

Integration technique using Laplace transforms.

A generalizized form of the Dirichlet integral.

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Abstract: the problem of integration technique over integrands of the form $f(t)/t^n$, can be solved by differentiation (n-times) by using Leibniz's rule to get rid of t^n , that leads to integrate back (n-times) to end the game, wich it's harder than the original problem. This work focuses on the derivation of the formula (*Espil's theorem*) wich is a perfect tool to avoid that hard task. It allows to change the difficult n iterated integrals into a more outstanding easier problem wich consists of n-1 derivatives. The *Espil's theorem* is a generalization of the Dirichlet integral.

Key words: Integration, Dirichlet integral ,Laplace Transform, iterated intergals, multiple integration, integration by parts, Espil's theorem

1 Hypothesis

H1) Let $f: [0; \infty) \rightarrow \mathbb{R}$ be a continuous piecewise function admitting n-2 derivatives at the origin.

H2) $\int_0^{\infty} \frac{f(t)}{t^n} dt$ is convergent $\forall n \in \mathbb{N}$

H3) $\lim_{t \rightarrow 0^+} \frac{f(t)}{t^n} = \gamma$, finite.

H4) let $f(t)$ as its first n-2 derivatives to be bounded functions.

2 Thesis

$$\int_0^{\infty} \frac{f(t)}{t^n} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{\Pi_{(n-1)}} \mathcal{L}(D^{n-1} f(t)) d\sigma$$

3 Demonstration

factor	D	I
(+1)	$f(t)$	t^{-n}
(-1)	$D^1 f(t)$	$\frac{(-1)t^{-n+1}}{(n-1)}$
(+1)	$D^2 f(t)$	$\frac{t^{-n+2}}{(n-1)(n-2)}$
(-1)	$D^3 f(t)$	$\frac{(-1)t^{-n+3}}{(n-1)(n-2)(n-3)}$
.....
$(-1)^k$	$D^k f(t)$	$\frac{(-1)^k t^{-n+k}}{\prod_{j=1}^k (n-j)}$
.....
$(-1)^{n-1}$	$D^{n-1} f(t)$	$\frac{(-1)^{n-1}}{(n-1)! t}$

$$\forall n \in \mathbb{N} \geq 2, \int_0^\infty \frac{f(t)}{t^n} dt = -\frac{f(t)t^{-n+1}}{(n-1)} - \frac{(D^1 f(t))t^{-n+2}}{(n-1)(n-2)} - \frac{(D^2 f(t))t^{-n+3}}{(n-1)(n-2)(n-3)} - \dots - \frac{(D^{n-2} f(t))}{(n-1)! t} \Big|_0^\infty + \int_0^\infty \frac{D^{n-1} f(t)}{(n-1)! t} dt$$

$$\int_0^\infty \frac{f(t)}{t^n} dt = -\lim_{t \rightarrow \infty} \sum_{k=1}^{n-1} \frac{(D^{k-1} f(t))t^{-n+k}}{\prod_{j=1}^k (n-j)} + \lim_{t \rightarrow 0^+} \sum_{k=1}^{n-1} \frac{(D^{k-1} f(t))t^{-n+k}}{\prod_{j=1}^k (n-j)} + \int_0^\infty \frac{1}{(n-1)!} \frac{D^{n-1} f(t)}{t} dt$$

remark: the first sum is canceled because $f(t)$ as well as its first $n-2$ derivatives are bounded functions (H4)

$$|D^{k-1} f(t)| \leq M_{k-1}, \forall k \in [1; n-1]$$

$$0 \leq \lim_{t \rightarrow \infty} \left| \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))t^{-n+k}}{\prod_{j=1}^k (n-j)} \right| = \lim_{t \rightarrow \infty} \left| \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))}{t^{n-k} \prod_{j=1}^k (n-j)} \right| \leq \lim_{t \rightarrow \infty} \sum_{k=1}^{n-1} \frac{|D^{k-1}f(t)|}{t^{n-k} \prod_{j=1}^k (n-j)} \leq$$

$$\lim_{t \rightarrow \infty} \sum_{k=1}^{n-1} \frac{M_{k-1}}{t^{n-k} \prod_{j=1}^k (n-j)} = 0$$

$$\lim_{t \rightarrow \infty} \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))t^{-n+k}}{\prod_{j=1}^k (n-j)} = 0$$

remark: the second sum is canceled because $\lim_{t \rightarrow 0^+} \frac{f(t)}{t^n} = \gamma$ existst (H3) and by using the L'Hopital's rule n-2 times (H1)

