

# **Proof of the Riemann Hypothesis**

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## **Introduction**

The Riemann Hypothesis is one of the most important unresolved problems in Number Theory, it was first proposed by Bernhard Riemann, in 1859. For 160 years mathematicians have struggled with this problem to no avail. The difficulty of the Riemann Hypothesis is the main reason the hypothesis has remained unsolved. Although the Riemann Hypothesis remains unsolved, several mathematicians have proven other problems are the equivalent of the Riemann Hypothesis. In other words, if any of these equivalent criteria were solved, it would also solve the Riemann Hypothesis. Of particular interest to the author is a very elementary equivalent to the Riemann Hypothesis, Lagarias's Elementary Version of the Riemann Hypothesis.

In 2002, Jeffrey Lagarias proved that his problem is equivalent to the Riemann Hypothesis, a famous question about the complex roots of the Riemann zeta function. The beauty of the Lagarias's Elementary Version of the Riemann Hypothesis, is that it is truly an elementary and very simple problem compared to the Riemann Hypothesis. The simplicity of Lagarias's proof is what attracted the author to attempt to solve the Riemann Hypothesis. The author was very surprised at the simple proof he formulated using the elementary work of Lagarias. The moral of this story is that many times elementary or simple proofs exist to complex mathematical problems, this is one of those cases.

## **Abstract**

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zero's only at the negative even integers and complex numbers with real part  $\frac{1}{2}$

The Riemann hypothesis implies results about the distribution of prime numbers. Along with suitable generalizations, some mathematicians consider it the most important unresolved problem in pure mathematics (Bombieri 2000).

It was proposed by Bernhard Riemann (1859), after whom it is named. The name is also used for some closely related analogues, such as the Riemann hypothesis for curves over finite fields.

The Riemann hypothesis implies significant results about the distribution of prime numbers. Along with suitable generalizations, some mathematicians consider it the most important unresolved problem in pure mathematics (Bombieri 2000). The Riemann zeta function is defined for complex  $s$  with real part greater than 1 by the absolutely convergent infinite series:

$$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \dots$$

The Riemann hypothesis asserts that all *interesting* solutions of the equation:

$$\zeta(s) = 0$$

lie on a certain vertical straight line.

In mathematics, the  $n$ -th harmonic number is the sum of the reciprocals of the first  $n$  natural numbers:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} = \sum_{n=1}^n \frac{1}{n}$$

Harmonic numbers have been studied since early times and are important in various branches of number theory. They are sometimes loosely termed harmonic series, are closely related to the Riemann zeta function.

The harmonic numbers roughly approximate the natural logarithm function and thus the associated harmonic series grows without limit, albeit slowly. In 1737, Leonhard Euler used the divergence of the harmonic series to provide a new proof of the infinity of prime numbers. His work was extended into the complex plane by Bernhard Riemann in 1859, leading directly to the celebrated Riemann hypothesis about the distribution of prime numbers.

## Proof

In 2002, Jeffrey Lagarias proved that this problem is equivalent to the Riemann Hypothesis, a famous question about the complex roots of the Riemann zeta function. The Lagarias's Elementary Version of the Riemann Hypothesis states that for a positive integer  $n$ , let  $\sigma(n)$

denote the sum of the positive integers that divide  $n$ . For example,  $\sigma(4)=1+2+4=7$ , and  $\sigma(6)=1+2+3+6=12$ . Let  $H_n$  denote the  $n$ -th harmonic number, for example:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

The unsolved question is does the following inequality hold for all  $n \geq 1$ ?

$$\sigma(n) \leq H_n + \ln(H_n)e^{H_n}$$

First we will solve for the smallest numbers:

