SOME HEREDITARY PROPERTIES OF THE E-J GENERALIZED CESÀRO MATRICES

H. C. RHALY JR.

Abstract. A countable subcollection of the Endl-Jakimovski generalized Cesàro matrices of positive order is seen to inherit posinormality, coposinormality, and hyponormality from the Cesàro matrix of the same order.

1. Introduction

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then $A \in B(H)$ is said to be is *posinormal* (see [1]) if

$$AA^* = A^*PA$$

for some positive operator $P \in B(H)$, and A is *coposinormal* A^* is posinormal. The operator $A \in \mathcal{B}(H)$ is *hyponormal* if

$$< (A^*A - AA^*)f, f > \geq 0$$

for all $f \in H$. Hyponormal operators are necessarily posinormal but need not be coposinormal, as the unilateral shift on ℓ^2 illustrates. Here it will be demonstrated that some of the Endl-Jakimovski generalized Cesàro matrices of order greater than or equal to 1, as described in the next section, inherit these properties from the corresponding Cesàro matrix of the same order.

2. Inherited Properties for Some E-J Generalized Cesàro Matrices

The main tool will be the following theorem, which is an abbreviated version of [3, Theorem 2.2] merged with [4, Theorem 3.2].

Theorem 2.1. *Suppose that $T \in B(\ell^2)$ is a lower triangular infinite matrix and U is the unilateral shift. Then $T' :\equiv U^*TU$ inherits each of the following properties from T:*
(a) posinormality,
(b) coposinormality, and
(c) hyponormality.

Proof. (a). See [3, Theorem 2.2].
(b). See [4, Theorem 3.2].
(c). See [3, Theorem 2.2]. □

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As seen in [5], the Endl-Jakimovski generalized Cesàro matrices of order β, denoted $(C^{(\alpha)},\beta)$ for $\alpha,\beta \geq 0$, are the lower triangular infinite matrices given by

$$(C^{(\alpha)},\beta)_{ij} = \frac{\Gamma(i+\alpha+1)}{\Gamma(i-j+1)} \cdot \frac{\beta\Gamma(i-j+\beta)}{\Gamma(i+\alpha+\beta+1)}. \tag{2.1}$$

If U denotes the unilateral shift, then

$$(U^*(C^{(\alpha)},\beta)U)_{ij} = (C^{(\alpha)},\beta)_{i+1,j+1} =$$

$$\frac{\Gamma((i+1)+\alpha+1)}{\Gamma((i+1)-(j+1)+1)} \cdot \frac{\beta\Gamma((i+1)-(j+1)+\beta)}{\Gamma((i+1)+\alpha+\beta+1)} =$$

$$\frac{\Gamma(i+(\alpha+1)+1)}{\Gamma(i-j+1)} \cdot \frac{\beta\Gamma(i-j+\beta)}{\Gamma(i+(\alpha+1)+\beta+1)} = (C^{(\alpha+1)},\beta)_{ij}$$

From this and induction it follows that the posinormality and hyponormality of $(C^{(\alpha)},\beta) \in B[\ell^2)$ would guarantee the posinormality and hyponormality, respectively, of $(C^{(\alpha+k)},\beta)$ for each positive integer k; similary, the coposinormality of $(C^{(\alpha)},\beta)$ would guarantee the coposinormality of $(C^{(\alpha+k)},\beta)$ for each positive integer k.

Proposition 2.2. *Fix $\beta \geq 0$.*

(a) *If $(C^{(\alpha)},\beta)$ is posinormal (coposinormal, hyponormal) for fixed $\alpha > -1$, then $(C^{(\alpha+k)},\beta)$ is posinormal (coposinormal, hyponormal, respectively) for each positive integer k.*
(b) *Fix $x > -1$. If $(C^{(\alpha)},\beta)$ is posinormal (coposinormal, hyponormal) for all $\alpha \in [x, x+1)$, then $(C^{(\alpha)},\beta)$ is posinormal (coposinormal, hyponormal, respectively) for all $\alpha \in [x,\infty)$.*

Proof. Both parts are clear from Theorem 2.1 and the intervening discussion. □

In [6] it was shown that the generalized Cesàro matrix $(C^{(\alpha)},3)$ is hyponormal for $\alpha \in \{0,1\} \cup [2,\infty)$. If a proof is later found that includes $\alpha \in (0,1)$, it would automatically extend to $\alpha \in (1,2)$ by Proposition 2.2(b).

Theorem 2.3. *If $\beta \geq 1$, the generalized Cesàro matrix $(C^{(k)},\beta)$ is coposinormal and hyponormal for each $k \in \mathbb{Z}+$.*

Proof. Note that $(C^{(0)},\beta)$, the Cesàro matrix of order $\beta \geq 1$, is already known to be coposinormal and hyponormal (see [2]). Apply Propositions 2.2(a), to see that $(C^{(k)},\beta)$ must also be coposinormal and hyponormal whenever $k \in \mathbb{Z}+$. □

The E-J generalized Cesàro matrices of order one, two, and three have already been studied regarding posinormality, coposinormality, and hyponormality; see, e.g., [5], [6]. The next example, however, provides some new information regarding these properties for some of the E-J generalized Cesàro matrices of order four.

Example 2.4. *Consider the matrix $M_\alpha :\equiv (C^{(\alpha)},4)$ whose entries (computed using equation (2.1)) are as follows:*

$$m_{ij} = \begin{cases} \frac{4(i+1-j)(i+2-j)(i+3-j)}{(i+1+\alpha)(i+2+\alpha)(i+3+\alpha)(i+4+\alpha)} & \text{for } 0 \leq j \leq i \\ 0 & \text{for } j > i. \end{cases}$$

From [2] *we know that* M_0*, the Cesàro matrix of order four (take* $\alpha = 0$*), is posinormal, coposinormal, and hyponormal. It follows from Theorem* 2.3 *that* M_k *(take* $\alpha = k$*) is also posinormal, coposinormal, and hyponormal for all* $k \in \mathbb{Z}+$.

References

[1] C. S. Kubrusly and B. P. Duggal, *On posinormal operators*, Adv. Math. Sci. Appl. **17** (2007), no. 1, 131-147.

[2] Trieu Le and H. C. Rhaly Jr. *Coposinormality of the Cesàro matrices*, Arch. Math. (Basel) **110** (2018), no. 2, 167–173.

[3] H. C. Rhaly Jr., *Heredity for triangular operators*, Bol. Soc. Parana. Mat., (3) **31** (2013), no. 2, 231-234.

[4] ———, *Posinormality, coposinormality, and supraposinormality for some triangular operators.* Sarajevo J. Math. **12(24)** (2016), no. 1, 125–140.

[5] ———, *Supraposinormality and hyponormality for the generalized Cesàro matrices of order two.* Oper. Matrices **12** (2018), no. 1, 133–142.

[6] ———, *The generalized Cesàro matrices of order three are supraposinormal on* ℓ^2, https://arxiv.org/abs/1903.07896.