties:*

(1) *If $S_N\subset G_N\subset U_N$ is easy, we obtain the easy liberation $S_N^+\subset G_N^+\subset U_N^+$.*
(2) *If we are in the case $H_N\subset G_N\subset U_N$, then we have $G_N^+=<G_N,H_N^+>$.*
(3) *We have the formula $G_N^+=<G_N,S_N>^+$.*

Proof. All this is explained in [8], the idea being as follows:

(1) This follows from the Tannakian formula $C_{<G,H>}=C_G\cap C_H$, and from [11].
(2) This follows from the well-known formula $H_N^+=<H_N,S_N^+>$.
(3) This is something trivial, coming from definitions. □

We will need as well the following notion, from [3]:

Definition 1.5. *The easy envelope of a compact group $S_N\subset G_N\subset U_N$ is the smallest compact group $S_N\subset\widetilde{G_N}\subset U_N$ containing G_N, and which is easy.*

We refer to [3] for more details regarding this notion, and for some explicit computations, for basic examples of compact Lie groups. Here we will only need the following formula, which is actually the Tannakian definition of the easy envelope:

$$C_{\widetilde{G_N}}=span(T_\pi|\pi\in P,T_\pi\in C_{G_N})$$

We have the following conjecture, recently made in [8]:

Conjecture 1.6 (Soft liberation conjecture, SLC)**.** *We have the formula*

$$G_N^+=(\widetilde{G_N})^+$$

valid for any compact group $G_N\subset U_N$, where tilde denotes the easy envelope.

As explained in [8], this is ultimately something about partitions. To be more precise, by using the above-mentioned description of $C_{\widetilde{G_N}}$, the SLC is equivalent to the following fact, for any $r\in\mathbb N$, any partitions $\pi_1,\ldots,\pi_r\in NC$, and any scalars $\lambda_1,\ldots,\lambda_r\neq0$:

$$\lambda_1T_{\pi_1}+\ldots+\lambda_rT_{\pi_r}\in C_{G_N}\implies T_{\pi_1},\ldots,T_{\pi_r}\in C_{G_N}$$

Such questions, however, can be quite difficult, as explained in [10]. This is further discussed in [8], along with the remark that, in case all this is too difficult, an alternative approach might come from using Lie group theory, a bit as in [10]. See [8].

Here is now a related conjecture, which is new:

Conjecture 1.7 (Absorption conjecture, AC)**.** *Assuming that $E_N\subset U_N^+$ is easy, and non-classical, then so is*

$$<G_N,E_N>$$

for any compact quantum group $G_N\subset U_N^+$.

As a first remark here, the fact that E_N is not classical is really needed, because with $E_N = S_N$ we can have as counterexample any non-easy intermediate compact group $S_N \subset G_N \subset U_N$. And there are indeed such groups, as for instance:

$$U_N^2 = \left\{ U \in U_N \Big| \det U = \pm 1 \right\}$$

In the free case, $S_N^+ \subset E_N \subset U_N^+$, our claim is that the AC is equivalent to:

Conjecture 1.8 (Freeness conjecture, FC). *Any intermediate quantum group*

$$S_N^+ \subset G_N \subset U_N^+$$

must be easy.

Indeed, assume first that the AC holds in the free case. Given $S_N^+ \subset G_N \subset U_N^+$, we can simply take $E_N = S_N^+$, and the AC tells us that $< G_N, E_N >= G_N$ is easy.

Conversely, assume that the FC holds. Given $S_N^+ \subset E_N \subset U_N^+$ and $G_N \subset U_N^+$ we have $S_N^+ \subset < G_N, E_N > \subset U_N^+$, and by the FC this quantum group is easy.

In order to further comment on these conjectures, and on the relation between them, we will need a negative statement, concerning counterexamples, as follows:

Theorem 1.9. *The following happen, with "c" standing for the classical version,*

1. *The soft liberation is not necessarily maximal, as a liberation,*
2. *The "strong SLC" formula $< G_N, S_N^+ >= (\widetilde{G_N^c})^+$ does not necessarily hold,*

due to the existence of certain quantum groups $B_N'^+ \subset B_N'^{++}$ and $K_N^+ \subset K_N^{++}$.

Proof. This is well-known, but in view of the importance of these counterexamples, let us discuss them in detail. We have two groups to be discussed, as follows:

$\underline{B_N'}$. This compact group, from [11], is $B_N' = B_N \times \mathbb{Z}_2$, where $B_N \subset O_N$ is the bistochastic group. Its category of partitions is known from [11] to be the category of singletons and pairings P_{12}', with an even number of singletons. The soft liberation of B_N' is then the easy quantum group $B_N'^+$ appearing from the category $NC_{12}' = P_{12}' \cap NC$.

Contrary to was was announced in [11], this liberation is not maximal, and we have a bigger quantum group, $B_N'^+ \subset B_N'^{++}$, corresponding to the smaller category $NC_{12}'^- \subset NC_{12}'$ coming from the fact that each pairing must connect $\circ - \bullet$, after rotating the partition on one line, and labelling the legs alternatively $\circ \bullet \circ \bullet \ldots$ We refer here to [32].

$\underline{K_N}$. This is the complex reflection group, $K_N = \mathbb{T} \wr S_N$, whose free liberation theory can be deduced from the general classification results in [29]. However, all this being quite technical, here are some explanations. First of all, K_N comes from the category $\mathcal{P}_{even}$ of partitions having even blocks, with $\#\circ = \#\bullet$ holding over each block, and its soft liberation $K_N = \mathbb{T} \wr_* S_N^+$ comes then from the category $\mathcal{NC}_{even} = \mathcal{P}_{even} \cap NC$.

Consider now the free complexification $K_N^+ \to K_N^{++}$, obtained by replacing $u \to zu$, with z being a Haar unitary, free from u. By [25] this complexification is easy as well, corresponding to the category $\mathcal{NC}_{even}^- \subset \mathcal{NC}_{even}$ coming from the fact that the partitions, when rotated on one line, must contain in each block alternating $\circ, \bullet$ symbols.

But this shows that we have an inclusion $K_N^+ \subset K_N^{++}$, which is not an isomorphism. In addition, it is elementary to check that the category $\mathcal{P}_{even}^- =< \mathcal{NC}_{even}^-, \mathcal{P}_2 >$ contains all 4 matching versions of $\sqcap\!\sqcap\!\sqcap$, and so equals $\mathcal{P}_{even}$ itself, which gives $(K_N^{++})^c = K_N$.

Summarizing, we have our "exotic" quantum groups $B_N'^+ \subset B_N'^{++}$ and $K_N^+ \subset K_N^{++}$, and these provide counterexamples to both (1) and (2), as claimed. □

We can now further comment on the relation between the SLC, AC, FC. The key connecting statement would be the "strong SLC" from Theorem 1.9 (2) above, but as explained there, this latter statement is wrong. We do not have a fix for this fact, although we believe that such a fix could come from a careful examination of [29].

To be more precise, the question is that of understanding whether the above constructions $G_N^+ \to G_N^{++}$ and $D \to D^-$ are of the "same type", and then if such things can be avoided, via some simple extra axiom. Indeed, imposing such an extra axiom could probably lead to a "strong SLC", making a clear link between the SLC, AC, FC.