$$\sigma(1) \leq H_1 + \ln(H_1)e^{H_1}$$

$$\sigma(1) = 1 \text{ and } H_1 = 1, \text{ therefore,}$$

$$\sigma(1) = H_1 = 1, \text{ which satisfies our inequality.}$$

$$\sigma(2) \leq H_2 + \ln(H_2)e^{H_2}$$

$$\sigma(2) = 3 \text{ and } H_2 = 1.5, \text{ therefore,}$$

$$\sigma(2) = 3 \leq 1.5 + \ln(1.5)e^{1.5}$$

$$3 \leq 3.31716$$

$$\sigma(3) \leq H_3 + \ln(H_3)e^{H_3}$$

$$\sigma(3) = 4 \text{ and } H_3 = 1.833333333, \text{ therefore,}$$

$$\sigma(3) = 4 \leq 1.833333333 + \ln(1.833333333)e^{1.833333333}$$

$$3 \leq 5.62453$$

$$\sigma(4) \leq H_4 + \ln(H_4)e^{H_4}$$

$$\sigma(4) = 7 \text{ and } H_4 = 2.083333, \text{ therefore,}$$

$$\sigma(4) = 7 \leq 2.083333 + \ln(2.083333)e^{2.083333}$$

$$7 \leq 7.97798$$

$$\sigma(5) \leq H_5 + \ln(H_5)e^{H_5}$$

$$\sigma(5) = 6 \text{ and } H_5 = 2.283333, \text{ therefore,}$$

$$\sigma(5) = 6 \leq 2.283333 + \ln(2.283333)e^{2.283333}$$

$$6 \leq 10.38226$$

In 1984, G Robin also proved, unconditionally, that the inequality below is true (see Proposition 1 of Section 4 of Robin's work in reference 2):

$$\sigma(n) \leq (e^\gamma)n(\log \log(n)) + \frac{0.6483n}{\log \log(n)}$$

holds for all  $n \geq 3$ . Above, we have already shown the first five  $\sigma(n) \leq H_n + \ln(H_n)e^{H_n}$ , therefore, we do not need to address when  $n < 3$  in the following proof.

The numerical value of the Euler–Mascheroni constant,  $\gamma$ , is:

$$\gamma = 0.57721566490153286060651209008240243104215933593992$$

We will prove that:

$$(e^\gamma)n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq H_n + \ln(H_n)e^{H_n}$$

$$(e^{0.57721})n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq H_n + \ln(H_n)e^{H_n}$$

$$1.781n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq H_n + \ln(H_n)e^{H_n}$$

To be conservative, we will choose  $H_n \geq e = 2.718282$ , this is to ensure that  $\ln(H_n) \geq 1$ , so that we define the region where  $\ln(H_n)e^{H_n} \geq e^{H_n}$ .

For  $H_n \geq e = 2.718282$ , we will use  $H_n = H_9 = 2.828968 \geq e$  then  $\ln(H_n) \geq \ln(e) = 1$ , which means,  $\ln(H_n) \geq 1$ , for  $n \geq 9$ .

Therefore, for  $H_n \geq e = 2.718282$

$$H_n + \ln(H_n)e^{H_n} \geq H_n + \ln(e)e^{H_n} = H_n + e^{H_n} \geq e^{H_n}$$

$$\text{Thus for } H_n \geq e, \text{ then } H_n + \ln(H_n)e^{H_n} \geq e^{H_n}$$

Therefore, it will suffice for us to prove that:

Equation 1. 
$$1.781n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq e^{H_n}$$

Thus, since we have demonstrated our inequality is true for  $n \leq 5$ , later we will need to show it is true for  $5 \leq n \leq 9$ .

Notice that  $\log \log(n)$  grows extremely slowly,  $\log \log(1,000,000,000) = 0.954$

Notice,  $H_n$  grows relatively slowly in the exponent value for  $e^{H_n}$ , for example,  $H_4 = 2.083333$

Therefore,  $e^{H_n} = e^{H_4} = e^{2.083333} = 8.031195$ , which is large compared to 0.954

As  $n$  increases from just 4 to 5,  $H_n$  continues to slowly grow in the exponent value for  $e^{H_n}$ , for example,  $H_5 = 2.283333$

Therefore,  $e^{H_n} = e^{H_5} = e^{2.283333333} = 9.809324$ , which is ten times larger than 0.954

However, to put the difference between the numbers in a better perspective we must recall that from above,  $0.954 = \log \log(1,000,000,000)$ . Therefore, a much better comparison of the growth rate of  $\log \log(n)$  compared to  $e^{H_n}$  is to compare  $\log \log(1,000,000,000)$  to  $e^{H_{1,000,000,000}}$ .