$$\lim_{t \rightarrow 0^+} \frac{f(t)}{t^n} = \lim_{t \rightarrow 0^+} \frac{D^1 f(t)}{n \cdot t^{n-1}} = \lim_{t \rightarrow 0^+} \frac{D^2 f(t)}{n(n-1) \cdot t^{n-2}} = \dots = \lim_{t \rightarrow 0^+} \frac{D^k f(t)}{t^{n-k} \prod_{j=0}^{k-1} (n-j)} = \lim_{t \rightarrow 0^+} \frac{D^{n-2} f(t)}{t^2 \prod_{j=0}^{n-3} (n-j)} = \gamma$$

$$\lim_{t \rightarrow 0^+} \frac{D^k f(t)}{t^{n-k}} = \gamma \cdot \prod_{j=0}^{k-1} (n-j) = C_k$$

$$\lim_{t \rightarrow 0^+} \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))t^{-n+k}}{\prod_{j=1}^k (n-j)} = \lim_{t \rightarrow 0^+} \sum_{k=1}^{n-1} \frac{D^{k-1}f(t)}{t^{n-k} \prod_{j=1}^k (n-j)} =$$

$$\lim_{t \rightarrow 0^+} \sum_{k=1}^{n-1} \frac{t \cdot (D^{k-1}f(t))}{t^{n-(k-1)} \prod_{j=1}^k (n-j)} = \lim_{t \rightarrow 0^+} \sum_{k=1}^{n-1} \frac{t \cdot C_{k-1}}{\prod_{j=1}^k (n-j)} = 0$$

$$\lim_{t \rightarrow 0^+} \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))t^{-n+k}}{\prod_{j=1}^k (n-j)} = 0$$

$$\therefore \int_0^{\infty} \frac{f(t)}{t^n} dt = \int_0^{\infty} \frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t} dt$$

remark : valid for limit case $n=1$

therefore the left integral converges (H2), so does the right one.

$$\int_0^{\infty} \frac{f(t)}{t^n} dt = \int_0^{\infty} \frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t} dt, \forall n \in \mathbb{N}$$

$$\lim_{\sigma \rightarrow 0} \int_0^{\infty} \frac{f(t)}{t^n} e^{-\sigma t} dt = \lim_{\sigma \rightarrow 0} \int_0^{\infty} \frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t} e^{-\sigma t} dt$$

definition: let $g(t) = \frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t}$ $\therefore G(\sigma) = \mathcal{L} \left[\frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t} \right] = \int_0^{\infty} \frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t} e^{-\sigma t} dt$

remark : $\lim_{\sigma \rightarrow \infty} G(\sigma) = 0$

$$\frac{D[G(\sigma)]}{d\sigma} = \int_0^{\infty} \partial_{\sigma} \frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t} e^{-\sigma t} dt = (-1) \int_0^{\infty} \frac{1}{(n-1)!} D^{n-1}f(t) e^{-\sigma t} dt = \frac{-1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)]$$

$$dG = \frac{-1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$\lim_{\sigma \rightarrow 0} \int_{G(\sigma)}^0 dG = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{-1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$0 - \lim_{\sigma \rightarrow 0} G(\sigma) = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{-1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$\lim_{\sigma \rightarrow 0} G(\sigma) = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$\lim_{\sigma \rightarrow 0} \int_0^{\infty} \frac{1}{(n-1)!} \frac{D^{n-1}f(t)}{t} e^{-\sigma t} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$\lim_{\sigma \rightarrow 0} \int_0^{\infty} \frac{f(t)}{t^n} e^{-\sigma t} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$\int_0^{\infty} \frac{f(t)}{t^n} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{(n-1)!} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$\int_0^{\infty} \frac{f(t)}{t^n} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{\Pi_{(n-1)}} \mathcal{L}[D^{n-1}f(t)] d\sigma$$

$$\int_0^{\infty} \frac{f(t)}{t^n} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{\Pi_{(n-1)}} \mathcal{L}[D^{n-1}f(t)] d\sigma = \lim_{\sigma \rightarrow 0} \int_{\sigma}^{\infty} \int_{\sigma_1}^{\infty} \dots \int_{\sigma_{n-1}}^{\infty} F(\sigma_n) d\sigma_n \dots d\sigma_2 d\sigma_1$$

(2017)