There is probably a relation here with the work in [21], [22] as well.

2. Hard liberation

We discuss here an alternative approach to the liberation operation, which is harder to perform, but which leads to more powerful consequences.

This is the "hard" liberation operation, obtained by using tori. The idea indeed, coming from [6], [7], [8] and from a number of preceding papers, notably [15], [16], [18], is to construct the liberation operation by using a free real torus, as follows:

$$G_N^+ =< G_N, T_N^+ > \quad , \quad T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$$

In practice, this is known to work for basic groups like O_N, U_N, but the situation in general is more complicated. For instance this does not work for the real and complex bistochastic groups B_N, C_N, whose diagonal torus collapses, and nor does it work for S_N, for the same reason. For these quantum groups the solution is by using carefully chosen spinned tori, as in [7], with the exact procedure being not axiomatized yet.

In what follows we will not get into such difficulties, and we will focus on the case of hard liberation using diagonal tori, which works for O_N, U_N, and also, to some reasonable extent, for the other two groups that we are interested in here, namely H_N, K_N.

Let us begin with the following definition:

Definition 2.1. *A compact Lie group liberation $G_N \subset G_N^\times$ is called hard liberation when*

$$G_N^\times =< G_N^c, T_N^\times >$$

where $T_N^\times \subset G_N^\times$ is the diagonal torus.

Here, and in what follows, the diagonal torus is by definition constructed by imposing the conditions $u_{ij} = 0$, for any $i \neq j$. The fact that we obtain indeed a noncommutative torus, or group dual, comes from the fact that the elements $g_i = u_{ii}$ satisfy $\Delta(g_i) = g_i \otimes g_i$ in the quotient, and so are group-like, and generate a discrete group. See [12].

As a main statement regarding the hard liberation, we have:

Conjecture 2.2 (Hard liberation conjecture, HLC). *If $G_N^c \subset G_N^\circ$ is an easy hard liberation of an easy compact Lie group, then any intermediate liberation*

$$G_N \subset G_N^\times \subset G_N^\circ$$

is an easy hard liberation as well.

As we will see in what follows, verifying such things is in general non-trivial.

Summarizing, we have so far many interesting conjectures, which are related between them, usually in a non-trivial way. In what follows we will verify some of these conjectures in some basic cases, leaving the more technical verifications for some future work.

In order to further comment on these questions, let us introduce:

Definition 2.3. *A family $G = (G_N)$ with $G_N \subset U_N^+$ is called uniform when*

$$G_{N-1} = G_N \cap U_{N-1}^+$$

for any $N \geq 2$, with the embeddings $U_{N-1}^+ \subset U_N^+$ being given by $u \to diag(u, 1)$.

As a first remark, under this uniformity assumption, when assuming that G_{N-1} is not classical, G_N is not classical either. Thus, there is an integer $n \in \{2, 3, \ldots, \infty\}$ such that $G_1, \ldots, G_{n-1}$ are all classical, and then $G_n, G_{n+1}, \ldots$ are all non-classical.

Inspired from the work in [15], [16], [18], let us formulate now:

Proposition 2.4. *Assuming that $G = (G_N)$ is uniform, and letting $n \in \{2, 3, \ldots, \infty\}$ be minimal such that G_n is not classical, consider the following generation conditions:*

1. *Strong generation: $G_N =< G_N^c, G_n >$, for any $N > n$.*
2. *Usual generation: $G_N =< G_N^c, G_{N-1} >$, for any $N > n$.*
3. *Initial step generation: $G_{n+1} =< G_{n+1}^c, G_n >$.*

We have then (1) $\iff$ (2) $\implies$ (3), and (3) is in general strictly weaker.

Proof. All the implications and non-implications are elementary, as follows:

(1) $\implies$ (2) This follows from $G_n \subset G_{N-1}$ for $N > n$, coming from uniformity.

(2) $\implies$ (1) By using twice the usual generation, and then the uniformity, we have:

$$\begin{aligned} G_N &= < G_N^c, G_{N-1} > \\ &= < G_N^c, G_{N-1}^c, G_{N-2} > \\ &= < G_N^c, G_{N-2} > \end{aligned}$$

Thus we have a descent method, and we end up with the strong generation condition.

(2) $\implies$ (3) This is clear, because (2) at $N = n + 1$ is precisely (3).

(3) $\not\Longrightarrow$ (2) In order to construct counterexamples here, simplest is to use group duals. Indeed, with $G_N=\widehat{\Gamma_N}$ and $\Gamma_N=<g_1,\ldots,g_N>$, the uniformity condition from Definition 2.3 tells us that we must be in a projective limit situation, as follows:

$$\Gamma_1\leftarrow\Gamma_2\leftarrow\Gamma_3\leftarrow\Gamma_4\leftarrow\ldots\quad,\quad\Gamma_{N-1}=\Gamma_N/<g_N=1>$$

But with this picture in hand, the result is clear. Indeed, assuming for instance that Γ_2 is given and not abelian, there are many ways of completing the sequence, and so the uniqueness coming from the generation condition in (2) can only fail. $\square$

In relation now with the HLC, let us introduce as well:

Proposition 2.5. *Assuming that $G=(G_N)$ is uniform, and letting $n\in\{2,3,\ldots,\infty\}$ be as above, consider the following conditions, where $I_N\subset G_N$ is the diagonal torus:*

1. *Strong hard liberation: $G_N=<G_N^c,I_n>$, for any $N\geq n$.*
2. *Technical condition: $G_N=<G_N^c,I_{N-1}>$ for any $N>n$, and $G_n=<G_n^c,I_n>$.*
3. *Hard liberation: $G_N=<G_N^c,I_N>$, for any N.*
4. *Initial step hard liberation: $G_n=<G_n^c,I_n>$.*

We have then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

Proof. Our first claim is that when assuming that $G=(G_N)$ is uniform, the family of diagonal tori $I=(I_N)$ follows to be uniform as well. In order to prove this claim, observe first that the definition of the diagonal torus can be reformulated as follows:

$$I_N=G_N\cap\widehat{F_N}$$

WIth this picture in hand, the uniformity claim for $I=(I_N)$ comes from that of $G=(G_N)$, and from that of $\widehat{F}=(\widehat{F_N})$, which is trivial, as follows:

$$\begin{aligned}I_N\cap U_{N-1}^+&=(G_N\cap\widehat{F_N})\cap U_{N-1}^+\\&=(G_N\cap U_{N-1}^+)\cap(\widehat{F_N}\cap U_{N-1}^+)\\&=G_{N-1}\cap\widehat{F_{N-1}}\\&=I_{N-1}\end{aligned}$$

Thus our claim is proved, and this gives the various implications in the statement. $\square$

Let us discuss now to understand the relationship between the above conditions. In the group dual case, the simplest example to look at is the free real torus, $G=(T_N^+)$. Here, with respect to the $3+4=7$ conditions that we have, the last 2 conditions trivially hold, and the first 5 conditions all require $T_3^+=<T_3,T_2^+>$, which is wrong. Indeed, in order to see this latter fact, consider the following discrete group:

$$\Gamma=\left\langle a,b,c\Big|a^2=b^2=c^2=1,[a,b]=[a,c]=1\right\rangle$$

We have then $T_3\subset\widehat{\Gamma}$ and $T_2^+\subset\widehat{\Gamma}$ as well, and so $<T_3,T_2^+>\subset\widehat{\Gamma}$. On the other hand we have $\Gamma\neq\mathbb Z_2^{*3}$, and so $\widehat{\Gamma}\neq T_3^+$, and we conclude that we have $<T_3,T_2^+>\neq T_3^+$.