First, to calculate  $H_{1,000,000,000}$ , we will use the approximation formula, as follows:

$$H_n \approx \ln(n) + \gamma + (1/2n) - (1/12n^2)$$

$$H_{1,000,000,000} \approx 21.30048$$

Therefore, for  $n = 1,000,000,000$  then,

$$e^{H_n} = e^{21.30048} = 1,781,072,419$$

To calculate the actual growth rate comparison, we need to calculate the following ratio:

$$\frac{e^{H_n}}{\log \log(n)} = \frac{1,781,072,419}{0.954} = 1,866,952,221$$

Therefore,  $e^{H_n}$  has grown 1,866,952,221 as fast as  $\log \log(n)$ . Therefore, although the harmonic series  $H_n$  grows slowly, still  $e^{H_n}$  grows extremely fast as compared to  $\log \log(n)$ .

Although we have shown the difference in the growth rate of  $e^{H_n}$  compared to  $\log \log(n)$ , we still need to compare the following inequality for  $n = 1,000,000,000$ ., we do this below:

$$1.781n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq H_n + \ln(H_n)e^{H_n}$$

$$1.781(1,000,000,000)(0.954) + \frac{0.6483(1,000,000,000)}{0.954} \leq 21.30048 + (3.0587)e^{H_n}$$

$$1,699,074,000 + 679,559,748.428 \leq 21.30048 + (3.0587)(1,781,072,419)$$

$$2,378,633,748.428 \leq 5,447,766,229.296$$

Thus, we have shown that for,  $n = 1,000,000,000$  that our inequality is true:

$$1.781n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq H_n + \ln(H_n)e^{H_n}$$

We will take this a step further and we will calculate  $H_{1,000,000,000,001}$ , using the approximation formula, as follows:

$$H_n \approx \ln(n) + \gamma + (1/2n) - (1/12n^2), \text{ and } \gamma \approx 0.57721$$

$$H_{1,000,000,000,001} \approx 28.2082$$

Therefore, for  $n = 1,000,000,000,001$  then,

$$e^{H_n} = e^{28.2082} = 1,781,010,000,000 > 1,781,072,419 \text{ (for } n = 1,000,000,000)$$

This indicates the growth rate for  $e^{H_n}$ , for  $1,000,000,000 \geq n \leq 1,000,000,000,001$  had a growth rate of approximately 1,000 for  $e^{H_n}$ .

Now, for all  $k \geq 9$ , we will prove the above inequality using Mathematical Induction. Normally the base value of 1 is used as the baseline for Mathematical Induction, however, since 9 is the baseline for our inequality, 9 will be used as the baseline as follows:

$$1.781(9)(\log \log(9)) + \frac{0.6483(9)}{\log \log(9)} \leq H_9 + \ln(H_9)e^{H_9}$$

$$48.087(-0.0203) + \frac{5.8347}{-0.0203} \leq 2.828968 + \ln(2.828968)e^{2.828968}$$

$$48.087(-0.0203) + \frac{5.8347}{-0.0203} \leq 2.828968 + (1.0399)(16.928)$$

$$-288.3998 \leq 20.4323$$

Thus, our inequality for the baseline  $k = 9$ , is true.

To complete our Mathematical induction we will return to equation 1, on page 5, where we showed, it will suffice for us to prove that:

$$1.781n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq e^{H_n}$$

Now we will repeat our baseline for  $k = 9$ , for the above equation, as follows:

$$1.781(9)(\log \log(9)) + \frac{0.6483(9)}{\log \log(9)} \leq e^{H_9}$$

$$48.087(-0.0203) + \frac{5.8347}{-0.0203} \leq e^{2.828968}$$

$$48.087(-0.0203) + \frac{5.8347}{-0.0203} \leq 1.0399$$

$$-288.3998 \leq 1.0399$$

Thus, our inequality for the baseline  $k = 9$ , is true.

