# The Theorems of Rao–Blackwell and Lehmann–Scheffé, Revisited

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**Abstract**

It has been stated in the literature that for finding uniformly minimum-variance unbiased estimator through the theorems of Rao–Blackwell and Lehmann–Scheffé, the sufficient statistic should be complete; otherwise the discussion and the way of finding uniformly minimum-variance unbiased estimator should be changed, since the sufficiency assumption in the Rao–Blackwell and Lehmann–Scheffé theorems limits its applicability. So, it seems that the sufficiency assumptions should be expressed in a way that the uniformly minimum-variance unbiased estimator be derivable via the Rao–Blackwell and Lehmann–Scheffé theorems.

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# 1 Introduction

There are many books and papers about incomplete and complete minimal sufficient statistic, all of which indicate how to derive minimum-variance unbiased estimator (UMVUE). However, there is no uniform way for finding UMVUE and thus each researcher choose his own way to handle this problem. It is interesting to note that some of the researchers focus on the examples given by Rao (1973, problem 5.11) or Lehmann (1983, pp. 76-77), and try to solve these kinds of problems. It is noteworthy that the weak point of Rao–Blackwell (Rao 1945; Blackwell 1947) and Lehmann–Scheffé (Lehmann and Scheffé 1950, 1955) theorems are the sufficiency assumption. Thus, it is shown in this short note that by changing the way of impact of sufficiency, which seems to be new, the difficulty in using Rao–Blackwell and Lehmann-Scheffé theorems for deriving UMVUE can be overcome. We refer the readers to some of the following references,

- Mukhopadhyay (2000, p. 377, Subsection 7.6.1), ***"Does the Rao–Blackwell Theorem Lead to UMVUE?"***
- Peña and Rohatgi (1994, p. 243), ***"Could we use $T$ to Rao–Blackwellize an unbiased estimator $h$ of $\tau(\theta)$ to obtain the UMVUE?"***
- Galili and Meilison (2016, Abstract and Introduction ) ***"The Rao–Blackwell theorem offers a procedure for converting a crud unbiased estimator of a parameter $\theta$ into a better one, in fact unique and optimal if the improvement is based on minimal sufficient statistic that is complete"***
- Roussas (1997, p. 293, Subsection 12.4) ***"The Case Where Complete Sufficient Statistics Are Not Available or May Not Exist: Cramér–Rao Inequlity"***

- Yamada (1994, p. 59 , Chapter 4) ***"... Rao–Blackwellization of unbiased estimator does not necessarily provide a UMVUE estimator"***

- Lehmann and Casella (1998, p. 86, Example 1.8)

- Mood et al.(1974, pp. 330–331, Remark)

## 2 $\mathscr{H}$-Sufficient Statistic

**Definition 2.1** Let $X_1, \cdots, X_n$ be a random sample from the density $f(.;\theta)$ and $T = t(X_1, \cdots, X_n)$ be the unbiased estimator $a(\theta)$ in $\mathcal{U}$ . A statistic $\mathscr{H} = \mathcal{H}(X_1, \cdots, X_n)$ is defined to be an $\mathscr{H}$-Sufficient Statistic for $a(\theta)$ if and only if the conditional expectation of $T$ given $\mathscr{H}$ does not depend on $\theta$ for any statistic $T$ in $\mathcal{U}$.

⊛

Now, two theorems are stated as follows which cover the theorems of Rao–Blackwell and Lehmann-Scheffé.

**Theorem 2.2** *Let $W$ be any unbiased estimator of $a(\theta)$ and let $T$ be an $\mathscr{H}$-sufficient statistic for $a(\theta)$. Define $\Phi(T) = E(W|T)$. Then $E_\theta\Phi(T) = a(\theta)$ and $\mathrm{var}_\theta(\Phi(T)) \leqslant \mathrm{var}_\theta(W)$ for all $\theta$; that is, $\Phi(T)$ is a uniformly better unbiased estimator of $a(\theta)$.*

**Proof.** This can be proved very similarly to the Rao-Blackwell theorem (see Casela and Beger (2002)). ⊠

**Theorem 2.3** *Let $T$ be a complete and $\mathscr{H}$-sufficient statistic for a parameter $a(\theta)$, and let $\Phi(T)$ be any estimator based on $T$. Then $\Phi(T)$ is the unique best unbiased estimator for its expected valued (UMVUE).*

**Proof.** Similar to the Lehmann–Scheffé theorem ( see also Casela and Beger (2002)). ⊠

**Example 2.4 (Rao 1973; Lehmann 1983; Lehman & Casela 1998)** Let $X$ take on the values $-1, 0, \cdots$ with probabilities $P(X = -1) = p$, $P(X = k) = q^2 p^k$, $k = 0, 1, \cdots$, where $0 < p < 1$ and $q = 1 - p$. We note that $I_{\{0\}}(X)$ is a complete and $\mathscr{H}$-sufficient statistic for $q^2$, since

$$E\Big(I_{\{0\}}(X) + \alpha X | I_{\{0\}}(X)\Big) = I_{\{0\}}(X),$$

for any $\alpha \in \Re$. Hence, from theorem 2.3, $I_{\{0\}}(X)$ is the UMVUE for $q^2$ and thus $AI_{\{0\}}(X) + B$ is the UMVUE for $Aq^2 + B$. ⊛

Also, some other examples, such as the ones in the books of Shao (2003, p. 167, Example 3.7) and Rohatgi and Ehsanes (2015, p. 366, Example 10 ) satisfy our new condition.

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