Division by Zero Calculus in Complex Analysis

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Abstract: In this paper, we will introduce the division by zero calculus in complex analysis for one variable at the first stage in order to see the elementary properties.

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1 Division by zero calculus

The essence of the division by zero is given by the simple division by zero calculus. For this conclusion, see the papers in the references.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.1}$$

we **define** the identity

$$f(a) = C_0. \tag{1.2}$$

By considering derivatives in (1.1), we **define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

In addition, we will refer to the naturality of this division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus. We define the value of the function $f(z)$ at the singular point $z = a$ with the Cauchy integral.

The basic idea of the above may be considered that we can consider the value of a function by some mean value of the function.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [4] and also the references for recent related results.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan \frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp \frac{1}{x}(x = 0) = 1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [4],

he kindly sent the message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Meanwhile, on ZERO, S. K. Sen and R. P. Agarwal [29] published its long history and many important properties of zero. See also R. Kaplan [6] and E. Sondheimer and A. Rogerson [31] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [28] that

The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea
and that is represented geometrically with the horn torus model [4].

S. K. Sen and R. P. Agarwal [29] referred to the paper [7] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

“ Thou shalt not divide by zero ” remains valid eternally.

However, in [27] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

They stated in the book many meanings of zero over mathematics, deeply.

In this paper, we will introduce the division by zero calculus in complex analysis for one variable as the first stage in order to see the elementary properties.

2 Basic meanings of the division by zero calculus

The values of analytic functions at isolated singular points were given by the coefficients C_0 of the Laurent expansions (the first coefficients of the regular parts) as the division by zero calculus. Therefore, their property may be considered as arbitrary ones by any sift of the image complex plane. Therefore, we can consider the values as zero in any Laurent expansions by shifts, as normalizations. However, if by another normalizations, the Laurent expansions are determined, then the values will have their senses. We will firstly examine such properties for the Riemann mapping function.

Let D be a simply-connected domain containing the point at infinity having at least two boundary points. Then, by the celebrated theorem of Riemann, there exists a uniquely determined conformal mapping with a series expansion

$$W = f(z) = C_1 z + C_0 + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots, \quad C_1 > 0, \tag{2.1}$$

around the point at infinity which maps the domain D onto the exterior $|W| > 1$ of the unit disc on the complex W plane. We can normalize (2.1) as follows:

$$\frac{f(z)}{C_1} = z + \frac{C_0}{C_1} + \frac{C_{-1}}{C_1 z} + \frac{C_{-2}}{C_1 z^2} + \ldots.$$

Then, this function $\frac{f(z)}{C_1}$ maps D onto the exterior of the circle of radius $1/C_1$ and so, it is called the **mapping radius** of D. See [2, 33]. Meanwhile, from the normalization

$$f(z) - C_0 = C_1 z + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots,$$

by the natural shift C_0 of the image plane, the unit circle is mapped to the unit circle with center C_0. Therefore, C_0 may be called as **mapping center** of D. The function $f(z)$ takes the value C_0 at the point at infinity in the sense of the division by zero calculus and now we have its natural sense by the mapping center of D. We have considered the value of the function $f(z)$ as infinity at the point at infinity, however, practically it was the value C_0. This will mean that in a sense the value C_0 is the farthest point from the point at infinity or the image domain with the strong discontinuity.

The properties of mapping radius were investigated deeply in conformal mapping theory like estimations, extremal properties and meanings of the values, however, it seems that there is no information on the property of mapping center. See many books on conformal mapping theory or analytic function theory. See [33] for example.

From the fundamental Bierberbach area theorem, we can obtain the following inequality:

For analytic functions on $|z| > 1$ with the normalized expansion around the point at infinity

$$g(z) = z + b_0 + \frac{b_1}{z} + \cdots$$

that are univalent and take no zero point,

$$|b_0| \leq 2.$$

In our sense

$$g(\infty) = b_0.$$

See [13], Chapter V, Section 8 for the details.

3 Values of typical Laurent expansions

The values at singular points of analytic functions are represented by the Cauchy integral, and so for given functions, the calculations will be simple numerically, however, their analytical (precise) values will be given by using the known Taylor or Laurent expansions. In order to obtain some feelings for the values at singular points of analytic functions, we will see typical examples and fundamental properties.