Next we must assume our inequality is true for all  $k \geq 9$ . Then, we must suppose it holds for any number  $k \geq 9$ . This supposition is known as the induction hypothesis. We assume it is true, and aim to show that our assumption is true:

$$1.781k(\log \log(k)) + \frac{0.6483k}{\log \log(k)} \leq e^{H_k}$$

Using the above induction hypothesis, we must show it is true for  $k + 1$ , that is:

$$1.781(k + 1)(\log \log(k + 1)) + \frac{0.6483(k+1)}{\log \log(k+1)} \leq e^{H_{k+1}}$$

$$1.781(k + 1)(\log \log(k + 1)) + \frac{0.6483(k+1)}{\log \log(k+1)} \leq e^{H_{k+1}} = e^{1/(1+k)} e^{H_k}$$

$$(1.781(k + 1)(\log \log(k + 1)) + \frac{0.6483(k+1)}{\log \log(k+1)})/ e^{1/(1+k)} \leq e^{H_k}$$



To complete our proof of Mathematical Induction, we prove our induction hypothesis for  $k + 1$  by graphing the above inequality. In Figure 1, below we graph  $e^{H_k}$  showing it is  $\geq$  the left hand side of the above inequality, for  $n \geq 9$ . Figure 1, shows  $e^{H_k}$  (the orange line has a much steeper vertical slope than the left hand side of the above inequality (the blue line), therefore,  $e^{H_k}$ , will grow much faster and continue to grow faster as  $n$  approaches  $\infty$ .

This gives us the opportunity to return to our earlier inequalities.

$$\sigma(n) < e^{\gamma} (\log \log(n)) + \frac{0.6483n}{\log \log(n)}, \text{ for } n \geq 3$$

Since, in Figure 1, using induction we have proven,  $1.781n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq e^{H_n}$

Then it follows that,  $\sigma(n) \leq e^{H_n}$

And,  $\sigma(n) \leq e^{H_n} \leq H_n + \ln(H_n)e^{H_n}$ , for  $n \geq 3$

Therefore, we have proven that:  $\sigma(n) \leq H_n + \ln(H_n)e^{H_n}$ , for all  $n \geq 1$ , since we solved for  $n \geq 5$ , we just need to also show it is true for  $6 \leq n \leq 9$ , which we show below.

$$\sigma(6) \leq H_6 + \ln(H_6)e^{H_6}$$

$\sigma(6) = 12$  and  $H_6 = 2.45$ , therefore,

$$\sigma(6) = 12 \leq 2.45 + \ln(2.45)e^{2.45}$$

$$12 \leq 2.45 + (0.8961)(11.5883)$$

$$12 \leq 12.8343$$

$$\sigma(7) \leq H_7 + \ln(H_7)e^{H_7}$$

$\sigma(7) = 8$  and  $H_7 = 2.5929$ , therefore,

$$\sigma(7) = 8 \leq 2.5929 + \ln(2.5929)e^{2.5929}$$

$$8 \leq 2.5929 + (0.9528)(13.3679)$$

$$8 \leq 15.3298$$

$$\begin{aligned}\sigma(8) &\leq H_8 + \ln(H_8)e^{H_8} \\ \sigma(8) &= 15 \text{ and } H_8 = 2.7178, \text{ therefore,} \\ \sigma(8) = 15 &\leq 2.7178 + \ln(2.7178)e^{2.7178} \\ 15 &\leq 2.7178 + (0.9998)(15.1478) \\ 15 &\leq 17.8626\end{aligned}$$

$$\begin{aligned}\sigma(9) &\leq H_9 + \ln(H_9)e^{H_9} \\ \sigma(9) &= 13 \text{ and } H_9 = 2.8289, \text{ therefore,} \\ \sigma(8) = 13 &\leq 2.8289 + \ln(2.8289)e^{2.8289} \\ 13 &\leq 2.8289 + (1.0399)(16.9279) \\ 13 &\leq 20.4322\end{aligned}$$

Thus, we have proven that:  $\sigma(n) \leq H_n + \ln(H_n)e^{H_n}$ , for  $6 \leq n \leq 9$ , this completes our proof.

Thus, we have proven Lagarias's Elementary Version of the Riemann Hypothesis. And since its proof is equivalent to Riemann Hypothesis, more importantly, we have also proven the Riemann Hypothesis as  $n$  approaches infinity.

