Ramanujan value of Ln(x) when x tends to zero

Jesús Sánchez

Independent Researcher, Bilbao, Spain
Email: jesus.sanchez.bilbao@gmail.com
https://www.researchgate.net/profile/Jesus_Sanchez64
ORCID 0000-0002-5631-8195

Abstract

As we know, the natural logarithm at zero diverges, towards minus infinity:

$$\lim_{x \to 0} Ln(x) = -\infty$$

But, as happens with other functions or series that diverge at some points, it has a Ramanujan or Cauchy principal value (a finite value) associated to that point. In this case, it will be calculated to be:

$$\lim_{x \to 0} Ln(x) = -\gamma$$

Being γ the Euler-Mascheroni constant 0.577215... It will be shown that Ln(0) tends to the negative of the sum of the harmonic series (that of course, diverges). But the harmonic series has a Cauchy principal value that is γ, the Euler-Mascheroni constant. So the finite associated value to Ln(0) will be calculated as - γ .

Keywords

Natural Logarithm, divergent series, Ramanujan summation, Principal Cauchy Value, Euler-Mascheroni constant.

1. Introduction

In this paper, it will be calculated the Ramanujan value [1] of the Ln(x) when x tends to zero:

$$\lim_{x \to 0} Ln(x) = -\gamma \quad (1)$$

Being γ the Euler-Mascheroni constant 0.577215... [2]

2. Demonstration

We will use one of the series to calculate the natural logarithm [3]:

$$Ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \qquad (2 \geq x > 0) \qquad (2)$$

This series is not valid for x=0, but we will follow the same procedure that Ramanujan used in divergent series [4]. We will use the exact value where the limit applies (we will force the convergence in the divergence limit). In this case, it corresponds to x=0. Substituting in (2), we get:

$$\lim_{x \to 0} Ln(x) = -1 - \frac{1}{2} - \frac{1}{3} - \cdots \qquad (3)$$

This is exactly the negative of the harmonic series [5]:

$$\lim_{x \to 0} Ln(x) = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) \qquad (4)$$

The harmonic series diverge. But they have associated a Cauchy principal value [6][7][1] that is the Euler-Mascheroni constant [2]:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \gamma \qquad (5)$$

So, substituting in (4), we have:

$$\lim_{x \to 0} Ln(x) = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) = -\gamma \qquad (6)$$

As we wanted to prove.

3. Conclusions

Using a series of the natural logarithm and the fact that the Cauchy principal value of the harmonic series is the Euler-Mascheroni constant, we have calculated the Ramanujan value of the following equation:

$$\lim_{x \to 0} Ln(x) = -\gamma \qquad (1)$$

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12. Acknowledgements

To my family and friends.

13. References

[1] https://en.wikipedia.org/wiki/Ramanujan_summation

[2] https://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant

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[4] https://en.wikipedia.org/wiki/Srinivasa_Ramanujan

[5] https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)

[6] https://en.wikipedia.org/wiki/Cauchy_principal_value

[7] https://en.wikipedia.org/wiki/Riemann_zeta_function