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remark: As an alternative way to approach the same result we can make use of the Laplace transforms properties:

known

$$\mathcal{L}(D^{n-1}f(t)) = \sigma^{n-1}\mathcal{L}(f(t)) - \sigma^{n-2}f(0) - \sigma^{n-3}Df(0) - \dots - D^{n-2}f(0)$$

If we apply H3, $f(t)$ and “at least” its first $n-2$ derivatives cancel at the origin, because γ is finite.

$$f(0) = Df(0) = D^2f(0) = \dots = D^{n-2}f(0) = 0$$

\therefore

$$\mathcal{L}(D^{n-1}f(t)) = \sigma^{n-1}\mathcal{L}(f(t))$$

$$\lim_{s \rightarrow 0} \int_s^\infty \mathcal{L}(D^{n-1}f(t)) d\sigma = \lim_{s \rightarrow 0} \int_s^\infty \sigma^{n-1}\mathcal{L}(f(t)) d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^\infty \int_0^\infty \sigma^{n-1}f(t)e^{-\sigma t} dt d\sigma = \int_0^\infty f(t) \lim_{s \rightarrow 0} \int_s^\infty \sigma^{n-1}e^{-\sigma t} d\sigma dt =$$

$$\int_0^\infty f(t) \frac{(n-1)!}{t^n} dt = \lim_{s \rightarrow 0} \int_s^\infty \mathcal{L}(D^{n-1}f(t)) d\sigma =$$

$$\int_0^\infty f(t) \frac{\Pi_{(n-1)}}{t^n} dt = \lim_{s \rightarrow 0} \int_s^\infty \mathcal{L}(D^{n-1}f(t)) d\sigma =$$

$$\int_0^\infty \frac{f(t)}{t^n} dt = \lim_{s \rightarrow 0} \int_s^\infty \frac{1}{\Pi_{(n-1)}} \mathcal{L}(D^{n-1}f(t)) d\sigma$$

4 Examples

Example 4.1: $\int_0^{\infty} \frac{(\text{Sin}[t])^6}{t^6} dt = \frac{11\pi}{40}$ (WolframAlpha)

$$\int_0^{\infty} \frac{(\text{Sin}[t])^6}{t^6} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^5((\text{Sin}[t])^6))}{5!} d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^5((\frac{e^{it}-e^{-it}}{2i})^6))}{5!} d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^5(\frac{e^{i6t}-6e^{i4t}+15e^{i2t}-20+15e^{-i2t}-6e^{-i4t}+e^{-i6t}}{-64}))}{5!} d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \mathcal{L}\left(\frac{-81ie^{i6t}}{80} + \frac{4ie^{i4t}}{5} - \frac{ie^{i2t}}{16} + \frac{ie^{-i2t}}{16} - \frac{4ie^{-i4t}}{5} + \frac{81ie^{-i6t}}{80}\right) d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \mathcal{L}\left(\frac{81e^{i6t}}{80i} - \frac{4e^{i4t}}{5i} + \frac{e^{i2t}}{16i} - \frac{e^{-i2t}}{16i} + \frac{4e^{-i4t}}{5i} - \frac{81e^{-i6t}}{80i}\right) d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \mathcal{L}\left(\frac{81}{40} \text{Sin}[6t] - \frac{8}{5} \text{Sin}[4t] + \frac{1}{8} \text{Sin}[2t]\right) d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \frac{81}{40} \frac{6}{(\sigma^2+36)} - \frac{8}{5} \frac{4}{(\sigma^2+16)} + \frac{1}{8} \frac{2}{(\sigma^2+4)} d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{81}{40} \frac{1/6}{((\frac{\sigma}{6})^2+1)} - \frac{8}{5} \frac{1/4}{((\frac{\sigma}{4})^2+1)} + \frac{1}{8} \frac{1/2}{((\frac{\sigma}{2})^2+1)} d\sigma$$

$$\lim_{b \rightarrow \infty} \left(\frac{81}{40} \text{Arctan}\left(\frac{b}{6}\right) - \frac{8}{5} \text{Arctan}\left(\frac{b}{4}\right) + \frac{1}{8} \text{Arctan}\left(\frac{b}{2}\right)\right) - \lim_{s \rightarrow 0} \left(\frac{81}{40} \text{Arctan}\left(\frac{s}{6}\right) - \frac{8}{5} \text{Arctan}\left(\frac{s}{4}\right) + \frac{1}{8} \text{Arctan}\left(\frac{s}{2}\right)\right) =$$

$$= \frac{\pi}{2} \left(\frac{81}{40} - \frac{8}{5} + \frac{1}{8}\right) = \frac{11\pi}{40}$$

