Faux Proton Charge Smearing in Dirac Hydrogen by "Electron Zitterbewegung"

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Abstract

The commutator of the Dirac free-particle's velocity operator with its Hamiltonian operator is nonzero and independent of Planck's constant, which violates the quantum correspondence-principle requirement that commutators of observables must vanish when Planck's constant vanishes, as well as violating the absence of spontaneous acceleration of relativistic free particles. The consequent physically pathological "zitterbewegung" is of course completely absent when the natural relativistic square-root free-particle Hamiltonian operator is used; nevertheless the energy spectrum of that pathology-free natural relativistic square-root free-particle Hamiltonian is exactly matched by the positive-energy sector of the Dirac free-particle Hamiltonian's energy spectrum. Contrariwise, however, Foldy-Wouthuysen unitary transformation of the positive-energy sector of any hydrogen-type Dirac 4×4 Hamiltonian to 2×2 form reveals a "zitterbewegung"-induced "Darwin-term" smearing of the proton charge density which is completely absent in the straightforward relativistic extension of the corresponding hydrogen-type nonrelativistic Pauli 2×2 Hamiltonian. Compensating for an atomic proton's physically absent "electron zitterbewegung"-induced charge smearing would result in a misleadingly contracted impression of its charge radius.

Dirac kinematics: motion pathology, but positive-energy spectrum accuracy

The *natural* relativistic square-root free-particle quantum Hamiltonian operator,

$$\widehat{H} = \left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}, \tag{1a}$$

is diagonal in momentum representation by its nature. It implies the velocity operator,

$$(d\widehat{\mathbf{r}}/dt) = (-i/\hbar)\big[\widehat{\mathbf{r}},\, \widehat{H}\big] = (-i/\hbar)\Big[\widehat{\mathbf{r}},\, \left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}\Big] = \nabla_{\widehat{\mathbf{p}}}\left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}} = c\widehat{\mathbf{p}}\left(m^2c^2 + |\widehat{\mathbf{p}}|^2\right)^{-\frac{1}{2}}, \tag{1b}$$

and consequently the acceleration operator,

$$(d^2\widehat{\mathbf{r}}/dt^2) = (-i/\hbar)\big[(d\widehat{\mathbf{r}}/dt),\, \widehat{H}\big] = (-i/\hbar)\Big[c\widehat{\mathbf{p}}\left(m^2c^2 + |\widehat{\mathbf{p}}|^2\right)^{-\frac{1}{2}},\, \left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}\Big] = \mathbf{0}, \tag{1c}$$

which is consistent with *the absence of spontaneous acceleration of relativistic free particles*, i.e., Newton's First Law *remains valid in special relativity*. Two *crucial* relativistic characteristics of the Eq. (1b) free-particle velocity $(d\widehat{\mathbf{r}}/dt)$ are (1) that *its magnitude is less than c*,

$$|d\widehat{\mathbf{r}}/dt| = c|\widehat{\mathbf{p}}|\left(m^2c^2 + |\widehat{\mathbf{p}}|^2\right)^{-\frac{1}{2}} < c, \tag{1d}$$

and (2) that its asymptotic form for $|\widehat{\mathbf{p}}| \ll mc$ is *Newtonian*, i.e.,

$$(d\widehat{\mathbf{r}}/dt) \sim (\widehat{\mathbf{p}}/m) \text{ as } \widehat{\mathbf{p}} \to \mathbf{0}, \tag{1e}$$

which is *echoed* by the *Newtonian* asymptotic form for $|\widehat{\mathbf{p}}| \ll mc$ of the free-particle *kinetic-energy* operator,

$$\left(\widehat{H} - mc^2\right) = mc^2\Big(\left(1 + |\widehat{\mathbf{p}}/(mc)|^2\right)^{\frac{1}{2}} - 1\Big) \sim \left(|\widehat{\mathbf{p}}|^2/(2m)\right) \text{ as } \widehat{\mathbf{p}} \to \mathbf{0}. \tag{1f}$$