With these preliminaries in hand, we can now formulate our main theoretical observation on the subject, which is a statement related to the HLC, as follows:

Theorem 2.6. *Assuming that $G=(G_N)$ is uniform, and letting $n\in\{2,3,\ldots,\infty\}$ be as above, minimal such that G_n is not classical, the following conditions are equivalent,*

1. *Generation: $G_N=<G_N^c,G_{N-1}>$, for any $N>n$.*
2. *Strong generation: $G_N=<G_N^c,G_n>$, for any $N>n$.*
3. *Hard liberation: $G_N=<G_N^c,I_N>$, for any $N\geq n$.*
4. *Strong hard liberation: $G_N=<G_N^c,I_n>$, for any $N\geq n$.*

modulo their initial steps.

Proof. Our first claim is that generation plus initial step hard liberation imply the technical hard liberation condition. Indeed, the recurrence step goes as follows:

$$\begin{aligned}G_N&=<G_N^c,G_{N-1}>\\&=<G_N^c,G_{N-1}^c,I_{N-1}>\\&=<G_N^c,I_{N-1}>\end{aligned}$$

In order to pass now from the technical hard liberation condition to the strong hard liberation condition itself, observe that we have:

$$\begin{aligned}G_N&=<G_N^c,G_{N-1}>\\&=<G_N^c,G_{N-1}^c,I_{N-1}>\\&=<G_N^c,I_{N-1}>\end{aligned}$$

With this condition in hand, we have then as well:

$$\begin{aligned}G_N&=<G_N^c,G_{N-1}>\\&=<G_N^c,G_{N-1}^c,I_{N-2}>\\&=<G_N^c,I_{N-2}>\end{aligned}$$

This procedure can be of course be continued. Thus we have a descent method, and we end up with the strong hard liberation condition.

In the other sense now, we want to prove that we have $G_N=<G_N^c,G_{N-1}>$ at $N\geq n$. At $N=n+1$ this is something that we already have. At $N=n+2$ now, we have:

$$\begin{aligned}G_{n+2}&=<G_{n+2}^c,I_n>\\&=<G_{n+2}^c,G_{n+1}^c,I_n>\\&=<G_{n+2}^c,G_{n+1}>\end{aligned}$$

This procedure can be of course be continued. Thus, we have a descent method, and we end up with the strong generation condition. □

The above results remain of course quite theoretical. Still at the theoretical level, we believe that the uniformity condition and generation condition are best viewed together. The idea indeed is that given a family of compact quantum groups $G = (G_N)$ with $G_N \subset U_N^+$, we have a "ladder of cubes", formed by cubes as follows:

$$\begin{tikzcd}
 & U_{N-1}^+ \arrow[rr] & & U_N^+ \\
G_{N-1} \arrow[rr] \arrow[ur] & & G_N \arrow[ur] & \\
 & U_{N-1} \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
G_{N-1}^{class} \arrow[rr] \arrow[uu] \arrow[ur] & & G_N^{class} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Thus, as a natural problem, we have the question of investigating the $2 \times 6 = 12$ intersection and generation properties, for the faces of such cubes, either with $N \in \mathbb{N}$ arbitrary, or with $N \geq n$. It is quite unclear on what can be done here, but as a general idea, this could emerge on a notion of "super-strong uniformity", with findings refining those above. We intend to come back to these questions in some future work.

3. Reflection groups

In this section and in the next ones we discuss the various questions raised in sections 1-2 above, for the basic examples of compact Lie groups, namely H_N, K_N, O_N, U_N. The choice of these groups, widely known to be of key importance in quantum group theory, is best justified by the recent note [4]. It was shown indeed there that, under very strong axioms, these 4 groups and their liberations $H_N^+, K_N^+, O_N^+, U_N^+$ are the only ones left.

We start with a study of the hyperoctahedral group $H_N = \mathbb{Z}_2 \wr S_N$, and its liberations. The theory here is non-trivial, going back to the papers of Raum and Weber [26], [27], [28], and with some extra useful information coming from the more recent paper [1]. Our first purpose will be that of reviewing this material. Let us begin with:

Proposition 3.1. *We have a quantum group $H_N^{[\infty]} \subset H_N^+$, obtained via the relations $abc = 0$, for any $a \neq c$ on the same row or column of $u = (u_{ij})$. This quantum group is easy, and the corresponding category of partitions $P_{even}^{[\infty]}$ can be described as follows:*

1. *$P_{even}^{[\infty]}$ is the category generated by $\eta = \ker\binom{aab}{baa}$.*
2. *$P_{even}^{[\infty]} = \{\pi \in P_{even} | \sigma \in P_{even}^*, \forall \sigma \subset \pi\}$.*
3. *$P_{even}^{[\infty]} = \{\pi \in P_{even} | \varepsilon(\tau) = 1, \forall \tau \leq \pi\}$.*

Proof. All this is quite technical, and we refer to [26], [27], [28] and to [1] for the various unexplained notions and for details of the proof, the idea being as follows:

(1) This comes from the fact that η implements the relations in the statement. As a side remark here, η and $P_{even}^{[\infty]}$ were discovered in fact prior to $H_N^{[\infty]}$ itself, due to the fact that the relations $aab=baa$ trivially hold for the real reflection groups. See [28].

(2) This is related to $H_N^*\subset H_N^{[\infty]}$, with P_{even}^* being the category of partitions for H_N^*, consisting of the partitions having the property that when labelling counterclockwise the legs $\circ\bullet\circ\bullet\ldots$, each block has an equal number of black and white legs. See [28].

(3) This is something more recent, with $\varepsilon:P_{even}\to\{\pm1\}$ being the signature function, extending the signature of the usual permutations $S_\infty\subset P_{even}$. Besides giving a useful description of $P_{even}^{[\infty]}$, this formula shows that $H_N^{[\infty]}$ equals its own twist. See [1]. $\square$

Generally speaking, $H_N^{[\infty]}$ is the "main" intermediate liberation of H_N, with all the results in [1], [28] leading to this conclusion. Of key importance here is:

Proposition 3.2. *The easy liberations $H_N\subset H_N^\times\subset H_N^+$ fall into two classes:*

1. *Easy liberations $H_N\subset H_N^\times\subset H_N^{[\infty]}$.*
2. *Easy liberations $H_N^{[\infty]}\subset H_N^\times\subset H_N^+$, with $H_N^{[\infty]}$ excluded.*

Proof. This is something quite technical, whose proof comes from the classification work in [26], [27], [28]. The reasons for including $H_N^{[\infty]}$ into (1) are explained below. $\square$

