Proof that Fermat Prime Numbers are Infinite

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Abstract

Fermat prime is a prime number that are a special case, given by the binomial number of the form:

$$F_n = 2^{2^n} + 1, \text{ for } n \geq 0$$

They are named after Pierre de Fermat, a Frenchman of the 17th Century, Pierre de Fermat, effectively invented modern number theory virtually single-handedly, despite being a small-town amateur mathematician. Throughout his life he devised a wide range of conjectures and theorems. He is also given credit for early developments that led to modern calculus, and for early progress in probability theory.

The only known Fermat primes are:

$$F_0 = 3$$

$$F_1 = 5$$

$$F_2 = 17$$

$$F_3 = 257$$

$$F_4 = 65{,}537$$

It has been conjectured that there are only a finite number of Fermat primes, however, we will use the same technique the author used to prove that the Mersenne primes are infinite, to prove the Fermat primes are infinite.

Proof of Infinite Fermat Primes

The divergence of the harmonic series was independently proved by Johann Bernoulli in 1689 in a counter-intuitive manner (reference 1). His proof is worthy of deep study, as it shows the counter-intuitive nature of infinity. We will use Bernoulli's proof and apply it toward proving the Fermat prime numbers are infinite.

Let the finite set of, p, Fermat primes be listed in reverse order from the largest to smallest Fermat primes as follows:

$$n_1 = F_n = 2^{2^n} + 1 = \text{largest Fermat prime}$$

$$n_2 = F_n = 2^{2^{n-1}} + 1 = \text{second largest Fermat prime}$$

$$n_3 = F_n = 2^{2^{n-2}} + 1 = \text{third largest Fermat prime}$$

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$$n_p = F_p = 2^p - 1 = 2^0 - 1 = \text{smallest Fermat prime number} = 3$$

This reverse ordering of the finite set of Fermat prime numbers is key to our proof. We assume that the following Fermat prime reciprocal series have a finite sum, which we call S.

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_p} > \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_p} = S$$

Where, k is the denominator factor for the smallest Fermat prime number that exists in our finite set.

We now proceed to derive a contradiction in the following manner. First we rewrite each term occurring in S thus:

$$\frac{1}{3n_2} = \frac{2}{6n_2} = \frac{1}{6n_2} + \frac{1}{6n_2}\,,\ \frac{1}{4n_3} = \frac{3}{12n_3} = \frac{1}{12n_3} + \frac{1}{12n_3} + \frac{1}{12n_3}\,, \ldots,$$

Next we write the resulting fractions in an array as shown below:

$$
\begin{array}{ccccccccc}
\frac{1}{2n_1} & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} & \cdots \\
 & \frac{1}{6n_2} & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} & \cdots \\
 & & \frac{1}{12n_3} & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} & \cdots \\
 & & & \frac{1}{20n_4} & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} & \cdots \\
 & & & & \frac{1}{30n_5} & \frac{1}{42n_6} & \frac{1}{56n_7} & \cdots \\
 & & & & & \frac{1}{42n_6} & \frac{1}{56n_7} & \cdots \\
 & & & & & & \frac{1}{56n_7} & \cdots
\end{array}
$$

Note that the column sums are just the fractions of the Fermat primes; thus S is the sum of all the fractions occurring in the array. As Bernoulli did, we now sums the rows using the telescoping technique. Next we assign symbols to the row sums as shown below,

$$A = \frac{1}{2n_1} + \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots ,$$

$$B = \frac{1}{6n_2} + \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots ,$$

$$C = \frac{1}{12n_3} + \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots ,$$

$$D = \frac{1}{20n_4} + \frac{1}{30n_5} + \frac{1}{42n_6} + \frac{1}{56n_7} + \dots ,$$

We now rearrange as follows:

$$A = (\frac{1}{n_1} - \frac{1}{2n_1}) + (\frac{1}{2n_2} - \frac{1}{3n_2}) + (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + \dots$$

Since, $n_1 > n_2 > n_3 > n_4$

$$A = \frac{1}{n_1} + (\frac{1}{2n_2} - \frac{1}{2n_1}) + (\frac{1}{3n_3} - \frac{1}{3n_2}) + (\frac{1}{4n_4} - \frac{1}{4n_3}) + (\frac{1}{5n_5} - \frac{1}{5n_4}) + \dots$$

Since, $\left(\frac{1}{2n_2} - \frac{1}{2n_1}\right) > 0$, $\left(\frac{1}{3n_3} - \frac{1}{3n_2}\right) > 0$, $(\frac{1}{4n_4} - \frac{1}{4n_3}) > 0$, $(\frac{1}{5n_5} - \frac{1}{5n_4}) > 0$

Then, $A > \frac{1}{n_1}$

$$B = (\frac{1}{2n_2} - \frac{1}{3n_2}) + (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + (\frac{1}{5n_5} - \frac{1}{6n_5})\dots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *B*.

Then, $B > \frac{1}{2n_2}$

$$C = (\frac{1}{3n_3} - \frac{1}{4n_3}) + (\frac{1}{4n_4} - \frac{1}{5n_4}) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + (\frac{1}{6n_5} - \frac{1}{7n_5}) \dots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *C*.

Then, $C > \frac{1}{3n_3}$

$$D = (\frac{1}{4n_4} - \frac{1}{5n_4}) + \left(\frac{1}{5n_5} - \frac{1}{6n_5}\right) + \left(\frac{1}{6n_5} - \frac{1}{7n_5}\right) + \left(\frac{1}{7n_6} - \frac{1}{8n_6}\right) \ldots$$

Since, $n_1 > n_2 > n_3 > n_4$, the same rearranging that we did with *A* can be done with *D.*

Then, $D > \frac{1}{4n_4}$

and so on. Thus the sum *S,* which we had written in the form $A + B + C + D + \ldots$, turns out to be greater than

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots$$

At the start we had defined *S* to be the following finite series,

$$S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_p}$$

And we defined that, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots$

However, we just proved that $S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_p}$

However, this is a contradiction, since in the finite realm S can't be equal to and greater than $\frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_\mathrm{p}}$ at the same time. Therefore, S must be infinite.

Now we can rewrite the S, the Fermat prime series as,

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \frac{1}{4n_4} + \cdots > \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{kn_\mathrm{p}} = S$$

This implies that $S > S$

However, no finite number can satisfy such an equation. Therefore, we have a contradiction and must conclude that $S = \infty$**.** Remember our definition of S from the above series:

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_\mathrm{p}} > S = \infty$$

Therefore, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_\mathrm{p}} > \infty$

Therefore, we have proven that the reciprocal Fermat prime series diverges to infinity. Obviously, this cannot possibly happen if there are only finitely many Fermat prime reciprocals, therefore the Fermat prime reciprocals are infinite in number. Since the Fermat prime reciprocals are infinite in number, the Fermat prime numbers must be infinite as well.

This proof shows the shows the counter-intuitive nature of infinity, and why it has taken so long to prove the Fermat primes are infinite, as it is not obvious that the reciprocal Fermat prime series would diverge. The opposite is true, that it seems that the Fermat prime series would converge. For example, numerically it has been shown that only 5 Fermat prime have been found, however we have just proven that Fermat primes diverges to infinity extremely slow, slower than the Mersenne primes. Numerically the Fermat primes grow so fast that the reciprocals for the large primes are rounded off to zero numerically. However, our proof has shown that the infinitesimally small reciprocal Fermat primes add to infinity and should not have been rounded to zero. Numerically no computer could do these calculations to prevent rounding, additionally to date computers have only found the first 50 Fermat primes since they grow so rapidly.

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