However the *Dirac* relativistic free-particle quantum Hamiltonian operator $\widehat{H}_D$, which is given by,

$$\widehat{H}_D = \beta mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}}), \tag{2a}$$

flouts the Eq. (1f) *kinetic-energy's Newtonian asymptotic form because* $(\widehat{H}_D - mc^2) = (\beta - 1)mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}})$. The Eq. (2a) Dirac $\widehat{H}_D$ *as well* flouts the Eq. (1e) velocity's Newtonian asymptotic form because,

$$(d\widehat{\mathbf{r}}/dt) = (-i/\hbar)\big[\widehat{\mathbf{r}},\, \widehat{H}_D\big] = (-i/\hbar)\big[\widehat{\mathbf{r}},\, \beta mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}})\big] = \nabla_{\widehat{\mathbf{p}}}\left(\beta mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}})\right) = c\vec{\alpha}, \tag{2b}$$

and $c\vec{\alpha}$ *is completely independent of* $\widehat{\mathbf{p}}$. In fact, the Eq. (2a) Dirac $\widehat{H}_D$ *unphysically violates* Eq. (1d) since,

$$|d\widehat{\mathbf{r}}/dt| = c|\vec{\alpha}| = c\sqrt{(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2} = c\sqrt{1+1+1} = c\sqrt{3} = 1.732c > c. \tag{2c}$$

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The *physically unacceptable nature of the* Eq. (2a) *Dirac* $\widehat{H}_D$ *is further driven home by the fact that*,

$$\left[(d\widehat{\mathbf{r}}/dt),\, \widehat{H}_D\right] = \left[c\vec{\alpha},\, \beta mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}})\right] = 2mc^3\vec{\alpha}\beta + c((c\widehat{\mathbf{p}})\times(\vec{\alpha}\times\vec{\alpha})), \tag{2d}$$

is nonzero, yet independent of $\hbar$, which *flatly violates* the quantum correspondence-principle requirement *that commutators of observables such as* $(d\widehat{\mathbf{r}}/dt)$ *and* $\widehat{H}_D$ *must vanish when* $\hbar \to 0$. The *related* fact that,

$$(d^2\widehat{\mathbf{r}}/dt^2) = (-i/\hbar)\left[(d\widehat{\mathbf{r}}/dt),\, \widehat{H}_D\right] = (-2c/\hbar)\big(imc^2\vec{\alpha}\beta + (\vec{\sigma}\times(c\widehat{\mathbf{p}}))\big)\ \text{(because } \vec{\sigma} = (-i/2)(\vec{\alpha}\times\vec{\alpha})\text{)}, \tag{2e}$$

violates the absence of acceleration of relativistic free particles expressed by Eq. (1c). For a zero-momentum free Dirac electron, Eq. (2e) implies (*a physically nonexistent*) "zitterbewegung" spontaneous acceleration of the mind-boggling order of $10^{28}g$, where $g = 9.8$ m/s^2, the acceleration of gravity at the earth's surface.

Although the natural relativistic square-root free-particle Hamiltonian operator $\widehat{H} = (m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}$ is diagonal in momentum representation, that isn't the case for Dirac's free-particle Hamiltonian operator $\widehat{H}_D = \beta mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}})$ because $\vec{\alpha}$ isn't diagonal. However, Dirac's signature squared-Hamiltonian equality,

$$\big(\widehat{H}_D\big)^2 = \big(\widehat{H}\big)^2 = m^2c^4 + |c\widehat{\mathbf{p}}|^2, \tag{2f}$$

enables construction of a complete set of orthogonal eigenprojectors $P_D^{\pm}$ for $\widehat{H}_D$ that have the properties,

$$P_D^{\pm} \stackrel{\text{def}}{=} \tfrac{1}{2}\big(1 \pm \widehat{H}_D\widehat{H}^{-1}\big),\ \big(P_D^{\pm}\big)^2 = P_D^{\pm},\ P_D^{+}\,P_D^{-} = P_D^{-}\,P_D^{+} = 0,\ \big(P_D^{+} + P_D^{-}\big) = 1,\ \widehat{H}_D\big(P_D^{\pm}\big) = \pm\widehat{H}\big(P_D^{\pm}\big). \tag{2g}$$

The last two properties of the eigenprojectors $P_D^{\pm}$ produce the spectral decomposition of Dirac's $\widehat{H}_D$,

$$\widehat{H}_D = \widehat{H}_D\left(P_D^{+} + P_D^{-}\right) = \widehat{H}\left(P_D^{+}\right) - \widehat{H}\left(P_D^{-}\right), \tag{2h}$$

which reveals that although the spectrum of $\widehat{H}_D$ *starkly differs from the spectrum of* $\widehat{H}$ in that it has *an unphysical negative-energy sector* entirely alien to the spectrum of $\widehat{H}$, the *positive-energy sector* of the spectrum of $\widehat{H}_D$ *exactly matches the full spectrum of* $\widehat{H}$. Thus notwithstanding *the extremely unphysical characteristics of Dirac's free-particle Hamiltonian* $\widehat{H}_D$ *documented by* Eqs. (2b)–(2e) *and* (2h), *a resolutely blinkered focus on the positive-energy sector of the* $\widehat{H}_D$ *spectrum won't encounter those physics flaws*!