Let us first discuss the case (1). Given a reflection group $\mathbb Z_2^{*N}\to\Gamma\to\mathbb Z_2^N$ which is uniform, in the sense that each permutation $\sigma\in S_N$ produces a group automorphism, $g_i\to g_{\sigma(i)}$, we can associate to it a category of partitions $D=(D(k,l))$, as follows:

$$D(k,l)=\left\{\pi\in P(k,l)\Big|\ker\binom{i}{j}\leq\pi\implies g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}\right\}$$

Observe that we have $P_{even}^{[\infty]}\subset D\subset P_{even}$, with the inclusions coming respectively from $\eta\in D$, and from $\Gamma\to\mathbb Z_2^N$. Conversely, given a category of partitions $P_{even}^{[\infty]}\subset D\subset P_{even}$, we can associate to it a uniform reflection group $\mathbb Z_2^{*N}\to\Gamma\to\mathbb Z_2^N$, as follows:

$$\Gamma=\left\langle g_1,\ldots g_N\Big|g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l},\forall i,j,k,l,\ker\binom{i}{j}\in D(k,l)\right\rangle$$

As explained in [27], the correspondences $\Gamma\to D$ and $D\to\Gamma$ are bijective, and inverse to each other, at $N=\infty$. We have in fact the following result, from [26], [27], [28]:

Proposition 3.3. *We have correspondences between:*

1. *Uniform reflection groups $\mathbb Z_2^{*\infty}\to\Gamma\to\mathbb Z_2^\infty$.*
2. *Categories of partitions $P_{even}^{[\infty]}\subset D\subset P_{even}$.*
3. *Easy quantum groups $G=(G_N)$, with $H_N^{[\infty]}\supset G_N\supset H_N$.*

These correspondences will be denoted as $\Gamma\leftrightarrow P_{even}^\Gamma\leftrightarrow H_N^\Gamma$.

Proof. This is something quite technical. As an illustration, if we denote by $\mathbb{Z}_2^{\circ N}$ the quotient of $\mathbb{Z}_2^{*N}$ by the relations of type $abc = cba$ between the generators, we have:

$$\begin{tikzcd}
\mathbb{Z}_2^N \arrow[d, squiggly, no head] & \mathbb{Z}_2^{\circ N} \arrow[l] \arrow[d, squiggly, no head] & \mathbb{Z}_2^{*N} \arrow[l] \arrow[d, squiggly, no head] \\
H_N \arrow[r] & H_N^* \arrow[r] & H_N^{[\infty]}
\end{tikzcd}$$

In general, the various results follow from some combinatorial work. See [28]. $\square$

Regarding now the case (2) in Proposition 3.2, the result here, from [28], is:

Proposition 3.4. *Let $H_N^{[r]} \subset H_N^+$ be the easy quantum group coming from:*

$$\pi_r = \ker \begin{pmatrix} 1 & \dots & r & r & \dots & 1 \\ 1 & \dots & r & r & \dots & 1 \end{pmatrix}$$

Then $H_N^+ = H_N^{[1]} \supset H_N^{[2]} \supset H_N^{[3]} \supset \ldots \supset H_N^{[\infty]}$, and we obtain in this way all the intermediate easy quantum groups $H_N^{[\infty]} \subset G \subset H_N^+$, satisfying $G \neq H_N^{[\infty]}$.

Proof. For full details here, we refer to the paper [28]. $\square$

As a conclusion to all this, we have the following result, from [28]:

Proposition 3.5. *The easy quantum groups $H_N \subset G_N \subset H_N^+$, and the corresponding diagonal tori, are as follows,*

$$\begin{tikzcd}
H_N \arrow[r] & H_N^\Gamma \arrow[r] & H_N^{[\infty]} \arrow[r] & H_N^{[r]} \arrow[r] & H_N^+ \\
T_N \arrow[r] \arrow[u] & \widehat{\Gamma} \arrow[r] \arrow[u] & T_N^+ \arrow[r] \arrow[u] & T_N^+ \arrow[r] \arrow[u] & T_N^+ \arrow[u]
\end{tikzcd}$$

with the family H_N^Γ and the series $H_N^{[r]}$ being constructed as above.

Proof. The classification result follows by combining the above results, and the assertion about the diagonal tori is clear from definitions. See [28]. $\square$

With these results in hand, we can go back now to our hard liberation questions. Obviously, with our present hard liberation theory, based on blowing up the diagonal torus, we cannot get beyond $H_N^{[\infty]}$, due to Proposition 3.5 above. Thus, we have to focus on the quantum groups of type H_N^Γ. And here we have the following result:

Theorem 3.6. *The quantum groups H_N^Γ appear via hard liberation, as follows:*

$$H_N^\Gamma = < H_N, \widehat{\Gamma} >$$

In particular, we have the "master formula" $H_N^{[\infty]} = < H_N, T_N^+ >$.

Proof. We use the basic fact, from [27], and which is complementary to the easiness considerations above, that we have a crossed product decomposition as follows:

$$H_N^\Gamma = \widehat{\Gamma} \rtimes S_N$$

With this result in hand, we obtain that we have the missing inclusion, namely:

$$H_N^\Gamma \ = < S_N, \widehat{\Gamma} > \subset < H_N, \widehat{\Gamma} >$$

Finally, the last assertion is clear, by taking $\Gamma = \mathbb{Z}_2^{*N}$. Indeed, this group produces $H_N^{[\infty]}$, and the corresponding group dual is the free real torus T_N^+. □

As an interesting consequence of Theorem 3.6, let us record the following formula:

Proposition 3.7. *We have the following formula,*

$$span(P_{even}^\Gamma) = span(P_{even}) \cap C_{\widehat{\Gamma}}$$

where $C_{\widehat{\Gamma}}$ is the Tannakian category associated to $\widehat{\Gamma}$.

Proof. We use the Tannakian approach to the intersection and generation operations $\cap$ and $<,>$, which is summarized in the following well-known formulae, going back to [18], and widely used in the recent literature on the subject [4], [6], [7], [8]:

$$C_{G\cap H} = < C_G, C_H > \quad , \quad C_{<G,H>} = C_G \cap C_H$$

With these general formulae in hand, the generation formula in Theorem 3.6, namely $H_N^\Gamma = < H_N, \widehat{\Gamma} >$, reformulates in terms of Tannakian categories as follows:

$$C_{H_N^\Gamma} = C_{H_N} \cap C_{\widehat{\Gamma}}$$

But this is precisely the equality in the statement. □

In practice now, the category $C_{\widehat{\Gamma}}$ appearing in Proposition 3.7 above is given by the following well-known formula, for which we refer for instance to [7]:

$$C_{\widehat{\Gamma}}(k,l) = \left\{ T \in M_{N^l \times N^k}(\mathbb{C}) \Big| g_{i_1} \ldots g_{i_k} \neq g_{j_1} \ldots g_{j_l} \implies T_{j_1 \ldots j_l, i_1 \ldots i_k} = 0 \right\}$$

With this formula in hand, it is clear that the $\subset$ inclusion in Proposition 3.7 holds indeed, and that $\supset$ holds as well on P_{even}. However, having $\supset$ extended to the span of P_{even} looks like a difficult combinatorial question. Thus, as a philosophical conclusion, the crossed product results in [27] solve a difficult combinatorial question.

