What Was Division by Zero?; Division by Zero Calculus and New World

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Abstract: In this survey paper, we will introduce the importance of the division by zero and its great impact to elementary mathematics and mathematical sciences for some general people. For this purpose, we will give its global viewpoint in a self-contained manner by using the related references.

Key Words: Division by zero, division by zero calculus, differential equation, analysis, infinity, discontinuous, point at infinity, Laurent expansion, conformal mapping, stereographic projection, Riemann sphere, horn torus, elementary geometry, zero and infinity, $1/0 = 0/0 = z/0 = \tan(\pi/2) = 0$.

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1 Introduction

For the long history of division by zero, see [4, 29]. S. K. Sen and R. P. Agarwal [35] quite recentry referred to our paper [11] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book in the following way:

" Thou shalt not divide by zero " remains valid eternally.

However, in [32] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

For the long tradition on the division by zero, people may not be accepted our new results against many clear evidences on our division by zero, however, a physicist stated as follows:

Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.08:55).

The common sense on the division by zero with the long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on differential coefficients we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In this paper, in a new and definite sense, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$. In particular, we will introduce several fundamental concepts in calculus, Euclidean geometry, analytic geometry, complex analysis and differential equations. We will see new properties on the Laurent expansion, singularity, derivative, extension of solutions of differential equations beyond analytical and isolated singularities, and reduction problems of differential equations. On Euclidean geometry and analytic geometry, we will find new fields by the concept of the division by zero. We will give many concrete properties in mathematical sciences from the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics.

The contents are as follows.

8. Differential equations.
9. Euclidean spaces and division by zero calculus.
10. Analytic functions and division by zero calculus.
11. The Descartes circle theorem.
12. Horn torus models and division by zero calculus – a new world.

2 Division by zero

The division by zero with the mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [11] for example. However, we can state clearly and directly its essence as follows:

For a complex number α and the associated matrix A, the correspondence

$$\alpha = a_1 + ia_2 \longleftrightarrow A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$$

is homomorphism between the complex number field and the matrix field of 2×2.

For any matrix A, there exists a uniquely determined Moore-Penroze generalized inverse $A^\dagger$ satisfying the conditions, for complex conjugate transpose $*$,

$$AA^\dagger A = A,$$

$$A^\dagger AA^\dagger = A^\dagger A,$$

$$(AA^\dagger)^* = AA^\dagger,$$

and

$$(A^\dagger A)^* = A^\dagger A,$$

and it is given by, for $A \neq O$, not zero matrix,

$$A^\dagger = \frac{1}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \cdot \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A = O$, then $A^\dagger = O$.

In general, for a vector $x \in \mathbf{C}^n$, its Moore-Penrose generalized inverse $x^\dagger$ is given by

$$x^\dagger = \begin{cases} 0^* & \text{for } x = 0 \\ (x^* x)^{-1} x^* & \text{for } x \neq 0. \end{cases}$$

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [11, 38]:

Proposition 2.1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a,b)F(c,d) = F(ac,bd)$$

for all

$$a,b,c,d \in \mathbf{C}$$

and

$$F(a,b) = \frac{a}{b}, \quad a,b \in \mathbf{C}, b \neq 0.$$

Then, we obtain, for any $a \in \mathbf{C}$

$$F(a,0) = 0.$$

Proof. Indeed, we have

$$F(a,0) = F(a,0)1 = F(a,0)\frac{2}{2} = F(a,0)F(2,2) =$$

$$F(a \cdot 2, 0 \cdot 2) = F(2a,0) = F(2,1)F(a,0) = 2F(a,0).$$

Thus $F(a,0) = 2F(a,0)$ which implies the desired result $F(a,0) = 0$ for all $a \in \mathbf{C}$.

In the long mysterious history of the division by zero, this proposition seems to be decisive.

Indeed, Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Following Proposition 2.1, we should **define**

$$F(b,0) = \frac{b}{0} = 0,$$

and we should consider that for the mapping

$$W = f(z) = \frac{1}{z}, \tag{2.1}$$

the image $f(0)$ of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity on the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value at the origin by limiting like $+\infty$ and $-\infty$ or by the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotele. – For the related Greece philosophy, see [41, 42, 43]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([15, 19]) for example. Therefore, the division by zero will give great impact to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic idea for the space and universe.

The simple field structure containing division by zero was established by M. Yamada ([14]) in a natural way. For a simple introduction, H. Okumura [26] discovered the very simple essence that:

To divide by zero is to multiply by zero.

That is, for any complex numbers a and b, the general fraction(division) a/b may be defined as follows; for $b \neq 0$, with its inversion b^{-1}

$$\frac{a}{b} = ab^{-1}$$

and for $b = 0$

$$\frac{a}{b} = ab.$$

Then, the general fractions containing the division by zero form the Yamada field.

For the operator properties of the generalized fractions, see [38].

3 Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of unique determination of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{3.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{3.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i}, \tag{3.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{3.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

We will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{3.5}$$

we **define** the identity

$$f(a) = C_0. \tag{3.6}$$

Note that here, there is no problem on any convergence of the expansion (3.5) at the point $z = a$, because all terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$, when we use the result $1/0 = 0$.

Apart from the motivation, we define the division by zero calculus by (3.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

In addition, we will refer to an interesting viewpoint of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus.

We **define** the value of the function $f(z)$ at the singular point $z = a$ with the above Cauchy integral.

The basic idea of the above may be considered that we can consider the value of a function by a mean value of the function.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer

In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples to the author by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x = 0) = 1,$$

and others following the questions of the author. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019: 9: 45-10: 00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [6],

he kindly sent the message to the author as follows:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

4 We can divide the numbers and analytic functions by zero

In Section 2 and Section3, we will be interested in their precise relation from the logical point. We can derive from the motivations in Section 2, the division by zero calculus in Section 3. However, we will have some delicate logics for it. As the simple introduction and logical base on the division by zero we can start with the definition of the division by zero calculus in

Section 3. Indeed, we can develop our theory from the definition. In this sense, the division by zero calculus may be considered as a new axiom.

On this logic, the meaning (definition) of

$$\frac{1}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 1/z$. Similarly, the definition

$$\frac{0}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 0/z$.

In the division by zero, the essential problem was in the sense of the division by zero (**definition**) $z/0$. Many confusions and simple history of division by zero may be looked in [25].

In this section, in order to give the precise meaning of division by zero, we will give a simple and affirmative answer, for a famous rule that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero.

For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus

$$\frac{f(z)}{z^n} \tag{4.1}$$

at the point $z = 0$, for any positive integer n. This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \mid_{z=0} . \tag{4.2}$$

For example,

$$\frac{e^x}{x^n} \mid_{x=0} = \frac{1}{n!}.$$

This is the definition of our division by zero (general fraction). In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

The content of this subsection was presented by [32].

5 General division and usual division

Since the native division by zero $z/0$ in the sense that from $z/0 = X$ to $z = X \cdot 0$ is impossible for $z \neq 0$, we introduced its sense by the division by zero calculus. However, in our many formulas in mathematics and mathematical sciences we can see that they have the natural senses; that is for (4.2), we have:

$$\frac{f(z)}{z^n} \mid_{z=0} = \frac{f(0)}{0^n}.$$

However, this is, in general, not valid. Indeed, for the function $f(z) = \sin z$, we have

$$\frac{\sin z}{z} \mid_{z=0} = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, we have, by the division by zero calculus

$$\frac{\sin z}{z} \mid_{z=0} = 1.$$

For the functions $f(z) = 1/z$ and $g(z) = zf(z)$, we have $f(0) = 0$ and $g(0) = 1$ by the division by zero calculus, but we have another result in this way $g(0) = 0 \times f(0) = 0 \times 0 = 0$.

Here, we will show typical examples. See also [13, 15, 24, 27] for many examples.

5.1 Examples of $0/0 = 0$

The conditional probability $P(A|B)$ for the probability of A under the condition that B happens is given by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then, of course, $P(A \cap B) = 0$ and from the meaning, $P(A|B) = 0$ and so, $0/0 = 0$.

For the representation of inner product $\mathbf{A} \cdot \mathbf{B}$ in vectors

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

$$= \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}},$$

if $\mathbf{A}$ or $\mathbf{B}$ is the zero vector, then we see that $0 = 0/0$. In general, the zero vector is orthogonal for any vector and then, $\cos\theta = 0$.

For the differential equation

$$\frac{dy}{dx} = \frac{2y}{x},$$

we have the general solution with constant C

$$y = Cx^2.$$

At the origin $(0,0)$ we have

$$y'(0) = \frac{0}{0} = 0.$$

For three points a, b, c on a circle with its center at the origin on the complex z-plane with its radius R, we have

$$|a+b+c| = \frac{|ab+bc+ca|}{R}.$$

If $R = 0$, then $a, b, c = 0$ and we have $0 = 0/0$.

For a circle with its radius R and for an inscribed triangle with its side lengths a, b, c, and further for the inscribed circle with its radius r for the triangle, the area S of the triangle is given by

$$S = \frac{r}{2}(a+b+c) = \frac{abc}{4R}.$$

If $R = 0$, then we have

$$S = 0 = \frac{0}{0}$$

(H. Michiwaki: 2017.7.28.). We have the identity

$$r = \frac{2S}{a+b+c}.$$

If $a+b+c = 0$, then we have

$$0 = \frac{0}{0}.$$

For the distance d of the centers of the inscribed circle and circumscribed circle, we have the Euler formula

$$r = \frac{1}{2}R - \frac{d^2}{2R}.$$

If $R = 0$, then we have $d = 0$ and

$$0 = 0 - \frac{0}{0}.$$

For the second curvature

$$K_2 = \left((x'')^2 + (y'')^2 + (z'')^2\right)^{-1} \cdot \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

if $(x'')^2 + (y'')^2 + (z'')^2 = 0$; that is, for the case of lines, then $0 = 0/0$.

In a Hilbert space H, for a fixed member v and for a given number d we set

$$V = \{y \in H; (y, v) = d\}$$

and for fixed $x \in H$

$$d(x, V) := \frac{|(x, v) - d|}{\|v\|}.$$

If $v = 0$, then, $(y, v) = 0$ and d has to zero. Then, since $H = V$, we have

$$0 = \frac{0}{0}.$$

5.2 Examples of $1/0 = 0$

For constants a and b satisfying

$$\frac{1}{a} + \frac{1}{b} = k, \quad (\neq 0, \text{const.})$$

the function

$$\frac{x}{a} + \frac{y}{b} = 1$$

passes the point $(1/k, 1/k)$. If $a = 0$, then, by the division by zero, $b = 1/k$ and $y = 1/k$; this result is natural.

