Sobre una integral del tipo Frullani

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Resumen
En esta nota se presentan algunas fórmulas
relacionadas con una integral del tipo Frullani.

Introducción

En la referencia [1] , página 315 , aparece la integral (del tipo Frullani [2]):

$$\int_0^\infty \frac{(1+ax)^{-1/4}-(1+bx)^{-3/4}}{x}\,dx=\ln\left(\frac{b}{a}\right)+\pi \quad ,a,b>0 \tag{1}$$

Un caso particular es ($a=b=1$):

$$\pi=\int_0^\infty \frac{(1+x)^{-1/4}-(1+x)^{-3/4}}{x}\,dx \tag{2}$$

En esta nota mostramos algunas fórmulas relacionadas con la integral (2).

Fórmulas

$$\pi=\int_0^\infty \left((\cosh x)^{-1/4}-(\cosh x)^{-3/4}\right)\coth\left(\frac{x}{2}\right)dx \tag{3}$$

$$\frac{\pi}{2^{1/4}}=\int_0^{\pi/4}\tan(2x)\sqrt{\sec x}\left(\sqrt{2}-\sec x\right)dx+\int_0^\infty \tanh(2x)\sqrt{\operatorname{sech} x}\left(\sqrt{2}-\operatorname{sech} x\right)dx \tag{4}$$

$$\int_0^{\pi/4} \tan(2x)\sqrt{\sec x}\left(\sqrt{2}-\sec x\right)dx = 2^{7/4}\tan^{-1}\left(3+2\sqrt{2}-2\sqrt{4+3\sqrt{2}}\right) = \\ = 2^{-1/4}\left(\pi - 4\tan^{-1}\left(2^{-1/4}\right)\right) = 2^{-1/4}\left(-\pi + 4\tan^{-1}\left(2^{1/4}\right)\right) \tag{5}$$

$$\int_0^{\infty} \tanh(2x)\sqrt{\operatorname{sech} x}\left(\sqrt{2}-\operatorname{sech} x\right)dx = 2^{3/4}\tan^{-1}\left(2^{5/4}\left(\sqrt{2}+1\right)\right) = \\ = 2^{7/4}F\left(\frac{1}{4},\frac{1}{4},\frac{5}{4},-1\right) - \frac{2^{7/4}}{3}F\left(\frac{3}{4},\frac{3}{4},\frac{7}{4},-1\right) \tag{6}$$

Para $0 < a < 1 < b < \infty$, se tiene:

$$\pi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^n}{n \cdot n!}\left((3/4)_n - (1/4)_n\right) + \\ +4b^{-1/4}F\left(\frac{1}{4},\frac{1}{4},\frac{5}{4},-\frac{1}{b}\right) - \frac{4b^{-3/4}}{3}F\left(\frac{3}{4},\frac{3}{4},\frac{7}{4},-\frac{1}{b}\right) + \\ +4\sum_{n=0}^{\infty}\left\{(1+a)^{-n}\left(\frac{(1+a)^{-1/4}}{4n+1} - \frac{(1+a)^{-3/4}}{4n+3}\right) - (1+b)^{-n}\left(\frac{(1+b)^{-1/4}}{4n+1} - \frac{(1+b)^{-3/4}}{4n+3}\right)\right\} \tag{7}$$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{(\cosh x)^{1/2} - (\cosh x)^{-1/2}}{\sinh x}\,dx \tag{8}$$

$$\frac{\pi}{2} = \int_0^{\sqrt{2}-1}\left\{\cosh^{-1}\left(\frac{1-x^2+\sqrt{1-6x^2+x^4}}{2x^2}\right) - \cosh^{-1}\left(\frac{1-x^2-\sqrt{1-6x^2+x^4}}{2x^2}\right)\right\}dx \tag{9}$$

$$\frac{\pi}{2} = \int_0^{\pi/2} \frac{(\sec x)^{1/2} - (\sec x)^{-1/2}}{\sin x}\,dx \tag{10}$$

$$\frac{\pi}{2} = \int_0^{\pi/2} \frac{(\cos x)^{-1/2} - (\cos x)^{1/2}}{\sin x}\,dx \tag{11}$$

$$\frac{\pi}{2} = \int_0^{\infty} \sin^{-1}\left(\frac{2}{1+x^2+\sqrt{1+6x^2+x^4}}\right)dx \tag{12}$$

$$\frac{\pi}{2} = \int_0^{\infty}\left((\coth x)^{1/2} - (\coth x)^{-1/2}\right)dx \tag{13}$$

$$\frac{\pi}{2}=\int_0^\infty\left(\left(\tanh x\right)^{-1/2}-\left(\tanh x\right)^{1/2}\right)dx \tag{14}$$