The above results are basically what we need, in what follows. There are of course many other interesting questions regarding the quantum reflection groups, as follows:

(1) Hard liberation of $H_N^{[r]}$, and of H_N^+ itself. Here we cannot do this with the diagonal torus alone, and we must blow up some spinned tori as well. See [7], [8].

(2) Further classification questions. It is pehaps reasonable to conjecture that the intermediate quantum groups $H_N \subset H_N^\times \subset H_N^+$ must be easy. This looks difficult.

(3) Recurrent generation questions. This comes in relation with the considerations from section 2 above, and the subject is probably quite technical. See [15].

4. Complex reflections

We discuss here the same questions as before, this time for the complex reflection groups $K_N = \mathbb{T} \wr S_N$. The situation here is a bit different than the one for H_N, and for the groups O_N, U_N too, because the classification work for the easy quantum groups [11], [12], [23], [24], [28], [29] has avoided so far the classification of the easy liberations of K_N.

In the lack of this key ingredient, we can simply construct examples, by using our soft liberation operation, and then study them. Let us begin with:

Proposition 4.1. *We have easy quantum groups $K_N^\times$ as follows,*

$$\begin{tikzcd}
K_N \arrow[r] & K_N^\Gamma \arrow[r] & K_N^{[\infty]} \arrow[r] & K_N^{[r]} \arrow[r] & K_N^+ \\
H_N \arrow[r] \arrow[u] & H_N^\Gamma \arrow[r] \arrow[u] & H_N^{[\infty]} \arrow[r] \arrow[u] & H_N^{[r]} \arrow[r] \arrow[u] & H_N^+ \arrow[u]
\end{tikzcd}$$

obtained by soft intermediate liberation, $K_N^\times =< K_N, H_N^\times >$.

Proof. This is more of an empty statement, with perhaps the only thing to be justified being the fact that $K_N, K_N^{[\infty]}, K_N^+$, which are already known, appear indeed via soft liberation. But this latter fact follows by interesting categories, with input from [1], [29]. $\square$

In relation now with our hard liberation questions, we first have:

Proposition 4.2. *The diagonal tori of the quantum groups $K_N^\times$ are as follows,*

$$\begin{tikzcd}
K_N \arrow[r] & K_N^\Gamma \arrow[r] & K_N^{[\infty]} \arrow[r] & K_N^{[r]} \arrow[r] & K_N^+ \\
\mathbb{T}_N \arrow[r] \arrow[u] & \widehat{\Gamma}_c \arrow[r] \arrow[u] & \mathbb{T}_N^+ \arrow[r] \arrow[u] & \mathbb{T}_N^+ \arrow[r] \arrow[u] & \mathbb{T}_N^+ \arrow[u]
\end{tikzcd}$$

with $\Gamma \to \Gamma_c$ being a certain complexification operation, satisfying $< \mathbb{T}_N, \widehat{\Gamma} >\subset \widehat{\Gamma}_c$.

Proof. As a first observation, the results are clear and well-known for the endpoints K_N, K_N^+ and for the middle point $K_N^{[\infty]}$ as well. Indeed, these are, as explained in the proof of Proposition 4.1 above, previously known quantum groups, from [1], [29].

By functoriality it follows that the diagonal torus of $K_N^{[r]}$ must be the free complex torus $\mathbb{T}_N^+$, for any $r \in \mathbb{N}$, so we are done with the right part of the diagram.

Regarding now the left part of the diagram, concerning the quantum groups K_N^Γ, if we denote by $T_1(.)$ the diagonal torus, we have, by using [7]:

$$\begin{aligned} T_1(K_N^\Gamma) &= T_1(< K_N, H_N^\Gamma >) \\ &\supset < T_1(K_N), T_1(H_N^\Gamma) > \\ &= < \mathbb{T}_N, \widehat{\Gamma} > \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Observe that the above inclusion $< \mathbb{T}_N, \widehat{\Gamma} > \subset \widehat{\Gamma}_c$ fails to be an isomorphism, and this for instance for $\Gamma = \mathbb{Z}_2^{*N}$. However, the construction $\Gamma \to \Gamma_c$ can be in principle explicitely computed, for instance by using Tannakian methods. Indeed, our soft liberation formula $K_N^\Gamma =< K_N, H_N^\Gamma >$ translates into a Tannakian formula, as follows:

$$\mathcal{P}_{even}^\Gamma = \mathcal{P}_{even} \cap P_{even}^\Gamma$$

The problem is that of explicitely computing the category on the left, corresponding to K_N^Γ, and then of deducing from this a presentation formula for the associated diagonal torus $\widehat{\Gamma}_c$, by using methods from [7]. All this is probably related to [2], [21], [28].

Now back to the hard liberation question, we have the following result:

Theorem 4.3. *The quantum groups K_N^Γ appear via hard liberation, and this even in a stronger form, as follows:*

$$K_N^\Gamma =< K_N, \widehat{\Gamma} >$$

In particular, we have the formula $K_N^{[\infty]} =< K_N, T_N^+ >$.

Proof. This follows from Theorem 3.6 and Proposition 4.1. Indeed, we have:

$$\begin{aligned} K_N^\Gamma &= < K_N, H_N^\Gamma > \\ &= < K_N, H_N, \widehat{\Gamma} > \\ &= < K_N, \widehat{\Gamma} > \end{aligned}$$

Thus we have the formula in the statement, and the fact that this implies the fact that K_N^Γ appears indeed via hard liberation follows from Proposition 4.2 above.

Finally, with $\Gamma = \mathbb{Z}_2^{*N}$ we obtain from this the formula $K_N^{[\infty]} =< K_N, T_N^+ >$. □

Regarding the hard liberation question for the quantum groups $K_N^{[r]}$, the problem here is open, the difficulties being similar to those for the quantum groups $H_N^{[r]}$.

5. Orthogonal groups

In relation now with the orthogonal groups, the situation is much simpler, because the quantum groups $O_N \subset O_N^* \subset O_N^+$ are known from [12] to be the only easy liberations of O_N. In addition, it is known from [10] that the inclusion $O_N \subset O_N^*$ is maximal, in the sense that it has no intermediate object at all. Also, as explained in [10], the conjecture is that $O_N \subset O_N^* \subset O_N^+$ are the only liberations of O_N, not necessarily easy.

In order to discuss all this, we will need a technical result, as follows:

Proposition 5.1. *We have the generation formula*

$$O_N^+ =< O_N, H_N^{[\infty]} >$$

where $H_N^{[\infty]}$ is the liberation of H_N from section 3 above.

Proof. We use the Tannakian approach to the intersection and generation operations $\cap$ and $<,>$, explained in the proof of Proposition 3.7 above. According to the general formula $C_{<G,H>} = C_G \cap C_H$ there, the formula in the statement is equivalent to:

$$C_{O_N^+} = C_{O_N} \cap C_{H_N^{[\infty]}}$$

By easiness, we are led into the following combinatorial statement:

$$NC_2 = P_2 \cap P_{even}^{[\infty]}$$

In order to establish this latter formula, we use the explicit description of $P_{even}^{[\infty]}$ given in Proposition 3.1 (2) above, which is as follows:

$$P_{even}^{[\infty]} = \{\pi \in P_{even} | \sigma \in P_{even}^*, \forall \sigma \subset \pi\}$$

With this formula in hand, the fact that we have $NC_2 \subset P_2 \cap P_{even}^{[\infty]}$ is of course clear. This is in fact something that we already know, coming from $O_N^+ \supset< O_N, H_N^{[\infty]} >$.