We will consider the line $y = m(x-a)+b$ through a fixed point (a, b); $a, b > 0$ with its gradient m. We set $A(0, -am + b)$ and $B(a - (b/m), 0)$ that are common points with the line and both lines $x = 0$ and $y = 0$, respectively. Then,

$$\overline{AB}^2 = (-am + b)^2 + \left(a - \frac{b}{m}\right)^2.$$

If $m = 0$, then $A(0, b)$ and $B(a, 0)$, by the division by zero, and furthermore

$$\overline{AB}^2 = a^2 + b^2.$$

Then, the line AB is a corresponding line between the origin and the point (a, b). Note that this line has only one common point with both lines $x = 0$ and $y = 0$. Therefore, this result will be very natural in a sense. – Indeed, we can understand that the line $\overline{AB}$ is broken into two lines $(0, b) - (a, b)$ and $(a, b) - (a, 0)$, suddenly. Or, the line AB is one connecting the origin and the point (a, b).

The general line equation through fixed point (a, b) with its gradient m is given by

$$y = m(x - a) + b \tag{5.1}$$

or, for $m \neq 0$

$$\frac{y}{m} = x - a + \frac{b}{m}.$$

By $m = 0$, we obtain the equation $x = a$, by the division by zero. This equation may be considered as the cases $m = \infty$ and $m = -\infty$, and these cases may be considered by the strictly right logic with the division by zero.

By the division by zero, we can consider the equation (5.1) as a general line equation.

For the Newton's formula; that is, for a C^2 class function $y = f(x)$, the curvature K at the origin is given by

$$K = \lim_{x \to 0} \left| \frac{x^2}{2y} \right| = \left| \frac{1}{f''(0)} \right|,$$

we have for $f''(0) = 0$,

$$K = \frac{1}{0} = 0.$$

Recall the formula

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx = -\frac{2}{n},$$

for

$$n = \pm 1, \pm 2, ..., ...$$

Then, for $n = 0$, we have

$$b_0 = -\frac{2}{0} = 0.$$

5.3 Trigonometric functions

In order to see how elementary of the division by zero, we will see the division by zero in trigonometric functions as the fundamental object. Even the cases of triangles and trigonometric functions, we can derive new concepts and results.

Even the case

$$\tan x = \frac{\sin x}{\cos x},$$

we have the identity, for $x = \pi/2$

$$0 = \frac{1}{0}.$$

Note that from the inversion of the both sides

$$\cot x = \frac{\cos x}{\sin x},$$

for example, we have, for $x = 0$,

$$0 = \frac{1}{0}.$$

By this general method, we can consider many problems.

In the Lami's formula for three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfying

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0},$$

$$\frac{\|\mathbf{A}\|}{\sin\alpha} = \frac{\|\mathbf{B}\|}{\sin\beta} = \frac{\|\mathbf{C}\|}{\sin\gamma},$$

if $\alpha = 0$, then we obtain

$$\frac{\|\mathbf{A}\|}{0} = \frac{\|\mathbf{B}\|}{0} = \frac{\|\mathbf{C}\|}{0} = 0.$$

Here, of course, α is the angle of $\mathbf{B}$ and $\mathbf{C}$, β is the angle of $\mathbf{C}$ and $\mathbf{A}$, and γ is the angle of $\mathbf{A}$ and $\mathbf{B}$.

We will consider a triangle ABC with $BC = a, CA = b, AB = c$. Let θ be the angle of the side BC and the bisector line of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\tan\frac{A}{2}.$$

Of course, $\theta = \pi/2$; that is,

$$\tan\frac{\pi}{2} = 0.$$

Here, we used

$$\frac{2b}{0} = 0$$

and we did not consider that by the division by zero calculus

$$\frac{c+b}{c-b} = 1 + \frac{2b}{c-b}$$

and for $c = b$

$$\frac{c+b}{c-b} = 1.$$

In the Napier's formula

$$\frac{a+b}{a-b} = \frac{\tan(A+B)/2}{\tan(A-B)/2},$$

there is no problem for $a = b$ and $A = B$.

We have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\tan B}{\tan C}.$$

If $a^2+b^2-c^2=0$, then by the Pythagorean theorem $C=\pi/2$. Then,

$$0 = \frac{\tan B}{\tan\frac{\pi}{2}} = \frac{\tan B}{0}.$$

Meanwhile, for the case $a^2-b^2+c^2=0$, $B=\pi/2$, and we have

$$\frac{a^2+b^2-c^2}{0} = \frac{\tan\frac{\pi}{2}}{\tan C} = 0.$$

Let H be the perpendicular leg of A to the side BC and let E and M be the mid points of AH and BC, respectively. Let θ be the angle of EMB $(b>c)$. Then, we have

$$\frac{1}{\tan\theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B=C$, then $\theta=\pi/2$ and $\tan(\pi/2)=0$.

Thales' theorem

We consider a triangle BAC with $A(-1,0), C(1,0), \angle BOC = \theta; O(0,0)$ on the unit circle. Then, the gradients of the lines AB and CB are given by

$$\frac{\sin\theta}{\cos\theta+1}$$

and

$$\frac{\sin\theta}{\cos\theta-1},$$

respectively. We see that for $\theta=\pi$ and $\theta=0$, they are zero, respectively.

For many similar formulas, see [34].

5.4 Examples of Ctesibios and E. Torricelli

As a typical case, we recall

Ctesibios (BC. 286-222): *We consider a flow tube with some fluid. Then, when we consider some cut with a plane with area S and with velocity v of* *the fluid on the plane, by continuity, we see that for any cut plane, $Sv = C$; C : constant. That is,*

$$v = \frac{C}{S}.$$

When S tends to zero, the velocity v tends to infinity. However, for $S = 0$, the flow stopped and so, $v = 0$. Therefore, this example shows the division by zero $C/0 = 0$ clearly. Of course, in the situation, we have $0/0 = 0$, trivially.

We can find many and many similar examples, for example, in Archimedes' principle and Pascal's principle.

We will state one more example:

E. Torricelli (1608 -1646): *We consider some water tank and the initial high $h = h_0$ for $t = 0$ and we assume that from the bottom of the tank with a hole of area A, water is fall down. Then, by the law with a constant k*

$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h},$$

we have the equation

$$h(t) = \left(\sqrt{h_0} - \frac{k}{2A}\right)^2.$$

Similarly, of course, for $A = 0$, we have

$$h(t) = h_0.$$

5.5 Bhāskara's example – sun and shadow

We will consider the circle such that its center is the origin and its radius R. We consider the point S (sun) on the circle such that $\angle SOI = \theta$; $O(0,0), I(R,0)$. For fixed $d > 0$, we consider the common point $(-L, -d)$ of two line OS and $y = -d$. Then we obtain the identity

$$L = \frac{R\cos\theta}{R\sin\theta}d,$$

([7], page 77.). That is the length of the shadow of the segment of $(0,0) - (0,-d)$ onto the line $y = -d$ of the sun S.

When we consider $\theta \to +0$ we see that, of course

$$L \to \infty.$$

Therefore, Bhāskara considered that

$$\frac{1}{0} = \infty. \tag{5.2}$$

Even nowadays, our mathematics and many people consider so.

However, for $\theta = 0$, we have S=I and we can not consider any shadow on the line $y = -d$, so we should consider that $L = 0$; that is

$$\frac{1}{0} = 0. \tag{5.3}$$

Furthermore, for $R = 0$; that is, for S=O, we see its shadow is the point $(0, -d)$ and so $L = 0$ and

$$L = \frac{0 \cos\theta}{0 \sin\theta} d = 0;$$

that is

$$\frac{0}{0} = 0.$$

This example shows that the division by zero calculus is not almighty.

Note that both identities (5.2) and (5.3) are right in their senses. Depending on the interpretations of $1/0$, we obtain INFINITY and ZERO, respectively.

6 Division by zero calculus

We will see several typical results of the division by zero calculus.

6.1 Double natures of the zero point $z = 0$

Any line on the complex plane arrives at the point at infinity and the point at infinity is represented by zero. That is, a line, indeed, contains the origin;

the true line should be considered as the sum of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the point at infinity, however, it is represented by $z = 0$. Later, we will see this property by analytic geometry and the division by zero calculus in many situations.

However, for the general line equation

$$ax + by + c = 0,$$

by using the polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$r = \frac{-c}{a\cos\theta + b\sin\theta}.$$

When $a\cos\theta + b\sin\theta = 0$, by the division by zero, we have $r = 0$; that is, we can consider that the line contains the origin. We can consider so, in the natural sense. We can define so as a line with the compactification and the representation of the point at infinity - the ideal point.

For the envelop of the lines represented by, for constants m and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{6.1}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{6.2}$$

The origin of this parabolic function is excluded from the envelop of the linear functions, because the linear equations do not contain the y axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the x axis.

– This function may be considered as a function with zero gradient and passing the point at infinity; however, the point at infinity is represented by 0, the origin; that is, the line may be considered as the x axis. Furthermore, then we can consider the x axis as a tangential line of the parabolic function, because they are gradient zero at the point at infinity. –

Furthermore, we can say later that the x axis $y = 0$ and the parabolic function have the zero gradient at **the origin**; that is, in the reasonable sense the x axis is a tangential line of the parabolic function.

Indeed, we will see the surprising property that the gradient of the parabolic function at the origin is zero.

Anyhow, by the division by zero, the envelop of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the x axis through the point at infinity. Since the point at infinity is represented by zero, it will become the x axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the y axis $x = 0$ and this function is a native tangential line of the parabolic function. From these two tangential lines, we see that the origin has **double natures**; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

In addition, note that the tangential point of (6.2) for the line (6.1) is given by

$$\left(\frac{p}{m}, \frac{2p}{m}\right)$$

and it is $(0, 0)$ for $m = 0$.

We can see that the point at infinity is reflected to the origin; and so, the origin has the double natures; one is the native origin and another is the reflected one of the point at infinity.