For

$$f(z) = \frac{1}{\cos z - 1}, \quad f(0) = -\frac{1}{6}.$$

For

$$f(z) = \frac{\log(1+z)}{z^2}, \quad f(0) = \frac{-1}{2}.$$

For
$$f(z) = \frac{1}{z(z+1)}, \quad f(0) = -1.$$

For our purpose in the division by zero calculus, when a is an isolated singular point, we have to consider the Laurent expansion on $\{0 < r < |z-a| < R\}$ such that r may be taken arbitrary small r, because we are considering the function at a.

For
$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}, \quad f(i) = \frac{1}{4}.$$

For
$$f(z) = \frac{1}{\sqrt{(z+1)}-1}, \quad f(0) = \frac{1}{2}.$$

For the Bernoulli constants B_n, we have the expansions

$$\frac{1}{(\exp z)-1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}B_n}{(2n)!} z^{2n-1}$$

$$= \frac{1}{z} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2+4\pi^2 n^2}$$

and so, we obtain
$$\frac{1}{(\exp z)-1}(z=0) = -\frac{1}{2},$$
([23], page 444).

From the well-known expansion ([1], page 807) of the Riemann zeta function
$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2(s-1)^2 + ...,$$
we see that the Euler constant γ is the value at $s=1$; that is,

$$\zeta(1) = \gamma.$$

Meanwhile, from the expansion

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} C_k \frac{z^{2k-1}}{2k-1}$$

([1], 635 page 18.5.5), we have

$$\zeta(0) = 0.$$

From the representation of the Gamma function $\Gamma(z)$

$$\Gamma(z) = \int_1^\infty e^{-t} t^{z-1} dt + \sum_{n=0}^\infty \frac{(-1)^n}{n!(z+n)}$$

([23], page 472), we have

$$\Gamma(-m) = E_{m+1}(1) + \sum_{n=0, n\neq m}^\infty \frac{(-1)^n}{n!(-m+n)}$$

and

$$[\Gamma(z)\cdot(z+n)](-n) = \frac{(-1)^n}{n!}.$$

In particular, we obtain

$$\Gamma(0) = -\gamma,$$

by using the identity

$$E_1(z) = -\gamma - \log z - \sum_{n=1}^\infty \frac{(-1)^n z^n}{nn!}, \quad |\arg z| < \pi$$

([1], 229 page, (5.1.11)). Of course,

$$E_1(z) = \int_z^\infty e^{-t} t^{-1} dt.$$

From the recurrence formula

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

of the Psi (Digamma) function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

([1], 258), we have, for $z = 0, 1$,

$$\psi(0) = \psi(1) = -\gamma.$$

Note that

$$\psi(1+z) = -\gamma + \sum_{n=2}^{\infty}(-1)^n \zeta(n) z^{n-1}, \quad |z| < 1$$

$$= -\gamma + \sum_{n=}^{\infty} \frac{z}{n(n+z)}, \quad z \neq -1, -2, ...$$

([1], 259).

From the identities

$$\frac{\Gamma(z)}{\Gamma(z+1)} = \frac{1}{z},$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

note that their values are zero at $z = 0$.

From the reflection formula of the Psi (Digamma) function

$$\psi(1-z) = \psi(z) + \pi \frac{1}{\tan \pi z}$$

([1], 258), we have, for $z = 1/2$,

$$\tan \frac{\pi}{2} = 0.$$

From the expansions

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} C_k z^{2k-2}$$

and

$$\wp'(z) = \frac{-2}{z^3} + \sum_{k=2}^{\infty}(2k-2) C_k z^{2k-3}$$

([1], 623 page, 18.5.1. and 18.5.4), we have

$$\wp(0) = \wp'(0) = 0.$$

We can consider many special functions and the values at singular points. For example,

$$Y_{3/2}(z) = J_{-3/2}(z) = -\sqrt{\frac{2}{\pi z}}\left(\sin z + \frac{\cos z}{z}\right),$$

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sinh z,$$

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z},$$

and so on. They take the value zero at the origin, however, we can consider some meanings of the value.

Of course, the product property is, in general, not valid:

$$f(0)\cdot g(0) \neq (f(z)g(z))(0);$$

indeed, for the functions $f(z) = z + 1/z$ and $g(z) = 1/z + 1/(z^2)$

$$f(0) = 0, g(0) = 0, (f(z)g(z))(0) = 1.$$

For an analytic function $f(z)$ with a zero point a, for the inversion function

$$(f(z))^{-1} := \frac{1}{f(z)},$$

we can calculate the value $(f(a))^{-1}$ at the singular point a.