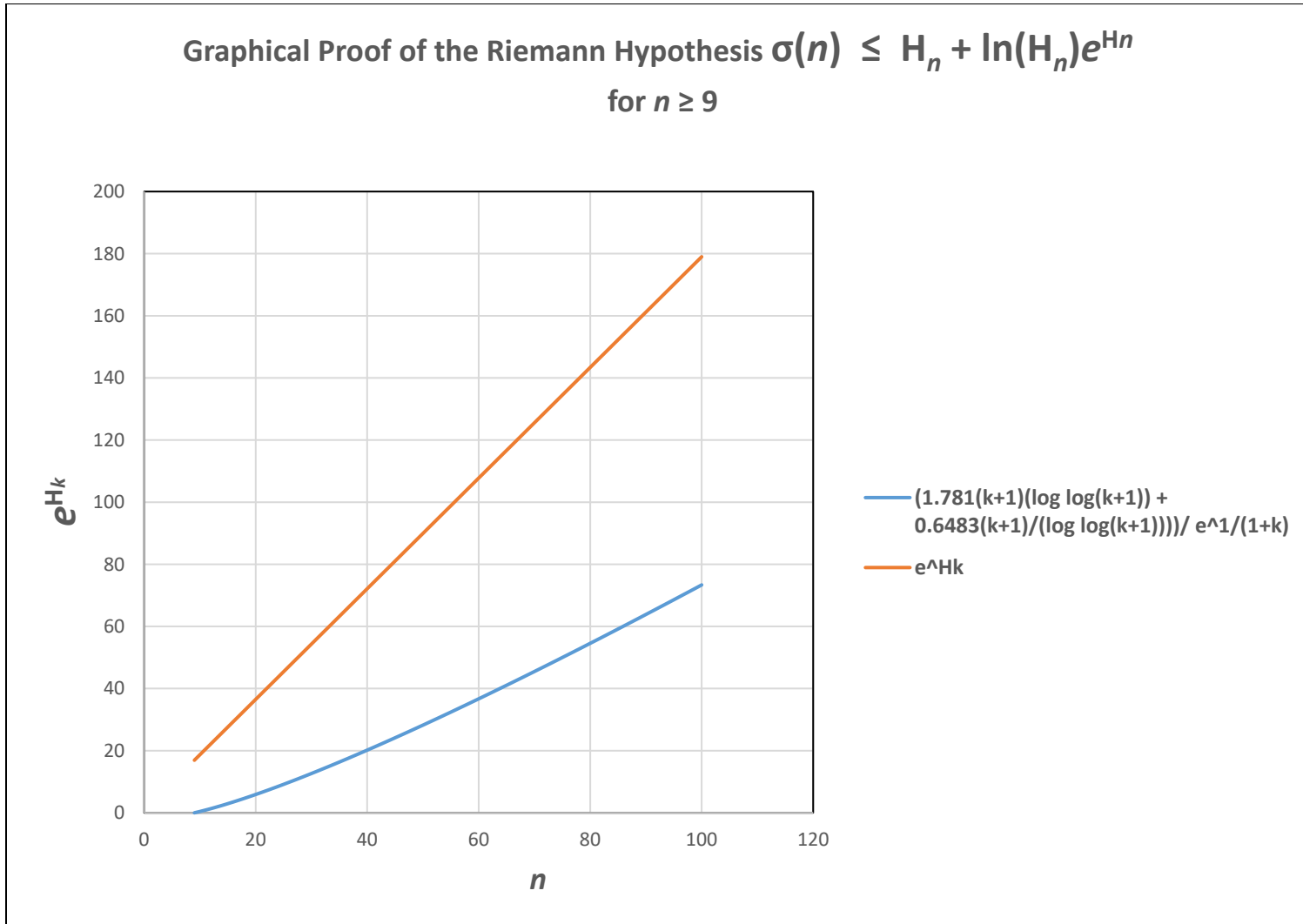
We now turn our focus towards proving this growth rate in general, for all  $n$ . Returning again to:

$$1.781n(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq e^{H_n}$$

All power functions, exponential functions, and logarithmic functions tend to  $\infty$  as  $x \Rightarrow \infty$ . But these three classes of functions tend to  $\infty$  at different rates. The main result we want to focus on is the following one;  $e^x$  grows faster than any power function while  $\log x$  grows slower than any power function (see reference 4). Notice the right hand side of our above inequality is an exponential function,  $e^{H_n}$ , which grows faster than any power function. The left side of our inequality is a combination of logarithmic functions and linear functions, and logarithmic functions grows slower than any power function. The linear functions on the left side are very small multiples of  $n$ , but they still grow much faster than  $\log \log(n)$ , however it grows extremely slow compared to  $e^{H_n}$ . Reference 4, provides proof of  $e^x$  growing the fastest and  $\log x$  growing the slowest, this provides proof of the above inequality for all  $n \geq 1$ .

