Polar Complex Integers

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Abstract. We introduce a special class of complex numbers, wherein their absolute values and arguments given in a polar coordinate system are integers and we introduce the corresponding class of the Optimization Problems: "Polar Complex Integer Optimization".

Keywords: complex plane; integer lattice; optimization; polar coordinate system

1. Introduction

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{ a + b\mathbf{i} \mid a, b \in \mathbf{Z} \}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2-dimensional integer lattice. The Gaussian integers form unique factorization domain: it is irreducible if and only if it is a prime(Gaussian primes). The field of Gaussian rationals consists of the complex numbers whose real and imaginary part are both rational(see, e.g., [5]).

Another well-known integral subclass of complex numbers are Eisenshtein integers: complex numbers of the form: $z = a + b\omega$, where a and b are integers and $\omega = e^u$, $u = 2\pi\mathbf{i}/3$. The Eisenshtein integers form a triangular lattice in the complex plane, in contrast with Gaussian integers, which form a square lattice in the complex plane. The Eisenstein integers form a commutative ring as well and similar to Gaussian integers form a Euclidean domain, which supposes unique factorization of Eisenshtein integers into Eisenshtein primes.

Similar integral subclasses can be defined for quaternions: Lipschitz and Hurwitz Integers(quaternions).

Quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers(see, e.g., [2], [3]).

The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$.

This norm makes it possible to define the distance $d(p, q) = \|p - q\|$, which makes **H** into a metric space.

Lipschitz Integer(quaternion) is defined as:

$$L := \{ q: q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a \in \mathbf{Z}, b \in \mathbf{Z}, c \in \mathbf{Z}, d \in \mathbf{Z} \}.$$

Lipschitz Integer(quaternion) is a quaternion, whose components are all integers.

Hurwitz Integer(quaternion) is defined as:

$$H := \{ q: q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{Z} + 1/2\}.$$

Thus, Hurwitz Integer(quaternion) is a quaternion, whose components are either all integers or all half-integers.

2. Polar Complex Integers

Let us introduce a new subclass of complex numbers and a new approach for their definition accordingly: Polar Complex Integers.

Its well-known for a complex number $z = \mathrm{Re}(z) + \mathrm{Im}(z)\mathbf{i} = a + \mathbf{i}b$, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $\mathbf{i}^2 = -1$, to use an alternative option for coordinates in the complex plane: polar coordinate system that uses the distant of the point z from the origin and the angle, subtended between the positive real axis and the line segment in a counterclockwise sense(see, e.g., [6], [7]).

The absolute value of the complex number: $r = |z|$ is the distance to the origin of the point, representing the complex number z in the complex plane.

The argument of z: φ, is the angle of the radius with the positive real axis. Note that there are two notations of angle φ: in degree and in radian.

Together, r and φ gives another way of representing complex numbers, the polar form. Recovering the original rectangular co-ordinates from the polar form is done by the formula called trigonometric form:

$$z = r(\cos \varphi + \mathbf{i}\sin \varphi).$$

Recall that addition of two complex numbers can be done geometrically by constructing the corresponding parallelogram.

Given two complex numbers:

$z_1 = r_1 (\cos \varphi_1 + \mathbf{i} \sin \varphi_1)$ and $z_2 = r_2 (\cos \varphi_2 + \mathbf{i} \sin \varphi_2)$, multiplication of z_1 and z_2 in polar form is given by:

$$z_1 z_2 = r_1 r_2 (\cos (\varphi_1 + \varphi_1) + \mathbf{i} \sin (\varphi_1 + \varphi_1)).$$

Similarly, division is given by:

$$z_1 / z_2 = = r_1 / r_2 (\cos (\varphi_1 - \varphi_1) + \mathbf{i} \sin (\varphi_1 - \varphi_1)).$$

Using polar form, let us introduce the following new subclass of complex numbers : Polar Complex Integers:

$$\mathbf{P} := \{z: z = r(\cos \varphi + \mathbf{i} \sin \varphi) \mid z \in \mathbf{C}, r \in \mathbf{Z}, \varphi \in \mathbf{Z} \}.$$

Theorem 1. *Polar Complex Integers are closed under multiplication.*

Proof. It follows from the formula:

$$z_1 z_2 = r_1 r_2 (\cos (\varphi_1 + \varphi_1) + \mathbf{i} \sin (\varphi_1 + \varphi_1)).$$ □

Theorem 2. *Polar Complex Integers are not closed under addition*.

