A note on the arbelos in Wasan geometry, Satoh's problem and a circle pattern

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Abstract. We generalize a problem in Wasan geometry involving an arbelos, and construct a self-similar circle pattern.

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1. Introduction

Let α, β and γ be circles of diameters AO, BO and AB, respectively, for a point O on the segment AB. The configuration consisting of the three circles and the radical axis of α and β are called an arbelos and the axis, respectively. The radii of α and β are denote by a and b, respectively, and the reflection in the axis is denoted by σ. In this note we generalize the following problem proposed by Satoh (佐藤幸吉定寄) in a sangaku presented in Iwate in 1850 (Yasutomi, 1987) (see Fig. 1).

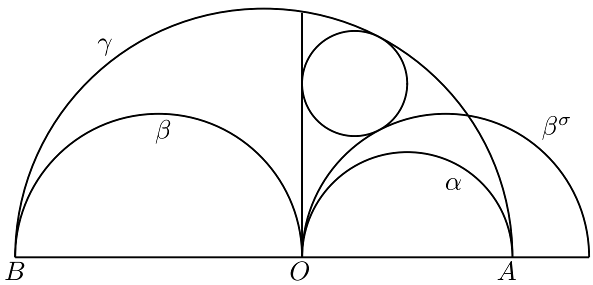

Figure 1.

Problem 1. Show that the two circles touching β^σ externally γ internally and the axis from the side opposite to B have radius $a/2$.

2. Generalization

Let $\alpha(n)$ (resp. $\beta(n)$) be the circle of radius na (resp. nb) touching the axis at O from the side opposite to A (resp. B) for a non-negative real number n. Let α_2 (resp. β_2) be one of the two circles touching $\beta(n)$ (resp. $\alpha(n)$) externally γ internally and the axis from the side opposite to B (resp. A), where the circles α_2 and β_2 lie on the same side of the segment AB. Problem 1 is the case $n = 1$ in the next theorem (see Fig. 2).

Theorem 1. *The circles α_2 and β_2 have radii $a/(n+1)$ and $b/(n+1)$, respectively, and touch at a point on the axis.*

Proof. We assume that D is the center of α_2, F is the foot of perpendicular from D to AB and $d = |DF|$. If r is the radius of α_2, then we get $(a+b-r)^2-((a-b)-r)^2 = (nb+r)^2 - (nb-r)^2 = d^2$ from the two right triangles formed by D, F and the center of γ, and D, F and the center of $\beta(n)$. Solving the equations, we have $r = a/(n+1)$ and $d = 2\sqrt{nab/(n+1)}$. Similarly β_2 has radius $b/(n+1)$. Since d is symmetric in a and b, the distance from the center of β_2 to AB also equals d. Therefore the circle α_2, β_2 touch at a point on the axis. □

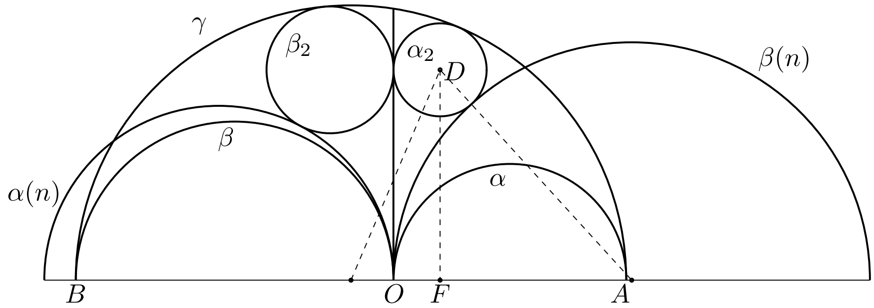

Figure 2.

Notice that if $n = 0$, then the circle $\beta(n)$ is a point circle and the circles α and α_2 coincide, also β and β_2 coincide.

3. A SELF-SIMILAR CIRCLE PATTERN

In this section we construct a self-similar circle pattern by Theorem 1. We now consider the case $n = 1$ in the theorem as in Problem 1. We denote the arbelos formed by α, β and γ by (α, β, γ), and denote the configuration consisting of (α, β, γ) and $(\alpha, \beta, \gamma)^\sigma$ by $\mathcal{C}_1$, which is symmetric in the axis (see Fig. 3). Notice that the axis and the segment AB are not included in $\mathcal{C}_1$. For a similar mapping τ, we call the figure $\mathcal{C}_1^\tau$ a symmetric arbelos of radius $|A^\tau B^\tau|$. Let us assume $|AB| = 1$.