Example 4.2: $\int_0^{\infty} \frac{(e^{-2t} - e^{-3t})^4}{t^4} dt = 0.100336374512$ (WolframAlpha)

$$\lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^3((e^{-2t} - e^{-3t})^4))}{3!} d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^3(e^{-8t} - 4e^{-9t} + 6e^{-10t} - 4e^{-11t} + e^{-12t}))}{3!} d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \mathcal{L}\left(\frac{-256}{3}e^{-8t} + 486e^{-9t} - 1000e^{-10t} + \frac{2662}{3}e^{-11t} - 288e^{-12t}\right) d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \frac{-256}{3(\sigma + 8)} + \frac{486}{(\sigma + 9)} - \frac{1000}{(\sigma + 10)} + \frac{2662}{3(\sigma + 11)} - \frac{288}{\sigma + 12} d\sigma =$$

$$\lim_{b \rightarrow \infty} \operatorname{Ln}\left(\frac{(b+9)^{486}(b+11)^{\frac{2662}{3}}}{(b+10)^{1000}(b+8)^{\frac{256}{3}}(b+12)^{288}}\right) - \lim_{s \rightarrow 0} \operatorname{Ln}\left(\frac{(s+9)^{486}(s+11)^{\frac{2662}{3}}}{(s+10)^{1000}(s+8)^{\frac{256}{3}}(s+12)^{288}}\right) =$$

$$\lim_{s \rightarrow 0} \operatorname{Ln}\left(\frac{(s+10)^{1000}(s+8)^{\frac{256}{3}}(s+12)^{288}}{(s+9)^{486}(s+11)^{\frac{2662}{3}}}\right) = \operatorname{Ln}\left(\frac{(10)^{1000}(8)^{\frac{256}{3}}(12)^{288}}{(9)^{486}(11)^{\frac{2662}{3}}}\right) = 0.100336374512$$

$$\int_0^{\infty} \frac{(e^{-2t} - e^{-3t})^4}{t^4} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^3((e^{-2t} - e^{-3t})^4))}{3!} d\sigma = 0.100336374512$$

Example 4.3: $\int_0^{\infty} \frac{(1-\cos[t])^2}{t^4} dt = \frac{\pi}{6}$ (WolframAlpha)

$$\int_0^{\infty} \frac{(1-\cos[t])^2}{t^4} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^3((1-\cos[t])^2))}{3!} d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^3(1-2\cos[t]+(\cos[t])^2))}{6} d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^3(1-2\cos[t]+(\cos[t])^2))}{6} d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(D^3(\frac{3}{2}-2\cos[t]+\frac{1}{2}\cos[2t]))}{6} d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \frac{\mathcal{L}(-2\sin[t]+4\sin[2t])}{6} d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \left(\frac{-1}{3(\sigma^2+1)} + \frac{4}{3(\sigma^2+4)} \right) d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \left(\frac{-1}{3(\sigma^2+1)} + \frac{1}{3(\frac{\sigma}{2})^2+1} \right) d\sigma =$$

$$\lim_{s \rightarrow 0} \int_s^{\infty} \left(\frac{-1}{3(\sigma^2+1)} + \frac{2}{3\left(\frac{\sigma}{2}\right)^2+1} \right) d\sigma = \lim_{b \rightarrow \infty} \frac{-1}{3} \text{ArcTan}[b] + \frac{2}{3} \text{ArcTan}\left[\frac{b}{2}\right] = \left(-\frac{1}{3} + \frac{2}{3}\right) \frac{\pi}{2} = \frac{\pi}{6}$$

Example 4.4: $\int_0^{\infty} \frac{\sin[t]}{t} dt = \frac{\pi}{2}$ (The Dirichlet integral)

$$\int_0^{\infty} \frac{\sin[t]}{t} dt = \lim_{s \rightarrow 0} \int_s^{\infty} \mathcal{L}(\sin[t]) d\sigma = \lim_{s \rightarrow 0} \int_s^{\infty} \frac{1}{(\sigma^2+1)} d\sigma = \lim_{b \rightarrow \infty} \text{ArcTan}[b] = \frac{\pi}{2}$$

5 Conclusion: this theorem allows to integrate special functions by using Laplace transforms. For low values of n, in the factor t^n we could solve the problem integrating few times the laplace transform of f(t), but when n goes high it turns very tough, in many cases impossible to solve this way, that's why this generalization formula for the Dirichlet integral is needed.