6.2 Difficulty in Maple for specialization problems

For the Fourier coefficients a_n

$$a_n = \int t \cos n\pi t dt = \frac{\cos n\pi t}{n^2\pi^2} + \frac{t}{n\pi} \cos n\pi t,$$

we obtain, by the division by zero calculus,

$$a_0 = \frac{t^2}{2}.$$

Similarly, for the Fourier coefficients a_n

$$a_n = \int t^2 \cos n\pi t dt = \frac{2t}{\pi^2 n^2} \cos n\pi t - \frac{2}{n^3\pi^3} \sin n\pi t + \frac{t^2}{n\pi} \sin n\pi t,$$

we obtain

$$a_0 = \frac{t^3}{3}.$$

For the Fourier coefficients a_k of a function

$$\frac{a_k \pi k^3}{4}$$

$$= \sin(\pi k)\cos(\pi k) + 2k^2\pi^2 \sin(\pi k)\cos(\pi k) + 2\pi(\cos(\pi k))^2 - \pi k,$$

for $k = 0$, we obtain, by the division by zero calculus, immediately

$$a_0 = \frac{8}{3}\pi^2.$$

We have many such examples.

6.3 Ratio

On the real x line, we fix two different points $P_1(x_1)$ and $P_2(x_2)$ and we will consider the point, with a real number r

$$P(x; r) = \frac{x_1 + r x_2}{1 + r}.$$

If $r = 1$, then the point $P(x; 1)$ is the mid point of two points P_1 and P_2 and for $r > 0$, the point P is on the interval (x_1, x_2). Meanwhile, for $-1 < r < 0$, the point P is on $(-\infty, x_1)$ and for $r < -1$, the point P is on $(x_2, +\infty)$. Of course, for $r = 0$, $P = P_1$. We see that when r tends to $+\infty$ and $-\infty$, P tends to the point P_2. We see the pleasant fact that by the division by zero calculus, $P(x, -1) = P_2$. For this fact we see that for all real numbers r correspond to all real line points.

In particular, we see that in many text books at the undergraduate course the formula is stated as a parameter representation of the line through two pints P_1 and P_2. However, if we do not consider the case $r = -1$ by the division by zero calculus, the classical statement is not right, because the point P_2 can not be considered.

For fixed two vectors $OA = a$ and $OB = b$ $(a \neq b)$, we consider two vectors $OA' = a' = \lambda a$ and $OB' = b' = \mu b$ with parameters λ and μ. Then, the common point x of the two lines AB and $A'B'$ is represented by

$$x = \frac{\lambda(1-\mu)a + \mu(\lambda - 1)b}{\lambda - \mu}.$$

For $\lambda = \mu$, we should have $x = 0$, by the division by zero. However, by the division by zero calculus, we have the curious result

$$x = (1 - \mu)a + \mu b.$$

On the real line, the points $P(p), Q(1), R(r), S(-1)$ form a harmonic range of points if and only if

$$p = \frac{1}{r}.$$

If $r = 0$, then we have $p = 0$ that is now the representation of the point at infinity (H. Okumura: 2017.12.27.)

6.4 Equalites and inequalities

In the identity

$$\frac{1}{x} - \frac{{}_nC_1}{x+1} + \frac{{}_nC_2}{x+2} + \cdots + (-1)^n \frac{{}_nC_n}{x+n}$$
$$= \frac{n!}{x(x+1)(x+2)\cdots(x+n)},$$

from the singular points, we obtain many identities, for example, from $x = 0$, we obtain the identity

$$-{}_nC_1 + \frac{{}_nC_2}{2} + \cdots + (-1)^n \frac{{}_nC_n}{n}$$
$$= -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

We can derive many identities in this way.

For the rational equation

$$\frac{2(x-1)}{(x-1)(x+1)} = 1,$$

we obtain the natural solution $x = 1$ by the division by zero calculus. However, we did not consider so; that is, there is no solution for the equation.

For the equation

$$\frac{x-4y+2z}{x}=\frac{2x+7y-4z}{y}=\frac{4x+10y-6z}{z}=k,$$

from $k=1$, we have the solution with parameter λ

$$x=y=\lambda, \quad z=2\lambda.$$

We obtain also the natural solution

$$x=y=z=0.$$

However, then $k=0$.

For the problem

$$f'(x)=\frac{1}{(x-1)(x-2)}<0,$$

we have the solution

$$1<x<2$$

in the usual sense. However, note that by the division by zero calculus

$$f(1)=-1$$

and

$$f(2)=-1.$$

Therefore, we have the solution

$$1\leq x\leq 2.$$

Meanwhile, we know

Growth Lemma ([28], 267 page) *For the polynomial*

$$P(z)=a_0+a_1z+...+a_nz^n(a_0,a_n\neq 0,n>1)$$

we have the inequality with a sufficient r, for $|z|\geq r$

$$\frac{|a_n|}{2}|z|^n\leq|P(z)|\leq\frac{3|a_n|}{2}|z|^n.$$

At the point at infinity, since $P(z)$ takes the value a_0, the inequality is not valid more.

Therefore, for inequalities, for the values of singular points by means of the division by zero calculus, we have to check the values, case by case.

6.5 Discontinuity in geometrical meanings

The division by zero calculus shows an interesting discontinuity in the sense of geometrical properties. We will show typical cases.

- The area $S(x)$ surrounded by two x, y axes and the line passing a fixed point $(a, b), a, b > 0$ and a point $(x, 0)$ is given by

  $$S(x) = \frac{bx^2}{2(x-a)}.$$

  For $x = a$, we obtain, by the division by zero calculus, the very interesting value

  $$S(a) = ab.$$

- For example, for fixed point $(a, b); a, b > 0$ and fixed a line $y = (\tan\theta)x, 0 < \theta < \pi$, we will consider the line $L(x)$ passing two points (a, b) and $(x, 0)$. Then, the area $S(x)$ of the triangle surrounded by three lines $y = (\tan\theta)x$, $L(x)$ and the x axis is given by

  $$S(x) = \frac{b}{2}\frac{x^2}{x-(a-b\cot\theta)}.$$

  For the case $x = a - b\cot\theta$, by the division by zero calculus, we have

  $$S(a - b\cot\theta) = b(a - b\cot\theta).$$

  Note that this is the area of the parallelogram through the origin and the point (a, b) formed by the lines $y = (\tan\theta)x$ and the x axis.

- We consider the circle

  $$h(x^2 + y^2) + (1 - h^2)y - h = 0$$

  through the points $(-1, 0)$, $(1, 0)$ and $(0, h)$. If $h = 0$, then we have

  $$y = 0.$$

  However, from the equation

  $$x^2 + y^2 + \left(\frac{1}{h} - h\right) y - 1 = 0,$$

  by the division by zero, we have an interesting result

  $$x^2 + y^2 = 1.$$

- We consider the regular triangle with the vertices

$$(-a/2, \sqrt{3}a/2), (a/2, \sqrt{3}a/2).$$

Then, the area $S(h)$ of the triangle surrounded by the three lines that the line through $(0, h + \sqrt{3}a/2)$ and

$$(-a/2, \sqrt{3}a/2),$$

the line through $(0, h + \sqrt{3}a/2)$ and $(a/2, \sqrt{3}a/2)$ and the x- axis is given by

$$S(h) = \frac{\left(h + (\sqrt{3}/2)a\right)^2}{2h}.$$

Then, by the division by zero calculus, we have, for $h = 0$,

$$S(0) = \frac{\sqrt{3}}{2}a^2.$$

- Similarly, we will consider the cone formed by the rotation of the line

$$\frac{kx}{a(k+h)} + \frac{y}{k+h} = 1$$

and the x, y plane around the z- axis ($a, h > 0$, and a, h are fixed). Then, the volume $V(x)$ is given by

$$V(k) = \frac{\pi}{3} \frac{a^2(k+h)^3}{k^2}.$$

Then, by the division by calculus, we have the reasonable value

$$V(0) = \pi a^2 h.$$

6.6 Sato hyperfunctions

As a typical example in A. Kaneko ([9], page 11) in the theory of hyperfunction theory we see that for non-integers λ, we have

$$x_+^\lambda = \left[\frac{-(-z)^\lambda}{2i \sin \pi\lambda}\right] = \frac{1}{2i \sin \pi\lambda}\{(-x + i0)^\lambda - (-x - i0)^\lambda\}$$

where the left hand side is a Sato hyperfunction and the middle term is the representative analytic function whose meaning is given by the last term. For an integer n, Kaneko derived that

$$x_+^n = \left[-\frac{z^n}{2\pi i}\log(-z)\right],$$

where log is a principal value on $\{-\pi < \arg z < +\pi\}$. Kaneko stated there that by taking a finite part of the Laurent expansion, the formula is derived. Indeed, we have the expansion, around an integer n,

$$\begin{aligned} &\frac{-(-z)^\lambda}{2i\sin\pi\lambda} \\ &= \frac{-z^n}{2\pi i}\frac{1}{\lambda - n} - \frac{z^n}{2\pi i}\log(-z) \\ &- \left(\frac{\log^2(-z)z^n}{2\pi i \cdot 2!} + \frac{\pi z^n}{2i \cdot 3!}\right)(\lambda - n) + ... \end{aligned}$$

([9], page 220).

However, we can derive this result from the Laurent expansion, immediately, by the division by zero calculus.

Meanwhile, M. Morimoto derived this result by using the Gamma function with the elementary means in [16], pages 60-62.

7 Derivatives of functions

On derivatives, we obtain new concepts, from the division by zero calculus. At first, we will consider the fundamental properties. From the viewpoint of the division by zero, when there exists the limit, at x

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \infty \tag{7.1}$$

or

$$f'(x) = -\infty, \tag{7.2}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{7.3}$$

This property was derived from the fact that the gradient of the y axis is zero; that is,

$$\tan\frac{\pi}{2} = 0. \tag{7.4}$$

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1-x^2},$$

$$y' = \frac{-x}{\sqrt{1-x^2}},$$

and so,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0.$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos\theta, y = \sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = -\cot\theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right) \tag{7.5}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have

$$\cot 0 = 0.$$

The differential equation

$$y' = -\frac{x}{y}$$

with a general solution

$$x^2 + y^2 = a^2$$

is satisfied for all points of the solutions by the division by zero. However, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1$$

are not satisfied for the points $(-a, 0)$ and $(a, 0)$.

In many and many textbooks, we find the differential equations, however, they are not good in this viewpoint.

For the function $y = \log x$,

$$y' = \frac{1}{x}, \tag{7.6}$$

and so,

$$[y']_{x=0} = 0. \tag{7.7}$$

For the elementary ordinary differential equation

$$y' = \frac{dy}{dx} = \frac{1}{x}, \quad x > 0, \tag{7.8}$$

how will be the case at the point $x = 0$? From its general solution, with a general constant C

$$y = \log x + C, \tag{7.9}$$

we see that

$$y'(0) = \left[\frac{1}{x}\right]_{x=0} = 0, \tag{7.10}$$

that will mean that the division by zero $1/0 = 0$ is very natural.