Proof. Let us consider $z_1 = 0 + 1\mathbf{i}$ and $z_2 = 1 + 0\mathbf{i}$.

Even for degree notation, where $z_1 = 1(\cos 90^\circ + \mathbf{i} \sin 90^\circ)$ and $z_2 = 1(\cos 0^\circ + \mathbf{i} \sin 0^\circ)$, absolute value of $z_1 + z_2$ is an irrational number. □

Theorem 3. *Polar Complex Integers are not closed under division.*

Proof. It follows from the formula:

$$z_1 / z_2 = = r_1 / r_2 (\cos (\varphi_1 - \varphi_1) + \mathbf{i} \sin (\varphi_1 - \varphi_1)).$$ □

Corollary 1. *Polar Complex Integers are mutually primes if and only if their absolute values are mutually primes*.

Similar to aforementioned Hurwitz integers let us introduce Polar Complex Hurwitz-like Integers:

$$\mathbf{PH} := \{z: z = r(\cos \varphi + \mathbf{i} \sin \varphi) \mid z \in \mathbf{C}, r \in \mathbf{Z} + 1/2 , \varphi \in \mathbf{Z} + 1/2 \},$$

and similar to aforementioned Gaussian Rationals, the corresponding set of Polar Complex Rationals can be introduced as well.

3. Optimization over subsets of Polar Complex Integers

It is well-known that an optimization problem can be represented in the following way:

given: a function $f: \mathbf{G} \to \mathbf{R}$ from some set $\mathbf{G}$ to the real numbers,
sought: an element $x_0 \in \mathbf{G}$ such that $f(x_0) \le f(x)$ for all $x \in \mathbf{G}$ ("minimization") or such that $f(x_0) \ge f(x)$ for all $x \in \mathbf{G}$ ("maximization").

Let us introduce a new class of Optimization problems, where $\mathbf{G}$ is some subset of the Polar Complex Integers $\mathbf{P}$ and $\mathbf{P}^n$ and target functions $f: \mathbf{P} \to \mathbf{R}$ and $f: \mathbf{P}^n \to \mathbf{R}$ are real-valued complex variable function: "Polar Complex Integer Optimization".

3.1. Polynomial Polar Complex Integer Optimization

pcop1 = **{** maximize $| c_n z^n + ... + c_1 z |$ subject to

$$| a_{1n} z^n + ... + a_{11} z | \le b_1,$$
$$... \qquad ... \qquad ...$$

$$| a_{mn}z^n + ... + a_{m1}z | \leq b_m,$$

$$z \in \mathbf{P},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_j \in \mathbf{C},$$

$$1 \leq i \leq m,\ 1 \leq j \leq n,\ n \in \mathbf{N},\ m \in \mathbf{N} \}.$$

(More sophisticated examples would contain rational meromorphic complex functions).