We next wish to ascertain the extent to which *a hydrogen-type* Dirac Hamiltonian, namely $(\widehat{H}_D + eA^0)$, can be expected to likewise yield positive-energy sector spectrum results that are physically correct. Just as we used the physically far more trustworthy $\widehat{H}$ to check the extent to which the positive-energy sector of the Dirac $\widehat{H}_D$ spectrum can be expected to yield physically correct results for the free particle, we shall check the use of the positive-energy sector of the Dirac hydrogen-type Hamiltonian $(\widehat{H}_D + eA^0)$ against the nonrelativistic Pauli Hamiltonian specialized to the particle's instantaneous rest frame, to which the four-vector potential $(A^0, \mathbf{0})$ is Lorentz transformed. A technical impediment to implementing such an approach is that the Dirac hydrogen-type Hamiltonian $(\widehat{H}_D + eA^0)$ involves, in addition to scalars, the 4×4 entities β and $\vec{\alpha}$, whereas the particle rest-frame Pauli Hamiltonian instead involves, in addition to scalars, *only* the 2×2 entity $\vec{\sigma}$. However *unitary Foldy-Wouthuysen transformation of* $(\widehat{H}_D + eA^0)$ addresses precisely this issue [1]: all dependence on $\vec{\alpha}$ in that transformation of $(\widehat{H}_D + eA^0)$ is specifically eliminated in favor of dependence on only β and $\vec{\sigma} = (-i/2)(\vec{\alpha}\times\vec{\alpha})$. Moreover, setting β to its $+1$ eigenvalue in that transformation *selects the desired positive-energy sector* (because the Foldy-Wouthuysen transformation is *unitary*, the Dirac Hamiltonian's energy spectrum *isn't altered*). Thus the Foldy-Wouthuysen transformation of the Dirac hydrogen-type Hamiltonian $(\widehat{H}_D + eA^0)$, with β set to $+1$, is to be compared to the nonrelativistic Pauli Hamiltonian specialized to the particle's instantaneous rest frame, to which the four-vector potential $(A^0, \mathbf{0})$ is Lorentz transformed. We shall carry out the program outlined in the foregoing sentence in the next section; we conclude *this* section with the instructive construction of the Foldy-Wouthuysen transformation *of the free-particle Dirac Hamiltonian* $\widehat{H}_D = \beta mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}})$. That transformation is *generated* by the normalized *product* of the pair of anticommuting terms which comprise $\widehat{H}_D$, namely by,

$$\xi \stackrel{\text{def}}{=} \left(\beta\vec{\alpha}\cdot\widehat{\mathbf{p}}/|\widehat{\mathbf{p}}|\right), \tag{3a}$$

which has the key properties of anticommuting with $\widehat{H}_D$ *and being anti-Hermitian*; an additional convenient property of ξ as it is defined above is that its square is equal to -1. Being anti-Hermitian, ξ *generates a family of unitary transformations of* $\widehat{H}_D$, which are parameterized by the angle θ, as follows,

$$\exp(\xi\theta/2)\widehat{H}_D\exp(-\xi\theta/2) = \exp(\xi\theta/2)\exp(\xi\theta/2)\widehat{H}_D = \exp(\xi\theta)\widehat{H}_D = (\cos\theta + \xi\sin\theta)\widehat{H}_D, \tag{3b}$$

where the first equality reflects the fact that $\widehat{H}_D$ *anticommutes with* ξ, and the third equality reflects the fact that the square of ξ is equal to -1. Inserting *the definitions of* ξ *and* $\widehat{H}_D$ into $(\cos\theta + \xi\sin\theta)\widehat{H}_D$ yields,

$$(\cos\theta + \xi\sin\theta)\widehat{H}_D = (\cos\theta + (\beta\vec{\alpha}\cdot\widehat{\mathbf{p}}/|\widehat{\mathbf{p}}|)\sin\theta)\left(\beta mc^2 + \vec{\alpha}\cdot(c\widehat{\mathbf{p}})\right) =$$
$$c\left(|\widehat{\mathbf{p}}|\cos\theta - mc\sin\theta\right)(\vec{\alpha}\cdot\widehat{\mathbf{p}}/|\widehat{\mathbf{p}}|) + c\beta\left(mc\cos\theta + |\widehat{\mathbf{p}}|\sin\theta\right). \tag{3c}$$