In addition, note that the function $y = \log x$ has infinite order derivatives and all values are zero at the origin, in the sense of the division by zero calculus.

However, for the derivative of the function $y = \log x$, we have to fix the sense at the origin, clearly, because the function is not differentiable in the usual sense, but it has a singularity at the origin. For $x > 0$, there is no problem for (7.8) and (7.9). At $x = 0$, we see that we can not consider the limit in the usual sense. However, for $x > 0$ we have (7.9) and

$$\lim_{x \to +0} (\log x)' = +\infty. \tag{7.11}$$

In the usual sense, the limit is $+\infty$, but in the present case, in the sense of the division by zero, we have the identity

$$\left[(\log x)'\right]_{x=0} = 0$$

and we will be able to understand its sense graphically.

By the new interpretation for the derivative, we can arrange the formulas for derivatives, by the division by zero. The formula

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} \tag{7.12}$$

is very fundamental. Here, we considered it for a local one to one correspondence of the function $y = f(x)$ and for nonvanishing of the denominator

$$\frac{dy}{dx} \neq 0. \tag{7.13}$$

However, if a local one to one correspondence of the function $y = f(x)$ is ensured like the function $y = x^3$ around the origin, we do not need the assumption (7.13). Then, for the point $dy/dx = 0$, we have, by the division by zero,

$$\frac{dx}{dy} = 0.$$

This will mean that the function $x = g(y)$ has the zero derivative and the tangential line at the point is a parallel line to the y- axis. In this sense the formula (7.12) is valid, even the case $dy/dx = 0$.

The derivative of the function

$$f(x) = \sqrt{x}(\sqrt{x} + 1)$$

$$f'(x) = \frac{1}{2\sqrt{x}}(\sqrt{x} + 1) + \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}}$$

is valid at even the origin by using the function $\frac{\sqrt{x}}{\sqrt{x}}$ (V. V. Puha: 2018, June). He derived such formulas by using the function x/x.

8 Differerential equations

From the viewpoint of the division by zero calculus, we will see many incompleteness mathematics, in particular, in the theory of differential equations at an undergraduate level; indeed, we have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics at the singular point itself. At the isolated singular point, we considered our mathematics with the limiting concept, however, the limiting value to the singular point and the value at the singular point of the function are, in general, different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. From this viewpoint, we will be able to consider differential equations even at singular points. We find many incomplete statements and problems in many undergraduate textbooks. In this section, we will point out the problems in concrete ways by examples.

This section is an arrangement of the paper [28] with new materials.

8.1 Missing a solution

For the differential equation

$$2xydx - (x^2 - y^2)dy = 0,$$

we have a general solution with a constant C

$$x^2 + y^2 = 2Cy.$$

However, we are missing the solution $y = 0$. By this expression

$$\frac{x^2 + y^2}{C} = 2y,$$

for $C = 0$, by the division by zero, we have the missing solution $y = 0$.

For the differential equation

$$x(y')^2 - 2yy' - x = 0,$$

we have the general solution

$$C^2x^2 - 2Cy - 1 = 0.$$

However, $x = 0$ is also a solution, because

$$xdy^2 - 2ydydx - xdx^2 = 0.$$

From

$$x^2 - \frac{2y}{C} - \frac{1}{C^2} = 0,$$

by the division by zero, we obtain the solution.

For the differential equation

$$2y = xy' - \frac{x}{y'},$$

we have the general solution

$$2y = Cx^2 - \frac{1}{C}.$$

For $C = 0$, we have the solution $y = 0$, by the division by zero.

8.2 Differential equations with singularities

For the differential equation

$$y' = -\frac{y}{x},$$

we have the general solution

$$y = \frac{C}{x}.$$

From the expression

$$xdy + ydx = 0,$$

we have also the general solution

$$x = \frac{C}{y}.$$

Therefore, there is no problem for the origin. Of course, $x = 0$ and $y = 0$ are the solutions.

For the differential equation

$$y' = \frac{2x - y}{x - y}, \tag{8.1}$$

we have the beautiful general solution with constant C

$$2x^2 - 2xy + y^2 = C. \tag{8.2}$$

By the division by zero calculus we see that on the whole points on the solutions (8.2) the differential equation (8.1) is satisfied. If we do not consider the division by zero, for $y = x(\neq 0)$, we will have a serious problem. However, for $x = y \neq 0$, we should consider that $y' = 0$, not by the division by zero calculus, but by $1/0 = 0$.

8.3 Continuation of solution

We will consider the differential equation

$$\frac{dx}{dt} = x^2 \cos t. \tag{8.3}$$

Then, as the general solution, we obtain, for a constant C

$$x = \frac{1}{C - \sin t}.$$

For $x_0 \neq 0$, for any given initial value (t_0, x_0) we obtain the solution satisfying the initial condition

$$x = \frac{1}{\sin t_0 + \frac{1}{x_0} - \sin t}. \tag{8.4}$$

If

$$\left| \sin t_0 + \frac{1}{x_0} \right| < 1,$$

then the solution has many poles and L. S. Pontrjagin stated in his book that the solution is disconnected at the poles and so, the solution may be considered as infinitely many solutions.

However, from the viewpoint of the division by zero, the solution takes the value zero at the singular points and the derivatives at the singular points are all zero; that is, the solution (8.4) may be understood as one solution.

Furthermore, by the division by zero, the solution (8.4) has its sense for even the case $x_0 = 0$ and it is the solution of (8.3) satisfying the initial condition $(t_0, 0)$.

We will consider the differential equation

$$y' = y^2.$$

For $a > 0$, the solution satisfying $y(0) = a$ is given by

$$y = \frac{1}{\frac{1}{a} - x}.$$

Note that the solution satisfies on the whole space $(-\infty, +\infty)$ even at the singular point $x = \frac{1}{a}$, in the sense of the division by zero, as

$$y'\left(\frac{1}{a}\right) = y\left(\frac{1}{a}\right) = 0.$$

8.4 Singular solutions

We will consider the differential equation

$$(1 - y^2)dx = y(1 - x)dy.$$

By the standard method, we obtain the general solution, for a constant C $(C \neq 0)$

$$\frac{(x-1)^2}{C} + y^2 = 1.$$

By the division by zero, for $C = 0$, we obtain the singular solution

$$y = \pm 1.$$

For the simple Clairaut differential equation

$$y = px + \frac{1}{p}, \quad p = \frac{dy}{dx},$$

we have the general solution

$$y = Cx + \frac{1}{C}, \tag{8.5}$$

with a general constant C and the singular solution

$$y^2 = 4x.$$

Note that we have also the solution $y = 0$ from the general solution, by the division by zero $1/0 = 0$ from $C = 0$ in (8.5).

8.5 Solutions with singularities

1). We will consider the differential equation

$$y' = \frac{y^2}{2x^2}.$$

We will consider the solution with an isolated singularity at a point a taking the value $-2a$ in the sense of division by zero.

First, by the standard method, we have the general solution, with a constant C

$$y = \frac{2x}{1 + 2Cx}.$$

From the singularity, we have, $C = -1/2a$ and we obtain the desired solution

$$y = \frac{2ax}{a - x}.$$

Indeed, from the expansion

$$\frac{2ax}{a - x} = -2a - \frac{2a^2}{x - a},$$

we see that it takes $-2a$ at the point a in the sense of the division by zero calculus. This function was appeared in ([14]).

2). We will consider the singular differential equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} - \frac{3}{x^2}y = 0. \tag{8.6}$$

By the series expansion, we obtain the general solution, for any constants a, b

$$y = \frac{a}{x^3} + bx. \tag{8.7}$$

We see that by the division by zero

$$y(0) = 0, y'(0) = b, y''(0) = 0.$$

The solution (8.7) has its sense and the equation is satisfied even at the origin. The value $y'(0) = b$ may be given arbitrary, however, in order to determine the value a, we have to give some value for the regular point $x \neq 0$. Of course, we can give the information at the singular point with the Laurent coefficient a, that may be interpreted with the value at the singular point zero. Indeed, the value a may be considered at the value

$$[y(x)x^3]_{x=0} = a.$$

3). Next, we will consider the Euler differential equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0.$$

We obtain the general solution, for any constants a, b

$$y = \frac{a}{x} + \frac{b}{x^2}.$$

This solution is satisfied even at the origin, by the division by zero and furthermore, all derivatives of the solution of any order are zero at the origin.

4). We will note that as the general solution with constants C_{-2}, C_{-1}, C_0

$$y = \frac{C_{-2}}{x^2} + \frac{C_{-1}}{x} + C_0,$$

we obtain the nonlinear ordinary differential equation

$$x^2y''' + 6xy'' + 6y' = 0.$$

5). For the differential equation

$$y' = y^2(2x - 3),$$

we have the special solution

$$y = \frac{1}{(x-1)(2-x)}$$

on the interval $(1, 2)$ with the singularities at $x = 1$ and $x = 2$. Since the general solution is given by, for a constant C,

$$y = \frac{1}{-x^2 + 3x + C},$$

we can consider some conditions that determine the special solution.

8.6 Solutions with an analytic parameter

For example, in the ordinary differential equation

$$y'' + 4y' + 3y = 5e^{-3x},$$

in order to look for a special solution, by setting $y = Ae^{kx}$ we have, from

$$y'' + 4y' + 3y = 5e^{kx},$$

$$y = \frac{5e^{kx}}{k^2 + 4k + 3}.$$

For $k = -3$, by the division by zero calculus, we obtain

$$y = e^{-3x} \left(-\frac{5}{2}x - \frac{5}{4} \right),$$

and so, we can obtain the special solution

$$y = -\frac{5}{2}xe^{-3x}.$$

For example, for the differential equation

$$y'' + a^2 y = b \cos \lambda x,$$

we have a special solution

$$y = \frac{b}{a^2 - \lambda^2} \cos \lambda x.$$

Then, for $\lambda = a$ (reasonance case), by the division by zero calculus, we obtain the special solution

$$y = \frac{bx \sin(ax)}{2a} + \frac{b \cos(ax)}{4a^2}.$$

We can find many examples.