For example, note that for the function

$$f(z) = z - \frac{1}{z},$$

$f(0) = 0, f(1) = 0$ and $f(-1) = 0$. Then, we have

$$(f(z))^{-1} = \frac{1}{2(z+1)} + \frac{1}{2(z-1)}.$$

Hence,

$$((f(z))^{-1})(z=0) = 0, ((f(z))^{-1})(z=1) = \frac{1}{4},$$

$$((f(z))^{-1}(z=-1) = -\frac{1}{4}.$$

Here, note that the point $z = 0$ is not a regular point of the function $f(z)$.

We, meanwhile, obtain that

$$\left(\frac{1}{\log x} \right)_{x=1} = 0.$$

Indeed, we consider the function $y = \exp(1/x), x \in \mathbf{R}$ and its inverse function $y = \frac{1}{\log x}$. By the symmetric property of the functions with respect to the function $y = x$, we have the desired result.

Here, note that for the function $\frac{1}{\log x}$, we can not use the Laurent expansion around $x = 1$, and therefore, the result is not trivial.

We shall refer to the trigonometric functions. See, for example, ([1], page 75) for the expansions.

From the expansion

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty} (-1)^{\nu} \left(\frac{1}{z - \nu\pi} + \frac{1}{\nu\pi} \right),$$

$$\left(\frac{1}{\sin z} \right)(0) = 0.$$

Meanwhile, from the expansion

$$\frac{1}{\sin^2 z} = \sum_{\nu=-\infty}^{\infty} \frac{1}{(z - \nu\pi)^2},$$

$$\left(\frac{1}{\sin^2 z} \right)(0) = \frac{2}{\pi^2} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{1}{3}.$$

From the expansion

$$\frac{1}{\cos z} = 1 + \sum_{\nu=-\infty}^{+\infty} (-1)^{\nu} \left(\frac{1}{z - (2\nu - 1)\pi/2} + \frac{2}{(2\nu - 1)\pi} \right),$$

$$\left(\frac{1}{\cos z} \right)\left(\frac{\pi}{2} \right) = 1 - \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{2\nu + 1} = 0.$$

Meanwhile, from the expansion

$$\frac{1}{\cos^2 z} = \sum_{\nu=-\infty}^{+\infty} \frac{1}{(z - (2\nu - 1)\pi/2)^2},$$

$$\left(\frac{1}{\cos^2 z}\right)\left(\frac{\pi}{2}\right) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

By the Laurent expansion and by the definition of the division by zero calculus, we note that:

Theorem: *For any analytic function $f(z)$ on $0 < |z| < \infty$, we have*

$$f(0) = f(\infty).$$

For a rational function

$$f(z) = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}; \quad a_0, b_0 \neq 0; \quad a_m, b_n \neq 0, m, n \geq 1$$

$$f(0) = f(\infty) = \frac{a_0}{b_0}.$$

Of course, here $f(\infty)$ is not given by any limiting $z \to \infty$, but it is the value at the point at ∞.

4 The derivatives of $n!$

Note that the identity $z! = \Gamma(z+1)$ and the Gamma function is a meromorphic function with isolated singular points on the entire complex plane. Therefore, we can consider the derivatives of the Gamma function even at isolated singular points, in our sense. This idea has a general concept for derivatives of discrete functions.

5 Values of domain functions

In this section, we will examine the values of typical domain functions at singular points. For a basic reference, see [13].

1). For the mapping

$$W = \frac{z}{1-z}$$

that maps conformally the unit disc $|z| < 1$ onto the half-plane $\{ReW > \frac{1}{2}\}$, we have

$$W(1) = -1.$$

2). For the Koebe function

$$W = \frac{z}{(1-z)^2}$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of $(-\infty, -\frac{1}{4})$ we have

$$W(1) = 0.$$

We can understand it as follows. The boundary point $z = 1$ of the unit disc is mapped to the point at infinity, however, the point is represented by zero. We can see the similar property, for many cases.