For the last expression of Eq. (3c) *to be the Foldy-Wouthuysen transformation of* $\widehat{H}_D$, the angle parameter θ must of course *be chosen such that the coefficient of* $(\vec{\alpha}\cdot\widehat{\mathbf{p}}/|\widehat{\mathbf{p}}|)$ *vanishes*. That is case for,

$$\theta = \arctan(|\widehat{\mathbf{p}}/(mc)|) \;\Rightarrow\; \cos\theta = \left(1 + |\widehat{\mathbf{p}}/(mc)|^2\right)^{-\frac{1}{2}} \text{ and } \sin\theta = |\widehat{\mathbf{p}}/(mc)|\left(1 + |\widehat{\mathbf{p}}/(mc)|^2\right)^{-\frac{1}{2}}, \tag{3d}$$

which when inserted into Eq. (3c) reveals the Foldy-Wouthuysen transformation of $\widehat{H}_D$ to be,

$$\beta\left(mc^2 + \left(|\widehat{\mathbf{p}}|^2/m\right)\right)\left(1 + |\widehat{\mathbf{p}}/(mc)|^2\right)^{-\frac{1}{2}} = \beta\left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}. \tag{3e}$$

The Eq. (3e) Foldy-Wouthuysen transformation of $\widehat{H}_D$ has the simple eigenprojector spectral decomposition,

$$\beta\left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}} = \left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}((1+\beta)/2) - \left(m^2c^4 + |c\widehat{\mathbf{p}}|^2\right)^{\frac{1}{2}}((1-\beta)/2), \tag{3f}$$

whose *positive-energy sector* is of course *selected* by setting β to its $+1$ eigenvalue.

Relativistic Pauli versus Dirac: quantum consequences of hydrogen potentials

In a single particle's instantaneous rest frame, *its nonrelativistic description exactly coincides with its relativistic description*. Moreover, *the action functional of a relativistic particle is Lorentz invariant*. Those two facts in principle enable the nonrelativistic Pauli Hamiltonian for a single spin-1/2 particle in an electromagnetic field *to be extended to the correct fully-relativistic Hamiltonian for that particle in that field*. A Hamiltonian *in closed form* isn't produced *when an external magnetic field is present*, but *a successive approximation scheme* is apparently satisfactory. This *physically impeccable route* which in principle *exists between the nonrelativistic Pauli Hamiltonian and its correct relativistic extension* eliminates the *guesswork* which entered into the creation of the Dirac Hamiltonian; that guesswork *produced the gross violations of physical principles pointed out in* Eqs. (2b)–(2e) *and* (2h).

We begin with the nonrelativistic Pauli Hamiltonian, *including*, for the purpose *of its upcoming relativistic extension*, a particle rest-mass term mc^2 which, *being constant*, affects neither the classical Hamiltonian equations of motion nor their quantum Heisenberg counterparts,

$$H = mc^2 + \left(|\mathbf{P} - (e/c)\mathbf{A}|^2/(2m)\right) + eA^0 - (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}). \tag{4a}$$

To obtain the nonrelativistic action S_{nr} which corresponds to this nonrelativistic Hamiltonian H, we need the Lagrangian L corresponding to H. The dependence of H on particle canonical momentum $\mathbf{P}$ is swapped in L for dependence on particle velocity $\dot{\mathbf{r}}$, which we obtain from the Heisenberg equation of motion,

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r}, H] = \nabla_{\mathbf{P}}H = (\mathbf{P} - (e/c)\mathbf{A})/m. \tag{4b}$$