8.7 Special reductions by division by zero of solutions

For the differential equation

$$y'' - (a+b)y' + aby = e^{cx}, c \neq a, b; a \neq b,$$

we have the special solution

$$y = \frac{e^{cx}}{(c-a)(c-b)}.$$

If $c = a(\neq b)$, then, by the division by zero calculus, we have

$$y = \frac{xe^{ax}}{a-b}.$$

If $c = a = b$, then, by the division by zero calculus, we have

$$y = \frac{x^2 e^{ax}}{2}.$$

For the differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0,$$

we obtain the general solution, for $\gamma^2 > 4mk$

$$x(t) = e^{-\alpha t}\left(C_1 e^{\beta t} + C_2 e^{-\beta t}\right)$$

with

$$\alpha = \frac{\gamma}{2m}$$

and

$$\beta = \frac{1}{2m}\sqrt{\gamma^2 - 4mk}.$$

For $m = 0$, by the division by zero calculus we obtain the reasonable solution $\alpha = 0$ and $\beta = -k/\gamma$.

We will consider the differential equation, for a constant K

$$y' = Ky.$$

Then, we have the general solution

$$y(x) = y(0)e^{Kt}.$$

For the differential equation

$$y' = Ky\left(1 - \frac{y}{R}\right),$$

we have the solution

$$y = \frac{y(0)e^{Kt}}{1 + \frac{y(0)(e^{Kt}-1)}{R}}.$$

If $R = 0$, then, by the division by zero, we obtain the previous result, immediately.

For the differential equation

$$x''(t) = -g + k(x'(t))^2$$

satisfying the initial conditions

$$x(0) = 0, x'(0) = V,$$

we have

$$x'(t) = -\sqrt{\frac{g}{k}}\tan(\sqrt{kg}t - \alpha),$$

with

$$\alpha = \tan^{-1}\sqrt{\frac{k}{g}}V$$

and the solution

$$x(t) = \frac{1}{k}\log\frac{\cos\left(\sqrt{kg}t - \alpha\right)}{\cos\alpha}.$$

Then we obtain for $k = 0$, by the division by zero calculus

$$x'(t) = -gt + V$$

and

$$x(t) = -\frac{1}{2}gt^2 + Vt.$$

We can find many and many such examples. However, note the following fact.

For the differential equation

$$y''' + a^2y' = 0,$$

we obtain the general solution, for $a \neq 0$

$$y = A \sin ax + B \cos ax + C.$$

For $a = 0$, from this general solution, how can we obtain the corresponding solution

$$y = Ax^2 + Bx + C,$$

naturally?

For the differential equation

$$y' = ae^{\lambda x}y^2 + afe^{\lambda x}y + \lambda f,$$

we obtain a special solution, for $a \neq 0$

$$y = -\frac{\lambda}{a}e^{-\lambda x}.$$

For $a = 0$, from this solution, how can we obtain the corresponding solution

$$y = \lambda f x + C,$$

naturally?

8.8 Open problems

As important open problems, we would like to propose them clearly. We have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics at the singular point itself. At the isolated singular point, we consider our mathematics with the limiting concept, however, the limiting values to the singular point and the values at the singular point in the sense of division by zero calculus are, in general, different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. We thus have a general open problem discussing our mathematics on a domain containing the singular point.

We referred to the reduction problem by concrete examples; there we found the delicate property. For this interesting property we expect some general theory.

9 Euclidean spaces and division by zero calculus

In this section, we will see the division by zero properties on the Euclidean spaces. Since the impact of the division by zero and division by zero calculus is widely expanded in elementary mathematics, here, elementary topics will be introduced as the first stage.

9.1 Broken phenomena of figures by area and volume

The strong discontinuity of the division by zero around the point at infinity will appear as the destruction of various figures. These phenomena may be looked in many situations as the universe one. However, the simplest cases are disc and sphere (ball) with their radius $1/R$. When $R \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $R = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts may be also looked by analytic geometry, as we see later. However, the results are clear already from the definition of the division by zero.

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$ (see [2]). We thus see that

$$\lim_{z\to\infty} z = \infty, \tag{9.1}$$

however,

$$[z]_{z=\infty} = 0, \tag{9.2}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (9.1) and (9.2) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x\to+\infty} x = +\infty, \quad \lim_{x\to-\infty} x = -\infty,$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0.$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [15], we gave many examples for this property.

In particular, in $z \to \infty$ in (9.1), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in (9.1) represents the limit by means of the ϵ - δ logic. That is, for any large number M, when we take for some large number N, we have, for $|z| > N$, $|z| > M$.

9.2 Parallel lines

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2.$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \right).$$

By the division by zero, we can understand that if $a_1 b_2 - a_2 b_1 = 0$, then the common point is always given by

$$(0, 0),$$

even two lines are the same.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)},$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle of OP ($P = (r, \theta)$ on the line) from the positive x axis. Then, if $\theta - \alpha = \pi/2$; that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics globally.

The distance d from the origin to the line determined by the two planes

$$\Pi_k : a_k x + b_k y + c_k z = 1, k = 1, 2,$$

is given by

$$d = \sqrt{\frac{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}.$$

If the two lines are coincident, then, of course, $d = 0$. However, if two planes are parallel, by the division by zero, $d = 0$. This will mean that any plane contains the origin as in a line.

9.3 Tangential lines and $\tan \frac{\pi}{2} = 0$

We looked the very fundamental and important formula $\tan \frac{\pi}{2} = 0$ in Section 6. In this subsection, for its importance we will furthermore see its geometrical meanings.

We consider the high $\tan \theta \left(0 \leq \theta \leq \frac{\pi}{2}\right)$ that is given by the common point of two lines $y = (\tan \theta)x$ and $x = 1$ on the (x, y) plane. Then,

$$\tan \theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$\tan \frac{\pi}{2} = 0,$$

by the division by zero. The result will show that, when $\theta = \pi/2$, two lines $y = (\tan \theta)x$ and $x = 1$ do not have a common point, because they are parallel in the usual sense. However, in the sense of the division by zero, parallel lines have the common point $(0, 0)$. Therefore, we can see the result $\tan \frac{\pi}{2} = 0$ following our new space idea.

We consider general lines represented by

$$ax + by + c = 0, \quad a'x + b'y + c' = 0.$$

The gradients are given by

$$k = -\frac{a}{b}, k' = -\frac{a'}{b'},$$

respectively. In particular, note that if $b = 0$, then $k = 0$, by the division by zero.

If $kk' = -1$, then the lines are orthogonal; that is,

$$\tan\frac{\pi}{2} = 0 = \pm\frac{k - k'}{1 + kk'},$$

which shows that the division by zero $1/0 = 0$ and orthogonality meets in a very good way.

Furthermore, even in the case of polar coordinates $x = r\cos\theta, y = r\sin\theta$, we can see the division by zero

$$\tan\frac{\pi}{2} = \frac{y}{0} = 0.$$

The division by zero may be looked even in the rotation of the coordinates.

We will consider a 2 dimensional curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and a rotation defined by

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta.$$

Then, we write, by inserting these (x, y)

$$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0.$$

Then,

$$H = 0 \Longleftrightarrow \tan 2\theta = \frac{2h}{a - b}.$$

If $a = b$, then, by the division by zero,

$$\tan\frac{\pi}{2} = 0, \quad \theta = \frac{\pi}{4}.$$

For $h^2 > ab$, the equation

$$ax^2 + 2hxy + by^2 = 0$$

represents 2 lines and the angle θ made by two lines is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}.$$

If $h^2 - ab = 0$, then, of course, $\theta = 0$. If $a + b = 0$, then, by the division by zero, $\theta = \pi/2$ from $\tan\theta = 0$.

For a hyperbolic function

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \quad a, b > 0$$

the angle θ made by two asymptotic lines $y = \pm(b/a)x$ is given by

$$\tan\theta = \frac{2(b/a)}{1-(b/a)^2}.$$

If $a = b$, then $\theta = \pi/2$ from $\tan\theta = 0$.

We consider the unit circle with its center at the origin on the (x, y) plane. We consider the tangential line for the unit circle at the point that is the common point of the unit circle and the line $y = (\tan\theta)x$ $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$. Then, the distance R_θ between the common point and the common point of the tangential line and x-axis is given by

$$R_\theta = \tan\theta.$$

Then,

$$R_0 = \tan 0 = 0,$$

and

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$R_{\pi/2} = \tan\frac{\pi}{2} = 0.$$

This example shows also that by the stereographic projection mapping of the unit sphere with its center at the origin $(0, 0, 0)$ onto the plane, the north pole corresponds to the origin $(0, 0)$.

In this case, we consider the orthogonal circle C_{R_θ} with the unit circle through at the common point and the symmetric point with respect to the x-axis with its center $((\cos\theta)^{-1}, 0)$. Then, the circle C_{R_θ} is as follows:

C_{R_0} is the point $(1,0)$ with curvature zero, and $C_{R_{\pi/2}}$ (that is, when $R_\theta = \infty$, in the common sense) is the y-axis and its curvature is also zero. Meanwhile, by the division by zero calculus, for $\theta = \pi/2$ we have the same result, because $(\cos(\pi/2))^{-1} = 0$.

The points $(\cos\theta, 0)$ and $((\cos\theta)^{-1}, 0)$ are the symmetric points with respect to the unit circle, and the origin corresponds to the origin.

In particular, the formal calculation

$$\sqrt{1 + R_{\pi/2}^2} = 1$$

is not good. The identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid formally for $\theta = \pi/2$.

This equation should be written as

$$\frac{\cos^2\theta}{\cos^2\theta} + \tan^2\theta = (\cos\theta)^{-2},$$

that is valid always.

Of course, as analytic functions, in the sense of the division by zero calculus, the identity is valid for $\theta = \pi/2$.

From the point at

$$x = \frac{1}{\cos\theta}$$

when we look the unit circle, we can see that the length $L(x)$ of the arc that we can see is given by

$$L(x) = 2\cos^{-1}\frac{1}{x}.$$

For $\theta = \pi/2$ that is for $x = 0$ we see that $L(x) = 0$.

We fix $B(0,1)$ and let $\angle ABO = \theta$ with $A(\tan\theta, 0)$. Let H be the point on the line BA such that two lines OH and AB are orthogonal. Then we see that

$$AH = \frac{\sin^2\theta}{\cos\theta}.$$

Note that for $\theta = \pi/2$, $AH = 0$.