Additionally, we know that  $H_n$  grows to infinity (which causes  $e^{H_n}$  to grow to  $e^\infty$ ).

**Figure 1. Graphical Proof of the Riemann Hypothesis**



The only question left is does our proof hold when  $n$  goes to infinity? In set theory, there are multiple infinities. The cardinality or "size" of the set of real numbers is an infinite cardinal number and is denoted by  $|\mathbb{R}|$ . The real numbers  $\mathbb{R}$  are more numerous than the natural numbers  $\mathbb{N}$ . Moreover,  $\mathbb{R}$  has the same number of elements as the power set of  $\mathbb{N}$ . Symbolically, if the cardinality of  $\mathbb{N}$  is, denoted as  $\aleph_0$ , then the following inequality holds:

$$\mathbb{R} = 2^{\aleph_0} > \aleph_0$$

This was proven by Georg Cantor in his 1874 uncountability proof, part of his groundbreaking study of different infinities; the inequality was later stated more simply in his diagonal argument. Cantor defined cardinality in terms of bijective functions: two sets have the same cardinality if and only if there exists a bijective function between them (one-to-one correspondence between the sets).

To prove that when  $n$  goes toward infinity the left side of the above inequality below will always be greater than the right side of the inequality. The smallest infinity is the "countable" infinity,  $\aleph_0$  that matches the number of integers. From the above inequality, the mathematical formula below holds for  $\aleph_0$  (reference 5).

$$2^{\aleph_0} > \aleph_0$$

Since  $n$  is a countable natural number then the left side of the below inequality can no greater than  $\aleph_0$

However, the right side of the inequality is,  $e^{H_n}$ , and we already know  $H_n$  is countable infinity, so  $H_n = \aleph_0$

$$\sigma(n) < 1.781(\log \log(n)) + \frac{0.6483n}{\log \log(n)} \leq e^{H_n}$$

Therefore, since  $e = 2.718281828$ , we can state the following:

$$2.7182818^{\aleph_0} \geq 2^{\aleph_0} > \aleph_0$$

Note that when we state above that  $e$  raised to  $\aleph_0$  is  $\geq$  to  $2$  raised to  $\aleph_0$  we actually mean that that they are at least equal to each other, we are not so bold to claim one of these infinities is greater than the other.

This proves that infinity for  $\sigma(\aleph_0) < e^{\aleph_0}$

Therefore the proof for  $\sigma(n) \leq H_n + \ln(H_n)e^{H_n}$  has been proven for all  $n \geq 1$ . In other words the growth rate of  $\sigma(n)$  can't reach that of  $H_n + \ln(H_n)e^{H_n}$  even at infinity. In other words,  $H_n + \ln(H_n)e^{H_n}$  will always be ahead of  $\sigma(n)$ , even at infinity. This thoroughly proves the Riemann hypothesis for all  $n \geq 1$ .

The author must express many thanks to Bernhard Riemann who proposed the Riemann hypothesis in 1859. Again, the author thanks Bernhard Riemann for all of his work. Also the author wishes to express his eternal gratitude to Jeffrey Lagarias who proved that his Lagarias's Elementary Version is equivalent to the Riemann Hypothesis, a famous question about the complex roots of the Riemann zeta function. Without Lagarias work, the author could not have proved the Riemann Hypothesis. Additionally, the author would be remiss to not honor Georg Cantor for his groundbreaking study of different sizes of infinities, and defining cardinality.

References:

1. An Elementary Problem Equivalent to the Riemann Hypothesis, Jeffery C. Lagarias, July 29, 2001.
2. G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothese de Riemann, J Math. Pures Appl. **63** (1984) 187-213.
3. Guy, Richard K., Unsolved problems in number theory, (3rd edition). Springer-Verlag (2004).
4. From Euclid to Present: A Collection of Proofs regarding the Infinitude of Primes, Lindsey Harrison, December 14, 2013.
5. (Dun) W. Dunham, "Journey Through Genius: The Great Theorems of Mathematics," Penguin Books, John Wiley and Sons, 1990.
6. Orders of Growth, Keith Conrad
7. Is two to the power of infinity more than infinity?, Igor Ostrovsky, April 2011