3.2. Linear Polar Complex Integer Optimization

pcop2a = { maximize $| c_1z_1 + ... + c_nz_n |$ subject to

$$| a_{11}z_1 + ... + a_{1n}z_n | \leq b_1,$$
$$... \quad ... \quad ...$$
$$| a_{m1}z_1 + ... + a_{mn}z_n | \leq b_m,$$

$$z_j \in \mathbf{P},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_j \in \mathbf{C},$$

$$1 \leq i \leq m,\ 1 \leq j \leq n,\ n \in \mathbf{N},\ m \in \mathbf{N} \}.$$

pcop2b = { maximize $| c_1z_1 + ... + c_nz_n |$ subject to

$$a_{11}z_1 + ... + a_{1n}z_n = b_1,$$
$$... \quad ... \quad ...$$
$$a_{m1}z_1 + ... + a_{mn}z_n = b_m,$$

$$z_j \in \mathbf{P},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{C},\ c_j \in \mathbf{C},$$

$$(A\mathbf{z} = b),$$

$$1 \leq i \leq m,\ 1 \leq j \leq n,\ n \in \mathbf{N},\ m \in \mathbf{N} \}.$$

3.3. Quadratic Polar Complex Integer Optimization

pcop3 = { maximize $| z_1^2 + ... + z_n^2 - \mathbf{i}z_1z_2 |$ subject to

$$| a_{11}z_1 + ... + a_{1n}z_n | \leq b_1,$$

... ...

$| a_{m1}z_1 + ... + a_{mn}z_n | \le b_m,$

$z_j \in \mathbf{P},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},$

$1 \le i \le m, 1 \le j \le n, n \in \mathbf{N},\ m \in \mathbf{N} \}.$

3.4. Non-Linear Polar Complex Integer Optimization

pcop4 = **{** maximize $| e^z - \sin(\pi z) |$ subject to

$| \cos(\pi z) | \le a, 0 \le \mathrm{Re}(z) \le 1, 0 \le \mathrm{Im}(z) \le 1,$

$z \in \mathbf{P}, a \in \mathbf{R} \}.$

3.5. Mixed-Real-Integer Polar Complex Optimization (MRIPCOP). (Similarly for the Polar Complex Hurwitz-like Integers and Polar Complex Rationals).

pcop5 = **{** minimize $| \mathbf{i}z_1^4 + z_2^2 | - x^2 + y^3t^2$ subject to

$xy \ge N,$

$a_1 \le | z_1 | \le b_1,$

$a_2 \le | z_2 | \le b_2,$

$a_3 \le x \le b_3,$

$a_4 \le y \le b_4,$

$a_5 \le t \le b_5,$

$z_1 \in \mathbf{C}, z_2 \in \mathbf{P},$

$x \in \mathbf{Z}, , y \in \mathbf{Z}, t \in \mathbf{R},$

$a_i \in \mathbf{R}, b_i \in \mathbf{R}, N \in \mathbf{N}, a_i > 0,$

$1 \le i \le 5 \}$.

Note that in addition, each such example may comprise complex conjugations as well.

4. Conclusions

We unveiled a special class of complex numbers, wherein their absolute values and arguments, given in a polar coordinate system are integers and we unveiled the corresponding class of the Optimization Problems: "Polar Complex Integer Optimization".

References

1. Floudas, C. A., Pardalos, P. M. (ed): Encyclopedia of Optimization, Springer, New York, (2009).

2. Gürlebeck, K., Sprößig, W.: Quaternionic analysis and elliptic boundary value problems, Birkhäuser, Basel, (1990).

3. Hamilton, W. R. (ed): Elements of Quaternions, Chelsea Publishing Company, (1969).

4. Hemmecke, R., Koppe, M., Lee, J., Weismantel, R.: Nonlinear Integer Programming, in: Junger, M., Liebling, T., Naddef, D., Pulleyblank, W., Reinelt, W., Rinaldi, G., Wolsey, L. (eds.): 50 Years of Integer Programming 1958‑2008: The Early Years and State-of-the-Art Surveys, pp. 561‑618, Springer-Verlag, Berlin, (2010).

5. Kleiner, I.: From Numbers to Rings: The Early History of Ring Theory, Elem. Math., 53 (1), pp. 18–35, Birkhäuser, Basel, (1998).

6. Scheidemann, V.: Introduction to complex analysis in several variables, Birkhauser, (2005).

7. Shaw, W. T.: Complex Analysis with Mathematica, Cambridge, (2006).