On the point $(p,q)(0 \le p,q \le 1)$ on the unit circle, we consider the tangential line $L_{p,q}$ of the unit circle. Then, the common points of the line $L_{p,q}$ with x-axis and y-axis are $(1/p, 0)$ and $(0, 1/q)$, respectively. Then, the area S_p of the triangle formed by three points $(0,0)$, $(1/p,0)$ and $(0,1/q)$ is given by

$$S_p = \frac{1}{2pq}.$$

Then,

$$p \longrightarrow 0; \quad S_p \longrightarrow +\infty,$$

however,

$$S_0 = 0$$

(H. Michiwaki: 2015.12.5.). We denote the point on the unit circle on the (x,y) plane with $(\cos\theta, \sin\theta)$ for the angle θ with the positive real line. Then, the tangential line of the unit circle at the point meets at the point $(R_\theta, 0)$ for $R_\theta = [\cos\theta]^{-1}$ with the x-axis for the case $\theta \neq \pi/2$. Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = \left[\cos\left(\frac{\pi}{2}\right)\right]^{-1} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$ (H. Michiwaki: 2015.12.5.).

The line through the points $(0,1)$ and $(\cos\theta, \sin\theta)$ meets the x axis with the point $(R_\theta, 0)$ for the case $\theta \neq \pi/2$ by

$$R_\theta = \frac{\cos\theta}{1 - \sin\theta}.$$

Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$.

Note also that

$$\left[1 - \sin\left(\frac{\pi}{2}\right)\right]^{-1} = 0.$$

9.4 Newton's method

The Newton's method is fundamental when we look for the solutions for some general equation $f(x) = 0$ numerically and practically. We will refer to its prototype case.

We will assume that a function $y = f(x)$ belongs to C^1 class. We consider the sequence $\{x_n\}$ for $n = 0, 1, 2, \ldots, n, \ldots$, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

When $f(x_n) = 0$, we have

$$x_{n+1} = x_n, \tag{9.3}$$

in the reasonable way. Even the case $f'(x_n) = 0$, we have also the reasonable result (9.3), by the division by zero.

9.5 Cauchy's mean value theorem

For the Cauchy mean value theorem; that is, for $f, g \in Differ(a, b)$, differentiable, and $\in C^0[a, b]$, continuous and if $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, then there exists $\xi \in (a, b)$ satisfying that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)},$$

we do not need the assumptions $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, by the division by zero. Indeed, if $g(a) = g(b)$, then, by the Rolle theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$. Then, both terms are zero and the equality is valid.

9.6 Length of tangential lines

We will consider the inversion $A(1/x, 0)$ of a point $X(x, 0), 0 < x < 1$ with respect to the unit circle with its center the origin. Then the length $T(x)$ of the tangential line AB ($B(x, \sqrt{1 - x^2})$) is given by

$$T(x) = \frac{1}{x}\sqrt{1 - x^2}.$$

For $x = 0$, by the division by zero calculus, we have

$$T(0) = 0$$

that was considered as $+\infty$.

We will consider a function $y = f(x)$ of C^1 class on the real line. We consider the tangential line through $(x, f(x))$

$$Y = f'(x)(X - x) + f(x).$$

Then, the length (or distance) $d(x)$ between the point $(x, f(x))$ and $\left(x - \frac{f(x)}{f'(x)}, 0\right)$ is given by, for $f'(x) \neq 0$

$$d(x) = |f(x)|\sqrt{1 + \frac{1}{f'(x)^2}}.$$

How will be the case $f'(x^*) = 0$? Then, the division by zero shows that

$$d(x^*) = |f(x^*)|.$$

Meanwhile, the x axis point $(X_t, 0)$ of the tangential line at (x, y) and y axis point $(0, Y_n)$ of the normal line at (x, y) are given by

$$X_t = x - \frac{f(x)}{f'(x)}$$

and

$$Y_n = y + \frac{x}{f'(x)},$$

respectively. Then, if $f'(x) = 0$, we obtain the reasonable results:

$$X_t = x, \quad Y_n = y.$$

9.7 Curvature and center of curvature

We will assume that a function $y = f(x)$ is of class C^2. Then, the curvature radius ρ and the center $O(x, y)$ of the curvature at point $(x, f(x))$ are given by

$$\rho(x, y) = \frac{(1 + (y')^2)^{3/2}}{y''}$$

and

$$O(x,y) = \left(x - \frac{1+(y')^2}{y''} y', y + \frac{1+(y')^2}{y''} \right),$$

respectively. Then, if $y'' = 0$, we have the results

$$\rho(x,y) = 0$$

and

$$O(x,y) = (x,y),$$

by the division by zero. They are reasonable.

We will consider a curve $\mathbf{r} = \mathbf{r}(s), s = s(t)$ of class C^2. Then,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \mathbf{t} = \frac{d\mathbf{r}(\mathbf{s})}{ds}, v = \frac{ds}{dt}, \frac{d\mathbf{t}(\mathbf{s})}{ds} = \frac{1}{\rho}\mathbf{n},$$

by the principal normal unit vector $\mathbf{n}$. Then, we see that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{t} + \frac{v^2}{\rho}\mathbf{n}.$$

If $\rho(s_0) = 0$: (consider a line case), then

$$\mathbf{a}(s_0) = \left[\frac{dv}{dt}\mathbf{t} \right]_{s=s_0}$$

and

$$\left[\frac{v^2}{\rho} \right]_{s=s_0} = \infty$$

will be funny. It will be the zero.

9.8 Our life figure

As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{ |\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2} \right\}.$$

We will consider a disc inscribed in the sector Δ_α whose center $(k,0)$ with its radius r. Then, we have

$$r = k \sin\alpha.$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately

$$[r]_{r=\infty} = 0.$$

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

9.9 H. Okumura's example

The suprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector Δ_α, we shall change the angle and we consider a fixed circle $C_a, a > 0$ with its radius a inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with its center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}.$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive y axis is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

These two subsections were taken from [15].

9.10 Interpretation by analytic geometry

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2, 3.$$

The area S of the triangle surrounded by these lines is given by

$$S = \pm\frac{1}{2} \cdot \frac{\triangle^2}{D_1 D_2 D_3},$$

where $\triangle$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and D_k is the co-factor of $\triangle$ with respect to c_k. $D_k = 0$ if and only if the corresponding lines are parallel. $\triangle = 0$ if and only if the three lines are parallel or they have a common point. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

Similarly we write lines by

$$M_k : a_{k1}x + a_{k2}y + a_{3k} = 0, k = 1, 2, 3.$$

The area S of the triangle surrounded by these lines is given by

$$S = \frac{1}{A_{11}A_{22}A_{33}} \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

where A_{kj} is the co-factor of a_{kj} with respect to the matrix $[a_{kj}]$. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

For a function

$$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{9.4}$$

the radius R of the circle $S(x, y) = 0$ is given by

$$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}.$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero. In this case, the circle is a line (degenerated).

The center of the circle (9.4) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a}\right).$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0$$

may be considered as the origin $(0, 0)$, by the division by zero.

We consider the functions

$$S_j(x, y) = a_j(x^2 + y^2) + 2g_jx + 2f_jy + c_j.$$

The distance d of the centers of the circles $S_1(x, y) = 0$ and $S_2(x, y) = 0$ is given by

$$d^2 = \frac{g_1^2 + f_1^2}{a_1^2} - 2\frac{g_1g_2 + f_1f_2}{a_1a_2} + \frac{g_2^2 + f_2^2}{a_2^2}.$$

If $a_1 = 0$, then by the division by zero

$$d^2 = \frac{g_2^2 + f_2^2}{a_2^2}.$$

Then, $S_1(x, y) = 0$ is a line and its center is the origin $(0, 0)$. Therefore, the result is very reasonable.

The distance d between two lines given by

$$\frac{x - a_j}{L_1} = \frac{y - b_j}{M_j} = \frac{z - c_j}{N_j}, \quad j = 1, 2,$$

is given by

$$d = \frac{\begin{vmatrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix}}{\sqrt{(M_lN_2 - M_2N_1)^2 + (N_lL_2 - N_2L_1)^2 + (L_lM_2 - L_2M_1)^2}}.$$

If two lines are parallel, then we have $d = 0$.

This subsection was taken from [12]. For more examples see it.

10 Analytic functions and division by zero calculus

The values of analytic functions at isolated singular points were given by the coefficients C_0 of the Laurent expansions (the first coefficients of the regular part) as the division by zero calculus. Therefore, their property may be considered as arbitrary ones by any sift of the image complex plane. Therefore, we can consider the values as zero in any Laurent expansions by shifts, as normalizations. However, if by another normalizations, the Laurent expansions are determined, then the values will have their senses. We will firstly examine such properties for the Riemann mapping function.

Let D be a simply-connected domain containing the point at infinity having at least two boundary points. Then, by the celebrated theorem of Riemann, there exists a uniquely determined conformal mapping with a series expansion

$$W = f(z) = C_1 z + C_0 + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \dots, \quad C_1 > 0, \tag{10.1}$$

around the point at infinity which maps the domain D onto the exterior $|W| > 1$ of the unit disc on the complex W plane. We can normalize (10.1) as follows:

$$\frac{f(z)}{C_1} = z + \frac{C_0}{C_1} + \frac{C_{-1}}{C_1 z} + \frac{C_{-2}}{C_1 z^2} + \dots.$$

Then, this function $\frac{f(z)}{C_1}$ maps D onto the exterior of the circle of radius $1/C_1$ and so, it is called the **mapping radius** of D. See [3, 39]. Meanwhile, from the normalization

$$f(z) - C_0 = C_1 z + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \dots,$$

by the natural shift C_0 of the image plane, the unit circle is mapped to the unit circle with center C_0. Therefore, C_0 may be called as **mapping center** of D. The function $f(z)$ takes the value C_0 at the point at infinity in the sense of the division by zero calculus and now we have its natural sense by the mapping center of D. We have considered the value of the function $f(z)$ as infinity at the point at infinity, however, practically it was the value C_0. This will mean that in a sense the value C_0 is the farthest point from the point at infinity or the image domain with the strong discontinuity.

The properties of mapping radius were investigated deeply in conformal mapping theory like estimations, extremal properties and meanings of the values, however, it seems that there is no information on the property of mapping center. See many books on conformal mapping theory or analytic function theory. See [39] for example.

From the fundamental Bierberbach area theorem, we can obtain the following inequality:

For analytic functions on $|z| > 1$ with the normalized expansion around the point at infinity

$$g(z) = z + b_0 + \frac{b_1}{z} + \cdots$$

that are univalent and take no zero point,

$$|b_0| \leq 2.$$

In our sense

$$g(\infty) = b_0.$$

See [17], Chapter V, Section 8 for the details.