Regarding the reverse inclusion, let $\pi \in P_2 \cap P_{even}^{[\infty]}$. If we assume that π has a crossing, then we have a basic crossing $\sigma \subset \pi$, and since we have $\sigma \notin P_{even}^*$, we obtain in this way a contradition. Thus our reverse inclusion is proved, and we are done. $\square$

As a comment here, the above result can be deduced as well from the classification reaults in [12], by using the fact that the quantum group $O_N^\times =< O_N, H_N^{[\infty]} >$ is easy, and is not classical, nor half-classical. However, all this is ultimately too complicated, and having a direct and clear proof as above is probably something quite useful.

In relation now with our hard liberation questions, we have:

Proposition 5.2. *The quantum groups O_N, O_N^*, O_N^+ all appear via hard liberation,*

$$O_N^\times =< O_N, T_N^\times >$$

where $T_N^\times \subset O_N^\times$ is the diagonal torus, equal respectively to T_N, T_N^*, T_N^+.

Proof. This is trivial for O_N, and known from [6] for O_N^*. In the case of O_N^+ the problem is more difficult, as explained in [8], but we have now the following complete proof:

$$\begin{aligned} O_N^+ &= < O_N, H_N^{[\infty]} > \\ &= < O_N, H_N, T_N^+ > \\ &= < O_N, T_N^+ > \end{aligned}$$

To be more precise here, the first formula is from Proposition 5.1, the second one is the 'master formula", from Theorem 3.6 above, and the last one is trivial. □

Let us go back now to the conjecture regarding $O_N \subset O_N^* \subset O_N^+$, which is probably the most interesting statement around. As already mentioned, it is known from [12] that $O_N \subset O_N^* \subset O_N^+$ are the unique easy liberations of O_N. In terms of our present formalism, this means that $O_N \subset O_N^* \subset O_N^+$ are the unique soft liberations of O_N.

Here is a related result, providing some slight advances on this question:

Theorem 5.3. *The basic orthogonal quantum groups, namely*

$$O_N \subset O_N^* \subset O_N^+$$

are the unique hard liberations of O_N.

Proof. A hard liberation of O_N must appear by definition as follows, for a certain real reflection group $\mathbb{Z}_2^{*N} \to \Gamma \to \mathbb{Z}_2^N$, whose dual is the diagonal torus of the liberation:

$$O_N^\Gamma =< O_N, \widehat{\Gamma} >$$

On the other hand, we have the following computation, based on Theorem 3.6, on the fact that the class of easy quantum groups is stable under $<,>$, and finally on [12]:

$$\begin{aligned} O_N^\Gamma &= < O_N, \widehat{\Gamma} > \\ &= < O_N, H_N, \widehat{\Gamma} > \\ &= < O_N, H_N^\Gamma > \\ &\in \{O_N, O_N^*, O_N^+\} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We believe that Theorem 5.3 can be further extended, by using the notion of spinned tori from [7], and the corresponding notions of hard liberation. However, the spinned version of the hard liberation is something which is not axiomatized yet.

6. Unitary groups

We are interested in what follows in the intermediate quantum groups $U_N \subset G \subset U_N^+$. A first construction of such quantum groups, from [9], [23], is as follows:

Proposition 6.1. *Associated to any $r \in \mathbb{N}$ is the quantum group $U_N \subset U_N^{(r)} \subset U_N^+$ coming from the category $\mathcal{P}_2^{(r)}$ of matching pairings having the property that $\#\circ = \#\bullet\,(r)$ holds between the legs of each string. These quantum groups have the following properties:*

(1) *At $r = 1$ we obtain the usual unitary group, $U_N^{(1)} = U_N$.*
(2) *At $r = 2$ we obtain the half-classical unitary group, $U_N^{(2)} = U_N^*$.*
(3) *For any $r|s$ we have an embedding $U_N^{(r)} \subset U_N^{(s)}$.*
(4) *In general, we have an embedding $U_N^{(r)} \subset U_N^r \rtimes \mathbb{Z}_r$.*
(5) *We have as well a cyclic matrix model $C(U_N^{(r)}) \subset M_r(C(U_N^r))$.*
(6) *In this latter model, $\int_{U_N^{(r)}}$ appears as the restriction of $tr_r \otimes \int_{U_N^r}$.*

Proof. This is something quite compact, summarizing the various findings from [9], [23]. Here are a few brief explanations on all this:

(1) This is clear from $\mathcal{P}_2^{(1)} = \mathcal{P}_2$, and from a well-known result of Brauer [17].

(2) This is because $\mathcal{P}_2^{(2)}$ is generated by the partitions with implement the relations $abc = cba$ between the variables $\{u_{ij}, u_{ij}^*\}$, used in [14] for constructing U_N^*.

(3) This simply follows from $\mathcal{P}_2^{(s)} \subset \mathcal{P}_2^{(r)}$, by functoriality.

(4) This is the original definition of $U_N^{(r)}$, from [9]. We refer to [9] for the exact formula of the embedding, and to [23] for the compatibility with the Tannakian definition.

(5) This is also from [9], more specifically it is an alternative definition for $U_N^{(r)}$.

(6) Once again, this is something from [9], and we will be back to it. □

Let us discuss now the second known construction of unitary quantum groups, from [24]. This construction uses an additive semigroup $D \subset \mathbb{N}$, but as pointed out there, using instead the complementary set $C = \mathbb{N} - D$ leads to several simplifications.

So, let us call "cosemigroup" any subset $C \subset \mathbb{N}$ which is complementary to an additive semigroup, $x, y \notin C \implies x + y \notin C$. The construction from [24] is then:

Proposition 6.2. *Associated to any cosemigroup $C \subset \mathbb{N}$ is the easy quantum group $U_N \subset U_N^C \subset U_N^+$ coming from the category $\mathcal{P}_2^C \subset P_2^{(\infty)}$ of pairings having the property $\#\circ - \#\bullet \in C$, between each two legs colored $\circ, \bullet$ of two strings which cross. We have:*

(1) *For $C = \emptyset$ we obtain the quantum group U_N^+.*
(2) *For $C = \{0\}$ we obtain the quantum group $U_N^\times$.*
(3) *For $C = \{0, 1\}$ we obtain the quantum group U_N^{**}.*
(4) *For $C = \mathbb{N}$ we obtain the quantum group $U_N^{(\infty)}$.*
(5) *For $C \subset C'$ we have an inclusion $U_N^{C'} \subset U_N^C$.*
(6) *Each quantum group U_N^C contains each quantum group $U_N^{(r)}$.*

Proof. Once again this is something very compact, coming from recent work in [24], with our convention that the semigroup $D \subset \mathbb{N}$ which is used there is replaced here by its complement $C = \mathbb{N} - D$. Here are a few explanations on all this:

(1) The assumption $C=\emptyset$ means that the condition $\#\circ-\#\bullet\in C$ can never be applied. Thus, the strings cannot cross, we have $\mathcal P_2^\emptyset=\mathcal{NC}_2$, and so $U_N^\emptyset=U_N^+$.