We must now *invert* the Eq. (4b) relation between $\dot{\mathbf{r}}$ and $\mathbf{P}$, thereby obtaining,

$$\mathbf{P} = m\dot{\mathbf{r}} + (e/c)\mathbf{A}, \tag{4c}$$

which we *insert* into the well-known relation of the Lagrangian L to the Hamiltonian H,

$$L = \dot{\mathbf{r}}\cdot\mathbf{P} - H\Big|_{\mathbf{P}=m\dot{\mathbf{r}}+(e/c)\mathbf{A}} = -mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(A^0 - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}). \tag{4d}$$

This nonrelativistic Lagrangian L immediately yields the nonrelativistic action S_{nr},

$$S_{\mathrm{nr}} = \int L dt = \int \left[-mc^2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - e(A^0 - (\dot{\mathbf{r}}/c)\cdot\mathbf{A}) + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B})\right] dt, \tag{4e}$$

which we now *specialize* to the particle's *instantaneous rest frame where its velocity* $\dot{\mathbf{r}} = \mathbf{0}$,

$$S = \int \left[-mc^2 - e(A')^0 + (e\hbar/(2mc))(\vec{\sigma}\cdot\mathbf{B}')\right] dt'. \tag{4f}$$

Taking the particle to be an electron, we now *furthermore* suppose the existence of *a proton*, which in its *own* rest frame *produces the hydrogen four-potential* $A^\mu = (A^0, \mathbf{0})$. If in that proton's rest frame, the electron's instantaneous velocity is $\dot{\mathbf{r}}$, then *in the electron's instantaneous rest frame* this hydrogen four-potential $A^\mu = (A^0, \mathbf{0})$ *is Lorentz transformed to*,

$$(A')^\mu = \gamma(|\dot{\mathbf{r}}/c|)\left(A^0, -(\dot{\mathbf{r}}/c)A^0\right), \text{ where } \gamma(|\dot{\mathbf{r}}/c|) \stackrel{\text{def}}{=} \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{-\frac{1}{2}}, \tag{4g}$$

and consequently, *in the electron's instantaneous rest frame*,

$$(A')^0 = \gamma(|\dot{\mathbf{r}}/c|)A^0 \text{ and } \mathbf{B}' = \nabla_{\mathbf{r}} \times \left[\gamma(|\dot{\mathbf{r}}/c|)\left(-(\dot{\mathbf{r}}/c)A^0\right)\right] = \gamma(|\dot{\mathbf{r}}/c|)\left(\mathbf{E} \times (\dot{\mathbf{r}}/c)\right), \tag{4h}$$

where $\mathbf{E} = -\nabla_{\mathbf{r}}A^0$. The Eq. (4h) effective magnetic field $\mathbf{B}' = \gamma(|\dot{\mathbf{r}}/c|)(\mathbf{E} \times (\dot{\mathbf{r}}/c))$ in the electron's instantaneous rest frame will cause its spin $(\hbar/2)\vec{\sigma}$ to precess in consonance with the presence of the energy term $(e\hbar/(2mc))(\vec{\sigma} \cdot \mathbf{B}')$ in the integrand of the Eq. (4f) instantaneous rest frame action functional. However, in case the electron *is as well undergoing acceleration* $\ddot{\mathbf{r}}$ *such that* $(\ddot{\mathbf{r}} \times \dot{\mathbf{r}}) \neq \mathbf{0}$, this analysis of the relativistic physics *is incomplete*: in that case the transformation between the coordinate systems *in addition entails a rotation of their coordinate axes relative to each other*—successive Lorentz boosts *in different directions* don't resolve into *only* a net Lorentz boost; a relative rotation of the coordinate axes of the two systems *always occurs in addition*. The *effect* of such a coordinate axis rotation often tends *to partially cancel out* the spin precession caused by a magnetic field $\mathbf{B}'$ which is induced by a particle's velocity $\dot{\mathbf{r}}$ through a longitudinal electric field $\mathbf{E} = -\nabla_{\mathbf{r}}A^0$, such as the magnetic field $\mathbf{B}' = \gamma(|\dot{\mathbf{r}}/c|)(\mathbf{E} \times (\dot{\mathbf{r}}/c))$ described by Eq. (4h). This phenomenon is especially pronounced for particles with spin *traveling in circles*, in which case *their centripetal acceleration is orthogonal to their velocity*, the situation that is *the most favorable* to relativistic generation of relative coordinate axis rotation via successive Lorentz boosts *in different directions*.