10.1 Values of typical Laurent expansions

The values at singular points of analytic functions are represented by the Cauchy integral, and so for given functions, the calculations will be simple numerically, however, their analytical (precise) values will be given by using the known Taylor or Laurent expansions. In order to obtain some feelings for the values at singular points of analytic functions, we will see typical examples and fundamental properties.

For

$$f(z) = \frac{1}{\cos z - 1}, \quad f(0) = -\frac{1}{6}.$$

For

$$f(z) = \frac{\log(1+z)}{z^2}, \quad f(0) = \frac{-1}{2}.$$

For

$$f(z) = \frac{1}{z(z+1)}, \quad f(0) = -1.$$

For our purpose in the division by zero calculus, when a is an isolated singular point, we have to consider the Laurent expansion on $\{0 < r <$

$|z-a|<R\}$ such that r may be taken arbitrary small r, because we are considering the function at a.

For

$$f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}, \quad f(i)=\frac{1}{4}.$$

For

$$f(z)=\frac{1}{\sqrt{(z+1)}-1}, \quad f(0)=\frac{1}{2}.$$

For the Bernoulli constants B_n, we have the expansions

$$\frac{1}{(\exp z)-1}=\frac{1}{z}-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{(2n)!}z^{2n-1}$$

$$=\frac{1}{z}-\frac{1}{2}+2z\sum_{n=1}^{\infty}\frac{1}{z^2+4\pi^2n^2}$$

and so, we obtain

$$\frac{1}{(\exp z)-1}(z=0)=-\frac{1}{2},$$

([28], page 444).

From the well-known expansion ([1], page 807) of the Riemann zeta function

$$\zeta(s)=\frac{1}{s-1}+\gamma-\gamma_1(s-1)+\gamma_2(s-1)^2+...,$$

we see that the Euler constant γ is the value at $s=1$; that is,

$$\zeta(1)=\gamma.$$

Meanwhile, from the expansion

$$\zeta(z)=\frac{1}{z}-\sum_{k=2}^{\infty}C_k\frac{z^{2k-1}}{2k-1}$$

([1], 635 page 18.5.5), we have

$$\zeta(0)=0.$$

From the representation of the Gamma function $\Gamma(z)$

$$\Gamma(z) = \int_1^\infty e^{-t} t^{z-1} dt + \sum_{n=0}^\infty \frac{(-1)^n}{n!(z+n)}$$

([28], page 472), we have

$$\Gamma(-m) = E_{m+1}(1) + \sum_{n=0, n\neq m}^\infty \frac{(-1)^n}{n!(-m+n)}$$

and

$$[\Gamma(z) \cdot (z+n)](-n) = \frac{(-1)^n}{n!}.$$

In particular, we obtain

$$\Gamma(0) = -\gamma,$$

by using the identity

$$E_1(z) = -\gamma - \log z - \sum_{n=1}^\infty \frac{(-1)^n z^n}{n n!}, \quad |\arg z| < \pi$$

([1], 229 page, (5.1.11)). Of course,

$$E_1(z) = \int_z^\infty e^{-t} t^{-1} dt.$$

From the recurrence formula

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

of the Psi (Digamma) function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

([1], 258), we have, for $z = 0, 1$,

$$\psi(0) = \psi(1) = -\gamma.$$

Note that

$$\psi(1+z) = -\gamma + \sum_{n=2}^\infty (-1)^n \zeta(n) z^{n-1}, \quad |z| < 1$$

$$= -\gamma + \sum_{n=}^{\infty} \frac{z}{n(n+z)}, \quad z \neq -1, -2, ...$$

([1], 259).

From the identity

$$\frac{1}{\psi(z+1) - \psi(z)} = z,$$

we have

$$\frac{1}{\psi(z+1) - \psi(z)}(z=0) = 0.$$

From the identities

$$\frac{\Gamma(z)}{\Gamma(z+1)} = \frac{1}{z},$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

note that their values are zero at $z = 0$

From the expansions

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} C_k z^{2k-2}$$

and

$$\wp'(z) = \frac{-2}{z^3} + \sum_{k=2}^{\infty} (2k-2) C_k z^{2k-3}$$

([1], 623 page, 18.5.1. and 18.5.4), we have

$$\wp(0) = \wp'(0) = 0.$$

We can consider many special functions and the values at singular points. For example,

$$Y_{3/2}(z) = J_{-3/2}(z) = -\sqrt{\frac{2}{\pi z}} \left(\sin z + \frac{\cos z}{z} \right),$$

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z,$$

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z},$$

and so on. They take the value zero at the origin, however, we can consider some meanings of the value.

Of course, the product property is, in general, not valid:

$$f(0) \cdot g(0) \neq (f(z)g(z))(0);$$

indeed, for the functions $f(z) = z + 1/z$ and $g(z) = 1/z + 1/(z^2)$

$$f(0) = 0, g(0) = 0, (f(z)g(z))(0) = 1.$$

For an analytic function $f(z)$ with a zero point a, for the inversion function

$$(f(z))^{-1} := \frac{1}{f(z)},$$

we can calculate the value $(f(a))^{-1}$ at the singular point a.

For example, note that for the function

$$f(z) = z - \frac{1}{z},$$

$f(0) = 0, f(1) = 0$ and $f(-1) = 0$. Then, we have

$$(f(z))^{-1} = \frac{1}{2(z+1)} + \frac{1}{2(z-1)}.$$

Hence,

$$((f(z))^{-1})(z = 0) = 0, ((f(z))^{-1})(z = 1) = \frac{1}{4},$$

$$((f(z))^{-1}(z = -1) = -\frac{1}{4}.$$

Here, note that the point $z = 0$ is not a regular point of the function $f(z)$.

We, meanwhile, obtain that

$$\left(\frac{1}{\log x}\right)_{x=1} = 0.$$

Indeed, we consider the function $y = \exp(1/x), x \in \mathbf{R}$ and its inverse function $y = \frac{1}{\log x}$. By the symmetric property of the functions with respect to the function $y = x$, we have the desired result.

Here, note that for the function $\frac{1}{\log x}$, we can not use the Laurent expansion around $x = 1$, and therefore, the result is not trivial.

In particular, note that the function $W = \exp(1/z)$ takes the **Picard's exceptional value** 1 at the origin $z = 0$, by the division by zero calculus.

Meanwhile, for the identity

$$\frac{a-b}{\log a - \log b},$$

for $a = b$, we should consider it in the following way. By substituting $\log a = A$ and $\log b = B$, from

$$\frac{\exp A - \exp B}{A - B},$$

by the division by zero calculus, we have the reasonable result for $A = B$,

$$\exp A = a.$$

However, substitution methods are very delicate. For example, for the function

$$w = \frac{1 + it}{1 - it},$$

for $t = -i$, by the division by zero calculus, we have a good value $w = -1$. However, from the representation $z = e^{i\alpha}$ we have

$$\frac{1+z}{1-z} = i \cot \frac{\alpha}{2}$$

and for $\alpha = 0$ and $z = 1$, we have the contradiction $-1 = 0$. By considering the way

$$\frac{1 + e^{i\alpha}}{1 - e^{i\alpha}}$$

and when we consider it by the division by zero calculus in connection with α for $\alpha = 0$, we have the right value 0.

By the Laurent expansion and by the definition of the division by zero calculus, we note that:

Theorem: *For any analytic function $f(z)$ on $0 < |z| < \infty$, we have*

$$f(0) = f(\infty).$$

For a rational function

$$f(z) = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}; \quad a_0, b_0 \neq 0; \quad a_m, b_n \neq 0, m, n \geq 1$$

$$f(0) = f(\infty) = \frac{a_0}{b_0}.$$

Of course, here $f(\infty)$ is not given by any limiting $z \to \infty$, but it is the value at the point at ∞.

10.2 The derivatives of $n!$:

We shall state an example.

Note that the identity $z! = \Gamma(z+1)$ and the Gamma function is a meromorphic function with isolated singular points on the entire complex plane. Therefore, we can consider the derivatives of the Gamma function even at isolated singular points, in our sense.

10.3 Domain functions

We shall state an example.

The Szegö kernel

For the Szegö kernel $K(z, \overline{u})$ and its adjoint L kernel $L(z, u)$ on a regular region D on the complex z plane, the function

$$f(z) = \frac{K(z, \overline{u})}{L(z, u)}$$

is the Ahlfors function on the domain D and it maps the domain D onto the unit disc $|w| < 1$ with one to the multiplicity of the connectivity of the domain D. From the relation $L(z, u) = -L(u, z)$, we see that $L(u, u) = 0$ in the sense of the division by zero calculus. Therefore, from the identity

$$L(z, u) = \frac{1}{2\pi(z - u)} + \frac{1}{2\pi} \int_{\partial D} \frac{K(u, \overline{\zeta})}{\zeta - z} |d\zeta|$$

([17], 390 page), we have the identity

$$\int_{\partial D} \frac{K(z,\overline{\zeta})}{\zeta - z} |d\zeta| = 0.$$

By this method, we can find many new identities.

11 The Descartes circle theorem

We recall the famous and beautiful theorem ([8, 36]):

Theorem (Descartes). *Let C_i $(i = 1, 2, 3)$ be circles touching to each other of radii r_i. If a circle C_4 touches the three circles, then its radius r_4 is given by*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{11.1}$$

As well-known, circles and lines may be looked as the same ones in complex analysis, in the sense of stereographic projection and with many reasons. Therefore, we will consider whether the theorem is valid for line cases and point cases for circles. Here, we will discuss this problem clearly from the division by zero viewpoint. The Descartes circle theorem is valid except for one case for lines and points for the three circles and for one exception case, we can obtain very interesting results, by the division by zero calculus.

We would like to consider all cases for the Descartes theorem for lines and point circles, step by step.

11.1 One line and two circles case

We consider the case in which the circle C_3 is one of the external common tangents of the circles C_1 and C_2. This is a typical case in this paper. We assume that $r_1 \geq r_2$. We now have $r_3 = 0$ in (11.1). Hence

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{0} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 \cdot 0} + \frac{1}{0 \cdot r_1}} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\sqrt{\frac{1}{r_1 r_2}}.$$

This implies

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

in the plus sign case. The circle C_4 is the incircle of the curvilinear triangle made by C_1, C_2 and C_3. In the minus sign case we have

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_2}} - \frac{1}{\sqrt{r_1}}.$$

In this case C_2 is the incircle of the curvilinear triangle made by the other three.

Of course, the result is known. The result was also well-known in Wasan geometry [40] with the Descartes circle theorem itself.