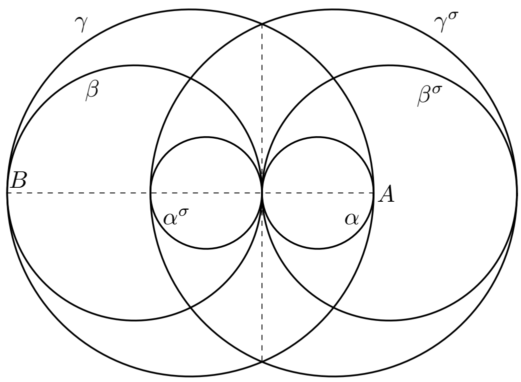

Figure 3: $\mathcal{C}_1$

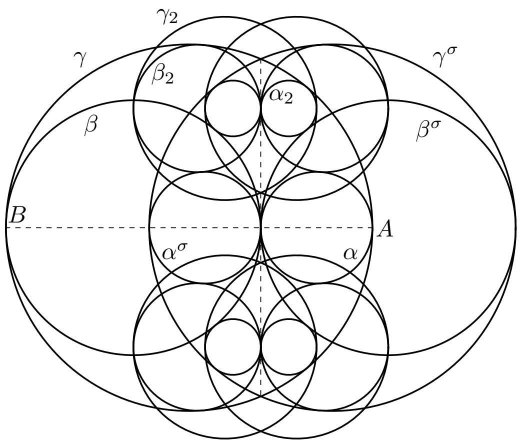

Figure 4: $\mathcal{C}_2$

Let γ_2 be the smallest circle touching α_2 and β_2 internally. Then the arbelos $(\alpha_2, \beta_2, \gamma_2)$ is similar to (α, β, γ) by Theorem 1. We call the figure consisting of $(\alpha_2, \beta_2, \gamma_2) \cup (\alpha_2, \beta_2, \gamma_2)^\sigma$ and its reflection in AB the two small copies of $\mathcal{C}_1$, which consists of two symmetric arbeloi of radius $1/2$. We define the two small copies of $\mathcal{C}_1^\tau$ for a similar mapping τ similarly. We denote the configuration consisting of $\mathcal{C}_1$ and its two small copies by $\mathcal{C}_2$ (see Fig. 4). If the figure $\mathcal{C}_k$ is constructed, which consists of $\mathcal{C}_1$, two symmetric arbeloi of radius $1/2$, four symmetric arbeloi of radius $1/2^2$, $\cdots$, 2^{k-1} symmetric arbeloi of radius $1/2^{k-1}$, then we add the two small copies of each of the 2^{k-1} symmetric arbeloi of radius $1/2^k$, and denote the resulting configuration by $\mathcal{C}_{k+1}$. By this construction, we can get $\mathcal{C}_n$ for any positive integer n. Then $\mathcal{C}_0 = \bigcup_{i\geq 1} \mathcal{C}_i$ is a self-similar circle configuration (see Fig. 5).

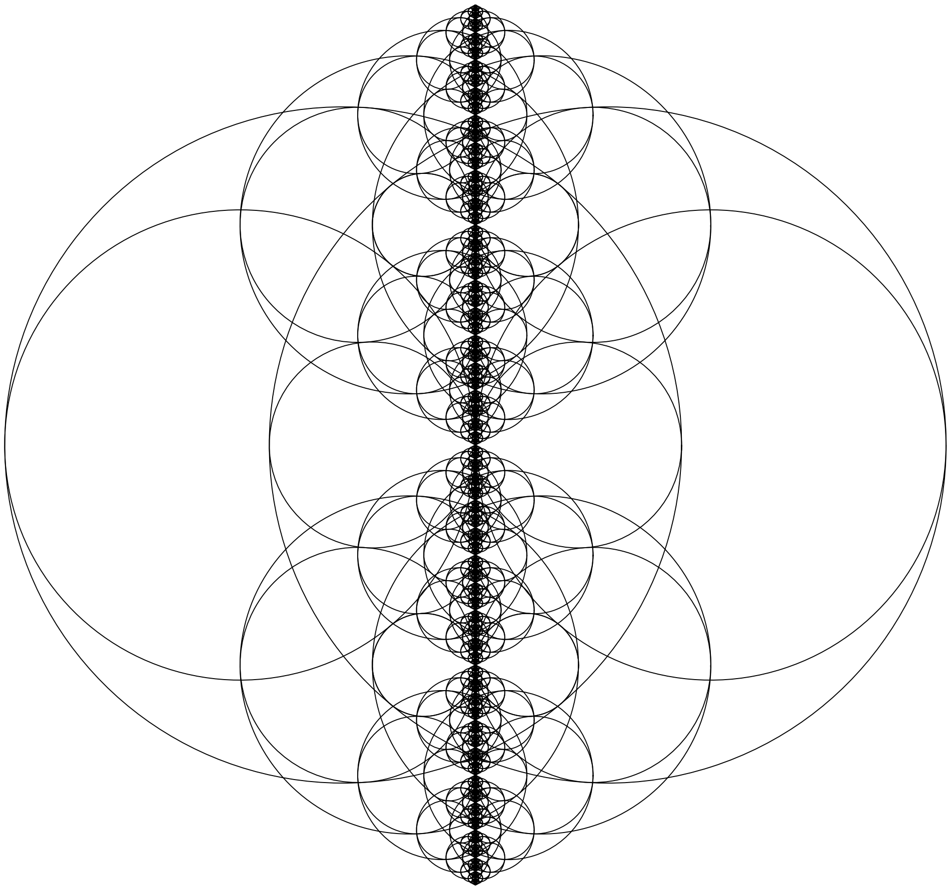

Figure 5: $\mathcal{C}_0$

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