(2) As explained in [24], here we obtain indeed the quantum group $U_N^\times$ from [13], constructed there by using the relations $ab^*c=cb^*a$, with $a,b,c\in\{u_{ij}\}$.

(3) This is also explained in [24], with U_N^{**} being the quantum group from [9], which is the biggest whose full projective version, in the sense there, is classical.

(4) Here the assumption $C=\mathbb N$ simply tells us that the condition $\#\circ-\#\bullet\in C$ in the statement is irrelevant. Thus, we have $\mathcal P_2^{\mathbb N}=\mathcal P_2^{(\infty)}$, and so $U_N^{\mathbb N}=U_N^{(\infty)}$.

(5) This is clear by functoriality, because $C\subset C'$ implies $\mathcal P_2^C\subset\mathcal P_2^{C'}$.

(6) This is clear from definitions, and from Proposition 6.1 above. $\square$

We have the following key result, from [24]:

Proposition 6.3. *The easy quantum groups $U_N\subset G\subset U_N^+$ are as follows,*

$$U_N\subset\{U_N^{(r)}\}\subset\{U_N^C\}\subset U_N^+$$

with the series covering U_N, and the family covering U_N^+.

Proof. This is something highly non-trivial, and we refer here to [24]. The general idea is that $U_N^{(\infty)}$ produces a dichotomy for the quantum groups in the statement, and this leads, via massive combinatorial computations, to the series and the family. See [23], [24]. $\square$

Observe that there is an obvious similarity here with the dichotomy in Proposition 3.2, for the liberations of H_N, coming from [28]. We will be back to this, later on.

In relation now with our liberation questions, we have:

Theorem 6.4. *The basic unitary quantum groups, U_N,U_N^*,U_N^+, appear via real and complex soft liberation, and via hard liberation as well, as follows:*

1. *If we set $K_N^\times=U_N^\times\cap K_N^+$, we have $U_N^\times=<U_N,K_N^\times>$.*
2. *In fact, if we set $H_N^\times=U_N^\times\cap H_N^+$, we have $U_N^\times=<U_N,H_N^\times>$.*
3. *In the free case, we have as well the formula $U_N^+=<U_N,H_N^{[\infty]}>$.*
4. *We have $U_N^\times=<U_N,I_N^\times>$, with $I_N^\times\subset U_N^\times$ being the diagonal torus.*

Proof. These results are trivial for U_N, and for U_N^*,U_N^+ the proofs are as follows:

(1) This is well-known since [2], coming from the following standard formulae:

$$\mathcal P_2^*=\mathcal P_2\cap\mathcal P_{even}^*\quad,\quad\mathcal{NC}_2=\mathcal P_2\cap\mathcal{NC}_{even}$$

(2) This enhances (1), by using the following standard formulae:

$$\mathcal P_2^*=\mathcal P_2\cap P_{even}^*\quad,\quad\mathcal{NC}_2=\mathcal P_2\cap NC_{even}$$

(3) This enhances (2) in the free case, and comes from Proposition 5.1, as follows:

$$\begin{aligned} U_N^+ &= < U_N, O_N^+ > \\ &= < U_N, O_N, H_N^{[\infty]} > \\ &= < U_N, H_N^{[\infty]} > \end{aligned}$$

(4) For U_N^* we have indeed the following computation, based on (2):

$$\begin{aligned} U_N^* &= < U_N, H_N^* > \\ &= < U_N, H_N, T_N^* > \\ &= < U_N, T_N^* > \\ &\subset < U_N, \mathbb{T}_N^* > \end{aligned}$$

For U_N^+ we can use a similar method, based on (3), as follows:

$$\begin{aligned} U_N^+ &= < U_N, H_N^{[\infty]} > \\ &= < U_N, H_N, T_N^+ > \\ &= < U_N, T_N^+ > \\ &\subset < U_N, \mathbb{T}_N^+ > \end{aligned}$$

Since the reverse inclusions are clear, this finishes the proof. $\square$

The above result is of course something quite elementary, and having the HLC proved for U_N, which would amount in proving (4) for all the quantum groups in Proposition 6.1 and Proposition 6.2, is still something which is far away, requiring lots of work.

For the quantum groups $U_N^{(r)}$ the corresponding reflection groups $K_N^{(r)} = U_N^{(r)} \cap K_N^+$ can be explicitely computed, because we have a diagram as follows:

$$\begin{tikzcd} U_N^r \rtimes \mathbb{Z}_r \arrow[r] & U_N^{(r)} \\ S_N^r \rtimes \mathbb{Z}_r \arrow[r] \arrow[u] & K_N^{(r)} \arrow[u] \end{tikzcd}$$

In other words, the construction $K_N \to K_N^{(r)}$ is similar to the construction $U_N \to U_N^{(r)}$, by applying the matrix model construction in [9]. Thus, our strategy of proof from Theorem 6.4, by doing soft liberation, and then hard liberation, looks viable.

For the quantum groups U_N^C, however, the situation is considerably more complicated, because the corresponding reflection groups $K_N^C = U_N^C \cap K_N^+$ seem to collapse to K_N, K_N^*, K_N^+. Thus, we are in need of a new method here. The problem is non-trivial already for $U_N^{(\infty)}, U_N^{**}, U_N^\times$, and we have no solution, even in these basic cases.

7. Duality results

As already noted above, there is an obvious similarity between the dictotomies from Proposition 3.2 and Proposition 6.3, and between the work in [26], [27], [28] and [23], [24] in general, regarding the easy liberations of H_N, and the easy liberations of U_N.

Our purpose here will be that of discussing this phenomenon. Our results will be quite modest. In fact, this is our third attempt of commenting on this, after [1], [2]. We believe however that all this is useful, at least providing some advertisement, and problems.

Let us begin with the following standard definition, coming from [1], [2]:

Definition 7.1. *We have "covariant" correspondences $H_N^\times \leftrightarrow U_N^\times$ between the liberations of H_N and the liberations of U_N, constructed as follows:*

(1) *To any $U_N^\times$ we can associate the quantum group $H_N^\times = U_N^\times \cap H_N^+$.*
(2) *To any $H_N^\times$ we can associate the quantum group $U_N^\times =< H_N^\times, U_N >$.*

These operations were introduced in [2], in a general setting, in the noncommutative geometry context. Observe that both correspondences are indeed covariant.

In practice now, in the easy case, we have the following result:

Proposition 7.2. *The operations $U_N^\times \to U_N^\times \cap H_N^+$ and $H_N^\times \to < H_N^\times, U_N >$ are both "controlled", in the easy case, by the corresponding quantum groups*

$$O_N^\times \in \{O_N, O_N^*, O_N^+\}$$

appearing via $U_N^\times \to U_N^\times \cap O_N^+$ and $H_N^\times \to < O_N, H_N^\times >$ respectively, and their images collapse to $\{H_N, H_N^*, H_N^+\}$ and $\{U_N, U_N^*, U_N^+\}$ respectively.