$$\frac{\pi}{2}=\int_0^\infty \coth^{-1}\left(\frac{2+x^2+x\sqrt{4+x^2}}{2}\right)dx \tag{15}$$

$$\frac{\pi}{2}=\int_0^\infty \tanh^{-1}\left(\frac{2}{2+x^2+x\sqrt{4+x^2}}\right)dx \tag{16}$$

$$\pi=\int_0^\infty \frac{e^{-x/4}-e^{-3x/4}}{1-e^{-x}}dx \tag{17}$$

$$\pi=\int_0^{1/2}\ln\left(\left(\frac{1+\sqrt{1-4x^2}}{2x}\right)^4\right)dx \tag{18}$$

$$\frac{\pi}{2}=\int_0^{\pi/2}\frac{\left(\sin x\right)^{-1/2}-\left(\sin x\right)^{1/2}}{\cos x}dx \tag{19}$$

$$\frac{\pi}{2}=\int_0^{\pi/2}\frac{\left(\csc x\right)^{1/2}-\left(\csc x\right)^{-1/2}}{\cos x}dx \tag{20}$$

$$\pi+\frac{1}{2}=\int_0^{1/2}\left(\frac{1}{\sqrt[5]{x+x^{5/2}\sqrt[5]{x+x^{5/2}\sqrt[5]{x+\ldots}}}}\right)^4 dx \tag{21}$$

Para $a>0$, se tiene:

$$\pi=4a^{1/4}F\left(\frac{1}{4},\frac{1}{4},\frac{5}{4},-a\right)-\frac{4}{3}a^{3/4}F\left(\frac{3}{4},\frac{3}{4},\frac{7}{4},-a\right)+ \\ +\sum_{n=1}^{\infty}\frac{\left(-1/4\right)_n-\left(-3/4\right)_n}{n\cdot n!}\left(\frac{1}{1+a}\right)^n F\left(1,n,n+1,\frac{1}{1+a}\right) \tag{22}$$

$$\pi=4a^{1/4}F\left(\frac{1}{4},\frac{1}{4},\frac{5}{4},-a\right)-\frac{4}{3}a^{3/4}F\left(\frac{3}{4},\frac{3}{4},\frac{7}{4},-a\right)+\sum_{n=1}^{\infty}\frac{\left(-1/4\right)_n-\left(-3/4\right)_n}{n!}I_n \tag{23}$$

donde

$$I_n = \begin{cases} \ln\left(1+\dfrac{1}{a}\right) & ,n=1 \\ \ln\left(1+\dfrac{1}{a}\right)+\displaystyle\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{k}\left(1-\left(\frac{a}{1+a}\right)^k\right) & ,n>1 \end{cases} \tag{24}$$

Para $0<a<1$, se tiene:

$$\pi = 4a^{1/4}F\left(1,\frac{1}{4},\frac{5}{4},a\right)-\frac{4}{3}a^{3/4}F\left(1,\frac{3}{4},\frac{7}{4},a\right)+\int_a^1\frac{x^{-3/4}-x^{-1/4}}{1-x}dx \tag{25}$$

$$\pi = 4a^{1/4}F\left(1,\frac{1}{4},\frac{5}{4},a\right)-\frac{4}{3}a^{3/4}F\left(1,\frac{3}{4},\frac{7}{4},a\right)+\sum_{n=1}^{\infty}\frac{(3/4)_n-(1/4)_n}{n\cdot n!}(1-a)^n \tag{26}$$

$$\begin{aligned}\pi = 4\left(\frac{1}{1+a}\right)^{1/4}F\left(1,\frac{1}{4},\frac{5}{4},\frac{1}{1+a}\right)-\frac{4}{3}\left(\frac{1}{1+a}\right)^{3/4}F\left(1,\frac{3}{4},\frac{7}{4},\frac{1}{1+a}\right)+\\ +\sum_{n=1}^{\infty}\frac{(-1/4)_n-(-3/4)_n}{n!}\left(\ln(1+a)+\sum_{k=1}^{n-1}\frac{(-a)^k}{k}\right)\end{aligned} \tag{27}$$

$$4\pi = \int_0^{\infty}\left((1+x)^{-5/4}-3(1+x)^{-7/4}\right)\ln x\,dx \tag{28}$$

Observación:

- $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592...$
- $F(a,b,c,x) = {}_2F_1(a,b;c;x)$ es la función hipergeométrica usual.
- $(x)_n = x(x+1)(x+2)...(x+n-1)$, $(x)_0 = 1$

Referencias

1. Bruce C. Berndt : Ramanujan's Notebooks Part I , New York,Springer-Verlag , 1985.
2. J. Arias-de Reyna: On the theorem of Frullani. Proc. Amer. Math. Soc.109,165-175,1990.