For an electron circling the proton at a speed much less than c in a bound state, "the Thomas half" rule of thumb for this Thomas precession phenomenon is that relativistic relative coordinate axis rotation *halves the spin precession effect produced by the* $\mathbf{B}'$ *of* Eq. (4h) inserted into the spin energy term $(e\hbar/(2mc))(\vec{\sigma} \cdot \mathbf{B}')$ of the integrand of Eq. (4f). However for an electron which is *not* in a bound state circling the proton, but is merely *being slightly deflected* (slightly elastically scattered) by the proton's longitudinal electric field $\mathbf{E} = -\nabla_{\mathbf{r}}A^0$, one would expect *negligible deviation* from the spin precession effect given by the $\mathbf{B}'$ of Eq. (4h) inserted into the spin energy term $(e\hbar/(2mc))(\vec{\sigma} \cdot \mathbf{B}')$ of the integrand of Eq. (4f).

Let us now work out the relativistic Lagrangian and Hamiltonian which follow from simply inserting the $(A')^0$ and $\mathbf{B}'$ of Eq. (4h) into the integrand of Eq. (4f), *bearing in mind* that this *ignores* the Thomas-precession consequence of particle acceleration not being parallel to particle velocity, and requires *correction* of its particle spin precession *prediction* which *ranges* from *negligible* for small-angle scattering to *"the Thomas half" rule of thumb* for bound states. The insertion of the $(A')^0$ and $\mathbf{B}'$ of Eq. (4h) into the integrand of Eq. (4f) yields,

$$S = \int \left[-mc^2 - \gamma(|\dot{\mathbf{r}}/c|)eA^0 + \gamma(|\dot{\mathbf{r}}/c|)(e\hbar/(2mc))(\vec{\sigma} \cdot (\mathbf{E} \times (\dot{\mathbf{r}}/c)))\right] dt', \tag{4i}$$

Since dt' in Eq. (4i) refers to time recorded by a clock traveling with the instantaneous particle rest frame, in terms of clock time dt recorded in the proton rest frame, $dt' = (1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} dt = (1/\gamma(|\dot{\mathbf{r}}/c|))dt$. Thus *the Lorentz-invariant relativistic action in terms of physical entities measured in the proton rest frame is*,

$$S_{\text{rel}} = \int \left[-mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - eA^0 + (e\hbar/(2mc))((\vec{\sigma} \times \mathbf{E}) \cdot (\dot{\mathbf{r}}/c))\right] dt, \tag{4j}$$

where we have *also interchanged* the "dot" $\cdot$ and "cross" $\times$ which occur in the last term of the integrand of Eq. (4i). Eq. (4j) immediately yields the corresponding relativistic Lagrangian,

$$L_{\text{rel}} = -mc^2(1 - |\dot{\mathbf{r}}/c|^2)^{\frac{1}{2}} - eA^0 + (e\hbar/(2mc))((\vec{\sigma} \times \mathbf{E}) \cdot (\dot{\mathbf{r}}/c)), \tag{4k}$$

which implies the canonical momentum,

$$\mathbf{P} = \nabla_{\dot{\mathbf{r}}} L_{\text{rel}} = m\dot{\mathbf{r}}(1 - |\dot{\mathbf{r}}/c|^2)^{-\frac{1}{2}} + (e\hbar/(2mc^2))(\vec{\sigma} \times \mathbf{E}), \tag{4l}$$

It is convenient to define kinetic momentum $\mathbf{p}$ in terms of canonical momentum $\mathbf{P}$ as,

$$\mathbf{p} \stackrel{\text{def}}{=} \left(\mathbf{P} - (e\hbar/(2mc^2))(\vec{\sigma} \times \mathbf{E})\right), \tag{4m}$$

which permits us to compactly *invert* Eq. (4l),

$$\dot{\mathbf{r}} = (\mathbf{p}/m)\left(1 + |\mathbf{p}/(mc)|^2\right)^{-\frac{1}{2}}. \tag{4n}$$