3). For the Joukowsky transform

$$W = \frac{1}{2}\left(\frac{1}{z} + z\right)$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of $[-1, 1]$ we have

$$W(0) = 0.$$

This correspondence will be curious in a sense. The origin that is an interior point corresponds to the boundary point of the origin. Should we consider the situation as in the case 2? - the image of the origin is the point at infinity and the point is represented by zero, the origin.

4). For the transform

$$W = \frac{z}{1-z^2}$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of the imaginary axis of $[+\infty, i/2]$ and $[-\infty, -i/2]$ we have

$$W(1) = -\frac{1}{4}, \quad W(-1) = \frac{1}{4},$$

by the method of Laurent expansion method, curiously. Should we consider the values at $z = 1$ and $z = -1$ as 0 from $1/0$ and $-1/0$ by the insertings $z = 1$ and $z = -1$ in the numerator and denominator?

5). For the conformal mapping $W = P(z; 0, v), |v| < 1$ of the unit disc onto the circular slit W plane that is normalized by $P(0; 0, v) = 0$ and

$$P(z : 0, v) = \frac{1}{z - v} + C_0 + C_0(z - v) + ...,$$

is given by, explicitly

$$P(z; 0, v) = \frac{1}{v(1 - |v|^2)} \frac{z(1 - \overline{v}z)}{z - v}$$

([13], 340 page). Then, we obtain

$$P(z : 0, v)|_{z=v} = C_0 = \frac{1 - 2|v|^2}{v(1 - |v|^2)},$$

at $z = v$ by the Laurent expansion method. By the constant C_0, we can consider as in the mapping center by shift of the image plane. We may also give the value for $z = v$ by

$$P(z : 0, v)|_{z=v} = \left[\frac{1}{v(1 - |v|^2)} \frac{z(1 - \overline{v}z)}{z - v}\right]_{z=v} = \frac{v(1 - |v|^2)}{0} = 0.$$

The circumstance is similar for the corresponding canonical conformal mapping $Q(z : 0, v)$ for the radial slit mapping.

6 The Szegö kernel

For the Szegö kernel $K(z, \overline{u})$ and its adjoint L kernel $L(z, u)$ on a regular region D on the complex z plane, the function

$$f(z) = \frac{K(z, \overline{u})}{L(z, u)}$$

is the Ahlfors function on the domain D and it maps the domain D onto the unit disc $|w| < 1$ with one to the multiplicity of the connectivity of the domain D. From the relation $L(z, u) = -L(u, z)$, we see that $L(u, u) = 0$ in the sense of the division by zero calculus. Therefore, from the identity

$$L(z, u) = \frac{1}{2\pi(z - u)} + \frac{1}{2\pi} \int_{\partial D} \frac{K(u, \overline{\zeta})}{\zeta - z} |d\zeta|$$

([13], 390 page), we have the very interesting identity

$$\int_{\partial D} \frac{K(z,\overline{\zeta})}{\zeta - z} |d\zeta| = 0.$$

In this method, we can find many new identities.

7 Sato hyperfunctions

As a typical example in A. Kaneko ([5], page 11) in the theory of hyperfunction theory we see that for non-integers λ, we have

$$x_+^\lambda = \left[\frac{-(-z)^\lambda}{2i\sin\pi\lambda}\right] = \frac{1}{2i\sin\pi\lambda}\{(-x+i0)^\lambda - (-x-i0)^\lambda\}$$

where the left hand side is a Sato hyperfunction and the middle term is the representative analytic function whose meaning is given by the last term. For an integer n, Kaneko derived that

$$x_+^n = \left[-\frac{z^n}{2\pi i}\log(-z)\right],$$

where log is a principal value on $\{-\pi < \arg z < +\pi\}$. Kaneko stated there that by taking a finite part of the Laurent expansion, the formula is derived. Indeed, we have the expansion, around an integer n,

$$\begin{aligned} &\frac{-(-z)^\lambda}{2i\sin\pi\lambda} \\ &= \frac{-z^n}{2\pi i}\frac{1}{\lambda - n} - \frac{z^n}{2\pi i}\log(-z) \\ &- \left(\frac{\log^2(-z)z^n}{2\pi i\cdot 2!} + \frac{\pi z^n}{2i\cdot 3!}\right)(\lambda - n) + ... \end{aligned}$$

([5], page 220).

However, we can now derive this result as the division by zero calculus, immediately.

Meanwhile, M. Morimoto derived this result by using the Gamma function with the elementary means in [12], pages 60-62.

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