Proof. We use here the standard fact, from [12], that the quantum groups $O_N \subset O_N^* \subset O_N^+$ are the unique easy liberations of the orthogonal group O_N. See [12].

With $O_N^\times = U_N^\times \cap O_N^+$, we have the following computation:

$$\begin{aligned}
H_N^\times &= U_N^\times \cap H_N^+ \\
&= U_N^\times \cap O_N^+ \cap H_N^+ \\
&= O_N^\times \cap H_N^+ \\
&\in \{H_N, H_N^*, H_N^+\}
\end{aligned}$$

Also, with $O_N^\times =< O_N, H_N^\times >$ this time, we have the following computation:

$$\begin{aligned}
U_N^\times &= < U_N, H_N^\times > \\
&= < U_N, O_N, H_N^\times > \\
&= < U_N, O_N^\times > \\
&\in \{U_N, U_N^*, U_N^+\}
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The above "covariant duality" is of course something quite deceiving, missing the whole point with the work in [26], [27], [28] and [23], [24]. However, technically speaking, this duality has some applications to classification problems, that we will discuss now.

Let us begin with an elementary statement, as follows:

Proposition 7.3. *We have intermediate quantum groups $H_N \subset G_N \subset U_N^+$ as follows,*

$$\begin{array}{ccccc} H_N & \longrightarrow & K_N & \longrightarrow & U_N \\ \uparrow & & \uparrow & & \uparrow \\ H_N^* & \longrightarrow & K_N^* & \longrightarrow & U_N^* \\ \uparrow & & \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N & \longrightarrow & U_N \end{array}$$

and this is an intersection and generation diagram, in the sense that any of its subsquare diagrams $A \subset B, C \subset D$ satisfies $A = B \cap C, < B, C >= D$.

Proof. The fact that we have a diagram as above is clear from definitions, and the intersection and generation properties follow from easiness. See [4]. □

In general now, any intermediate quantum group $H_N \subset G_N \subset U_N^+$ will appear inside the square, and we can therefore use some "2D orientation" methods in order to deal with it. To be more precise, we can use the following observation, from [4]:

Proposition 7.4. *Given an intersection and generation diagram $P \subset Q, R \subset S$ and an intermediate quantum group $P \subset G \subset S$, we have a diagram as follows:*

$$\begin{array}{ccccc} Q & \longrightarrow & < G, Q > & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow \\ G \cap Q & \longrightarrow & G & \longrightarrow & < G, R > \\ \uparrow & & \uparrow & & \uparrow \\ P & \longrightarrow & G \cap R & \longrightarrow & R \end{array}$$

In addition, G slices the square, in the sense that this is an intersection and generation diagram, precisely when $G =< G \cap Q, G \cap R >$ and $G =< G, Q > \cap < G, R >$.

Proof. This is indeed clear from definitions, because the intersection and generation diagram conditions are automatic for the upper left and lower right squares, as well as half of the generation diagram conditions for the lower left and upper right squares. See [4]. □

Now back to our classification problem, we have the following result:

Theorem 7.5. *The intermediate easy quantum groups $H_N \subset G_N \subset U_N^+$ which slice the square $H_N \subset H_N^+, U_N \subset U_N^+$, in the sense of Proposition 7.4, are as follows,*

$$\begin{array}{ccccc} H_N^+ & \longrightarrow & E_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ H_N^* & \longrightarrow & E_N^* & \longrightarrow & U_N^* \\ \uparrow & & \uparrow & & \uparrow \\ H_N & \longrightarrow & E_N & \longrightarrow & U_N \end{array}$$

with $H_N \subset E_N \subset U_N$ being an easy quantum group, and with E_N^*, E_N^+ being obtained via soft liberation, $E_N^* =< E_N, H_N^* >$ and $E_N^+ =< E_N, H_N^+ >$.

Proof. Assuming that $H_N \subset G_N \subset U_N^+$ is easy, and slices the square, its unitary version $G_N^u =< G_N, U_N >$ must be easy, and so is one of the easy quantum groups $U_N^\times$ from section 6 above. Now observe that the slicing condition tells us in particular that $U_N^\times$ appears via the duality in Proposition 7.2 from its real discrete version $H_N^\times = U_N^\times \cap H_N^+$. Thus by duality we must have $U_N^\times \in \{U_N, U_N^*, U_N^+\}$, and this gives the result. □

As an extra remark, when further imposing the uniformity condition from [4] the half-liberations dissapear, and we are left with the classical and free solutions, from [29].

Let us go back now to duality considerations, with the idea of "fixing" what we have, from Proposition 7.2. The classification results for H_N, U_N have some obvious similarity between them. We have indeed a family followed by a series, and a series followed by a family, and this suggests the existence of a "contravariant duality", as follows:

$$\begin{array}{ccccccc} U_N & \longrightarrow & U_N^{(r)} & \longrightarrow & U_N^C & \longrightarrow & U_N^+ \\ \vdots & & \vdots & & \vdots & & \vdots \\ H_N^+ & \longleftarrow & H_N^{[r]} & \longleftarrow & H_N^\Gamma & \longleftarrow & H_N \end{array}$$

As a first, naive attempt here, we could try to construct such a duality $H_N^\times \leftrightarrow U_N^\times$ by using a kind of "complementation formula", of the following type:

$$< H_N^\times, U_N^\times >= U_N^+$$

To be more precise, given a quantum group $H_N^\times$, we would like to define its dual $U_N^\times$ to be the "minimal" quantum group having the above property, and vice versa. Observe that such a correspondence $H_N^\times \leftrightarrow U_N^\times$ would be indeed contravariant.

In practice now, the main problem comes from the formula $U_N^+=<U_N,H_N^{[\infty]}>$ from Theorem 6.4. This formula shows indeed that our naive attempt presented above simply fails, because the dual of U_N would be $H_N^{[\infty]}$, instead of being H_N^+, as desired.

We believe however that all this should be doable, in the long run, with some further technical input from the classification program of Weber and al. on one hand, and from our soft and hard liberation program on the other, as both programs advance.

Needless to say, we believe that all this is important, and related to many things, and applications. In fact, in our opinion, if the easiness definition from [11] can be agreed upon as being the "Main Definition" in compact quantum groups, then such a duality would be probably the corresponding "Main Theorem" on compact quantum groups.

8. Open problems

We have seen that the soft and hard liberation leads to a fresh point of view on the liberations of O_N,U_N,H_N,K_N. Here are some open problems, in relation with this:

Problem 8.1. *Soft and hard liberation.*

As a key problem here, we have to understand the hard liberation above $H_N^{[\infty]}$, using the spinned tori from [7].

Problem 8.2. *Classification and maximality questions.*

The most basic question here is probably the one regarding the classification of the easy liberations of K_N, in the spirit of [23], [24].

Problem 8.3. *Covariant and contravariant dualities.*

The main problem here, and main problem in general, is that of constructing a contravariant duality between the liberations of H_N and the liberations of U_N.

Problem 8.4. *Noncommutative spheres and geometry.*

An interesting question here is whether the Laplacian theory from [19], [20] extends or not to the noncommutative spheres associated to the quantum groups $U_N^\times$.

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