11.2 Two lines and one circle case

In this case, the two lines have to be parallel, and so, this case is trivial, because then other two circles are the same size circles, by the division by zero $1/0 = 0$.

11.3 One point circle and two circles case

This case is another typical case for the theorem. Intuitively, for $r_3 = 0$, the circle C_3 is the common point of the circles C_1 and C_2. Then, there does not exist any touching circle of the three circles $C_j; j = 1, 2, 3$.

For the point circle C_3, we will consider it by limiting of circles attaching to the circles C_1 and C_2 to the common point. Then, we will examine the circles C_4 and the Descartes theorem.

We will need the following results:

For real numbers z, and $a, b > 0$, the point $(0, 2\sqrt{ab}/z)$ is denoted by V_z. H. Okumura and M. Watanabe gave the theorem in [18]:

Theorem 7. *The circle touching the circle α: $(x-a)^2 + y^2 = a^2$ and the circle β: $(x+b)^2 + y^2 = b^2$ at points different from the origin O and passing through $V_{z\pm1}$ is represented by*

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{11.2}$$

for a real number $z \neq \pm 1$.

The common external tangents of α and β can be expressed by the equations

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{11.3}$$

In Theorem 7, by setting $z = 1/w$, we will consider the case $w = 0$; that is, the case $z = \infty$ in the classical sense; that is, the circle C_3 is reduced to the origin.

We look for the circles C_4 attaching with three circles $C_j; j = 1, 2, 3$. We set

$$C_4 : (x - x_4)^2 + (y - y_4)^2 = r_4^2. \tag{11.4}$$

Then, from the touching property we obtain:

$$x_4 = \frac{r_1 r_2 (r_2 - r_1) w^2}{D},$$

$$y_4 = \frac{2r_1 r_2 \left(\sqrt{r_1 r_2} + (r_1 + r_2) w\right) w}{D}$$

and

$$r_4 = \frac{r_1 r_2 (r_1 + r_2) w^2}{D},$$

where

$$D = r_1 r_2 + 2\sqrt{r_1 r_2}(r_1 + r_2) w + (r_1^2 + r_1 r_2 + r_2^2) w^2.$$

By inserting these values to (11.4), we obtain

$$f_0 + f_1 w + f_2 w^2 = 0,$$

where

$$f_0 = r_1 r_2 (x^2 + y^2),$$

$$f_1 = 2\sqrt{r_1 r_2}\big((r_1 + r_2)(x^2 + y^2) - 2r_1 r_2 y\big)$$

and

$$f_2 = (r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2r_1 r_2 (r_2 - r_1) x - 4(r_1 + r_2) y + 4r_1^2 r_2^2.$$

By using the division by zero calculus for $w = 0$, we obtain, for the first, for $w = 0$, the second by setting $w = 0$ after dividing by w and for the third case, by setting $w = 0$ after dividing by w^2,

$$x^2 + y^2 = 0, \tag{11.5}$$

$$(r_1 + r_2)(x^2 + y^2) - 2r_1 r_2 y = 0 \tag{11.6}$$

and

$$\begin{aligned}(r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2r_1 r_2 (r_2 - r_1)x \\ -4r_1 r_2 (r_1 + r_2) y + 4r_1^2 r_2^2 = 0.\end{aligned} \tag{11.7}$$

Note that (11.6) is the circle with the radius

$$\frac{r_1 r_2}{r_1 + r_2} \tag{11.8}$$

and (11.7) is the circle whose radius is

$$\frac{r_1 r_2 (r_1 + r_2)}{r_1^2 + r_1 r_2 + r_2^2}.$$

When the circle C_3 is reduced to the origin, of course, the inscribed circle C_4 is reduced to the origin, then the Descartes theorem is not valid. However, by the division by zero calculus, then the origin of C_4 is changed suddenly for the cases (11.5), (11.6) and (11.7), and for the circle (11.6), the Descartes theorem is valid for $r_3 = 0$, surprisingly.

Indeed, in (11.1) we set $\xi = \sqrt{r_3}$, then (11.1) is as follows:

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\xi^2} \pm 2\frac{1}{\xi}\sqrt{\frac{\xi^2}{r_1 r_2} + \left(\frac{1}{r_1} + \frac{1}{r_2}\right)}.$$

and so, by the division by zero calculus at $\xi = 0$, we have

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2}$$

which is (11.8). Note, in particular, that the division by zero calculus may be applied in many ways and so, for the results obtained should be examined for some meanings. This circle (11.6) may be looked a circle touching the origin and two circles C_1 and C_2, because by the division by zero calculus

$$\tan \frac{\pi}{2} = 0,$$

that is a popular property.

Meanwhile, the circle (11.7) is the attaching circle with the circles C_1, C_2 and the beautiful circle with its center $((r_2 - r_1), 0)$ with its radius $r_1 + r_2$. Each of the areas surrounded by three circles C_1, C_2 and the circle of radius $r_1 + r_2$ is called an arbelos, and the circle (11.6) is the famous Bankoff circle of the arbelos. For $r_3 = -(r_1 + r_2)$, from the Descartes identity (10.4), we have (10.4). That is, when we consider that the circle C_3 is changed to the circle with its center $((r_2 - r_1), 0)$ with its radius $r_1 + r_2$, the Descartes identity holds. Here, the minus sign shows that the circles C_1 and C_2 touch C_3 internally from the inside of C_3.

11.4 Two point circles and one circle case

This case is trivial, because, the exterior touching circle is coincident with one circle.

11.5 Three points case and three lines case

In these cases we have $r_j = 0, j = 1, 2, 3$ and the formula (11.1) shows that $r_4 = 0$. This statement is trivial in the general sense.

As the solution of the simplest equation

$$ax = b, \tag{11.9}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (11.9) is impossible. The zero will represent some **impossibility**.

In the Descartes theorem, three lines and three points cases, we can understand that the attaching circle does not exist, or it is the point and so the Descartes theorem is valid.

This section is based on the paper [20].

12 Horn torus models and division by zero calculus – a new world

We recall the essence of the paper [6] for horn torus models.

We will consider the three circles represented by

$$\xi^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(\xi - \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2, \tag{12.1}$$

and

$$\left(\xi + \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2.$$

By rotation on the space (ξ, η, ζ) on the (x, y) plane as in $\xi = x, \eta = y$ around ζ axis, we will consider the sphere with $1/2$ radius as the Riemann sphere and the horn torus made in the sphere.

The stereographic projection mapping from (x, y) plane to the Riemann sphere is given by

$$\xi = \frac{x}{x^2 + y^2 + 1},$$

$$\eta = \frac{y}{x^2 + y^2 + 1},$$

and

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Of course,

$$\xi^2 + \eta^2 = \zeta(1 - \zeta),$$

and

$$x = \frac{\xi}{1 - \zeta}, y = \frac{\eta}{1 - \zeta}, \tag{12.2}$$

([2]).

The mapping from (x, y) plane to the horn torus is given by

$$\xi = \frac{2x\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

$$\eta = \frac{2y\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

and

$$\zeta = \frac{(x^2+y^2-1)\sqrt{x^2+y^2}}{(x^2+y^2+1)^2} + \frac{1}{2}.$$

This Puha mapping has a simple and beautiful geometrical correspondence. At first for the plane we consider the stereographic mapping to the Riemann sphere and next, we consider the common point of the line connecting the point and the center (0,0,1/2) and the horn torus. This is the desired point on the horn torus for the plane point.

The inversion is given by

$$x = \xi \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)} \tag{12.3}$$

and

$$y = \eta \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)} . \tag{12.4}$$

For the properties of horn torus with physical applications, see [5].

12.1 Conformal mapping from the plane to the horn torus with a modified mapping

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model (2018.8.18):

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

We can represent the direct Däumler mapping from the z plane onto the horn torus as follows (V. V. Puha: 2018.8.28.22:31):

With

$$\phi = 2\cot^{-1}(-\log|z|), \quad z = x + yi, \tag{12.5}$$

$$\xi = \frac{x \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}},$$

$$\eta = \frac{y \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}},$$

and

$$\zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}.$$

We have the inversion formula from the horn torus to the x, y plane:

$$x = \frac{\xi}{\sqrt{\xi^2+\eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2+\eta^2+\zeta^2)}}{\sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2}} \right\} \tag{12.6}$$

and

$$y = \frac{\eta}{\sqrt{\xi^2+\eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2+\eta^2+\zeta^2)}}{\sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2}} \right\}. \tag{12.7}$$

.

12.2 New world and absolute function theory

We will discuss on Däumler's horn torus model from some fundamental viewpoints.

First of all, note that in the Puha mapping and the Däumler mapping, and even in the classical stereographic mapping, we find the division by zero $1/0 = 0/0 = 0$. See [6] for the details.

12.2.1 What is the number system?

What are the numbers? What is the number system? For these fundamental questions, we can say that the numbers are complex numbers **C** and the number system is given by the Yamada field with the simple structure **as a field containing the division by zero.**

Nowadays, we have still many opinions on these fundamental questions, however, this subsection excludes all those opinions as in the above.

12.2.2 What is the natural coordinates?

We represented the complex numbers **C** by the complex plane or by the points on the Riemann sphere. On the complex plane, the point at infinity is the ideal point and for the Riemann sphere representation, we have to accept the **strong discontinuity**. From these reasons, the numbers and the numbers system should be represented by the Däumler's horn torus model that is conformally equaivalent to the extended complex plane.

12.2.3 What is a function? What is the graph of a function?

A function may be considered as a mapping from a set of numbers into a set of numbers.

The numbers are represented by Däumler's horn torus model and so, we can consider that a function, in particular, an analytic function can be considered as a mapping from Däumler's horn torus model into Däumler's horn torus model.

12.2.4 Absolute function theory

Following the above considerings, for analytic functions when we consider them as the mappings from Däumler's horn torus model into Däumler's horn torus model we would like to say that it is an **absolute function theory.**

For the classical theory of analytic functions, discontinuity of functions at singular points will be the serious problems and the theory will be quite different from the new mathematics, when we consider the functions on the Däumler's horn torus model. Even for analytic function theory on bounded domains, when we consider their images on Däumler's horn torus model, the results will be very interesting.

12.2.5 New mathematics and future mathematicians

The structure of Däumler's horn torus model is very involved and so, we will need some computer systems like MATHEMATICA and Isabelle/HOL system for our research activity. Indeed, for the analytical proof of the conformal mapping of Däumler, we had to use MATHEMATICA, already. Here, we will be able see some future of mathematicans.

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