With this and the aid of Eqs. (4k) and (4m), we obtain H_{rel} from L_{rel} via their standard relationship,

$$H_{\rm rel} = \dot{\mathbf{r}}\cdot\mathbf{P} - L_{\rm rel}\Big|_{\dot{\mathbf{r}}=(\mathbf{p}/m)(1+|\mathbf{p}/(mc)|^2)^{-\frac{1}{2}}} =$$
$$\left(m^2c^4 + |c\mathbf{p}|^2\right)^{\frac{1}{2}} + eA^0 = \left(m^2c^4 + |c\mathbf{P} - (e\hbar/(2mc))(\vec{\sigma}\times\mathbf{E})|^2\right)^{\frac{1}{2}} + eA^0. \tag{4o}$$

The $H_{\rm rel}$ of Eq. (4o) is appropriate for small angle scattering, but for bound states it needs to be brought into line with "the Thomas half" rule of thumb for spin precession by its modification to,

$$H_{\rm rel} = \left(m^2c^4 + |c\mathbf{P} - (e\hbar/(4mc))(\vec{\sigma}\times\mathbf{E})|^2\right)^{\frac{1}{2}} + eA^0. \tag{4p}$$

The Foldy-Wouthuysen transformation of the Dirac Hamiltonian ($\widehat{H}_D + eA^0$) with β set equal to $+1$ agrees with the $H_{\rm rel}$ of Eq. (4p) above, *except* for having an *additional* "Darwin term" [1] which is proportional to $-e(\hbar/(mc))^2(\nabla_{\mathbf{r}}\cdot\mathbf{E})$, and is therefore, by Coulomb's Law, proportional to $-e(\hbar/(mc))^2\rho$, where ρ is the proton's charge density. A term of such short range (the proton's charge density is around 50,000 times smaller than the hydrogen atom's Bohr radius) won't normally have a discernible effect on the hydrogen atomic physics, but *for experiments whose purpose is to determine the proton's charge radius via effects of the proton's charge density on the hydrogen atomic physics* [2], this "Darwin term" cannot be neglected. Although *there is no trace whatsoever of such a* "Darwin term" in the relativistically-extended Pauli physics treated above, in Dirac theory its existence is convincingly attributed to *averaged* smearing of the electron's potential energy term $eA^0(\mathbf{r})$ by the electron's "zitterbewegung" motion, which has a mean displacement $\langle\delta\mathbf{r}\rangle = \mathbf{0}$ with an isotropic root-mean-square magnitude $\langle|\delta\mathbf{r}|^2\rangle^{\frac{1}{2}} \approx (\hbar/(mc))$, yielding that [1],

$$\begin{gathered}\left\langle eA^0(\mathbf{r}+\delta\mathbf{r}) - eA^0(\mathbf{r})\right\rangle \approx e\left\langle\delta\mathbf{r}\cdot\left(\nabla_{\mathbf{r}}A^0(\mathbf{r})\right) + (1/2)\textstyle\sum_{i,j=1}^3 \delta r_i\,\delta r_j\left(\partial^2 A^0(\mathbf{r})/(\partial r_i\partial r_j)\right)\right\rangle \approx \\ e(1/6)\left\langle|\delta\mathbf{r}|^2\right\rangle\left(\nabla_{\mathbf{r}}^2 A^0(\mathbf{r})\right) \approx -e(1/6)(\hbar/(mc))^2\left(\nabla_{\mathbf{r}}\cdot\mathbf{E}\right).\end{gathered} \tag{5}$$

But *not only is there no trace whatsoever of such a* "Darwin term" in the relativistically-extended Pauli physics, we have seen in Eqs. (2d) and (2e) that Dirac's postulated anticommutation relations produce an egregious *violation of the correspondence principle of quantum mechanics*, and that this violation of the quantum mechanics correspondence principle *directly spawns the free-particle spontaneous acceleration "zitterbewegung" phenomenon*, which furthermore egregiously *violates the absence of spontaneous acceleration of relativistic free particles*. It is clear that the "zitterbewegung" phenomenon of the Dirac Hamiltonian *doesn't occur in actual physics*, and consequently *neither does the "Darwin term"* of the positive-energy sector of the Foldy-Wouthuysen transformed Dirac hydrogen-type Hamiltonian ($\widehat{H}_D + eA^0$). Compensating for an atomic proton's *physically absent* "Darwin term" charge smearing would result in a misleadingly contracted impression of its charge radius.

References

[1] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Eqs. (4.5)–(4.7), pp. 51–52.

[2] S. Schmidt et al., "The next generation of laser spectroscopy experiments using light muonic atoms", arXiv:1808.07240 [physics.atom-ph] (2018).