SOFT AND HARD LIBERATION OF COMPACT LIE GROUPS

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Abstract. We investigate the liberation question for the compact Lie groups, by using various "soft" and "hard" methods, based respectively on joint generation with a free quantum group, and joint generation with a free torus. The soft methods extend the "easy" methods, notably by covering groups like SO_N,SU_N, and the hard methods partly extend the soft methods, notably by covering the real and complex tori themselves.

Contents

Introduction

The liberation operation consists in removing, at the algebraic level, the commutativity relation $ab=ba$ between the coordinates. The removal operation is, however, something quite tricky. The story here goes back to an 1981 paper by Brown [15], constructing a free analogue U_N^{nc} of the unitary group U_N, according to the following formula:

$$C(U_N^{nc})=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1}\right)$$

This free analogue of U_N is something quite interesting, and some work on it, functional analytic and K-theoretical, was done later on by McClanahan [23]. Observe that U_N^{nc} has a comultiplication and a counit, defined respectively by $\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}$ and $\varepsilon(u_{ij})=\delta_{ij}$. However, U_N^{nc} has no antipode, because setting $S(u_{ij})=u_{ji}^*$ would require $u^t=(u_{ji})$ to be unitary as well, and in the noncommutative world, this is not automatic. Thus, U_N^{nc} is rather a semigroup-type object, and Brown was "wrong".

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The continuation of the story goes to the 1995 paper by Wang [30] where a smaller and better object U_N^+ was constructed, according to the following formula:

$$C(U_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)$$

Here we can define an antipode by $S(u_{ij})=u_{ji}^*$, and this is a compact quantum group in the sense of Woronowicz [35], [36], and a well-established object.

Be said in passing, Wang's paper [30] and its follow-up [28] with Van Daele were still not totally away from any controversy, because the quantum group U_N^+ was regarded there as being part of a more general family $\{U_F^+\}$, not subject to the key antipode axiom $S^2=id$, and whose mathematical and physical revelance remains quite unclear.

The quantum groups U_N^+ can be investigated by using the Temperley-Lieb algebra [27], with some further inspiration from the work of Connes [18], Jones [20] and Voiculescu [29], and this was done in [1], and then in a joint work with Collins [9].

In parallel, more liberations of compact Lie groups were constructed, starting with Wang's paper [31] on the quantum permutation group S_N^+. Once again, the whole story here has not been exactly error-proof. As an illustrating example, the quantum symmetry group of the N-dimensional hypercube is O_N^{-1}, and considering this is as a free analogue of H_N is an "error", of rather advanced type. Indeed, the correct analogue H_N^+ is the quantum symmetry group of the coordinate axes of $\mathbb{R}^N$, as explained in [6].

As a conclusion, a parallel between the classical and free world, notably with a free analogue of the Weingarten formula [17], [32], was beginning to take shape.

The representation theory results in [1] and related papers were unified with Brauer's theorem [14] and related theorems in the paper with Speicher [10]. The conclusion there was that the compact Lie groups $G_N\subset U_N$ can be liberated into quantum groups $G_N^+\subset U_N^+$ by "removing the crossings" from the associated Tannakian category. In order to do so, a key algebraic geometry assumption, called "easiness", is however needed.

We should mention that the easiness assumption is something which goes back to Weyl, Brauer and others, and which has popped out regularily, in various areas of mathematics and physics. The originality of [10] lies of course in axiomatizing and using the easiness in the noncommutative setting, although, as is always the case with mathematics, physicists usually know such things long in advance. It would be of course interesting to find out what the exact story with all this was, and who really invented the wheel.

Probably Weyl [33], or even Klein [21].

From a modern perspective, of the generation operation $<,>$, which has been folklore for some time, as is always the case with "unwritten foundations", and first appeared in [16], the "removing of the crossings" procedure from [10] amounts in setting:

$$G_N^+=<G_N,S_N^+>$$

We should mention that this formula, known more of less since around [10], has not been really exploited for long, due to fears with the $<\,,>$ operation, which is something quite complicated in the quantum group setting. A version of this formula was systematically used in the recent paper [3], in connection with some 3D problems.

Our first purpose here will be that of investigating this operation, in the general case, not necessarily easy, as a generalization of [10]. Following the work on the half-liberation in [4], we will call this operation "soft liberation". This operation is quite subtle, and as a conclusion of our work here, we will formulate a general conjecture, as follows:

$$G_N^+ = (\widetilde{G_N})^+$$

Here on the right $\widetilde{G_N}$ is the easy envelope of G_N, constructed in [2], and the liberation operation for this quantum group is the standard "easy" one, from [10].

We will discuss then the notion of "hard liberation". Things here are more tricky, and according to vast folklore knowledge, the idea would be to set:

$$G_N^{+r} =< G_N, T_N^+ > \quad , \quad G_N^{+c} =< G_N, \mathbb{T}_N^+ >$$

Here $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$ is the free real torus, also called free cube, and $\mathbb{T}_N^+ = \widehat{\mathbb{Z}^{*N}}$ is the free complex torus, which is the same as the abstract dual of the free group F_N.

We will prove here that for the groups O_N, U_N, the above real and complex hard liberation methods agree with the "standard" soft liberation method.

In general, however, our hard theory does not cover $G_N = S_N$ itself, and not even $G_N = H_N$. Here we must use spinned tori, as in [5], and we will comment on this.

The paper is organized as follows: in 1-2 we discuss the soft liberation operation, and in 3-4 we discuss the hard liberation operation, and we end with some open problems.

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1. Soft liberation

We use Woronowicz's quantum group formalism in [35], [36], under the extra assumption $S^2 = id$. Such quantum groups appear as closed subgroups $G_N \subset U_N^+$ of Wang's free unitary quantum group, constructed in [30], and studied in detail in [1].

We will heavily use the easy quantum group formalism from [10], sometimes in the modified unitary sense of [26]. This formalism appears as a quantum extension of Brauer's diagrammatic formalism in [14], via the Tannakian results in [36]. See [10].

Finally, we use the intersection and generation operations $\cap$ and $<,>$ for the closed subgroups of U_N^+, whose axiomatization and basic properties can be found in [16]. We also refer to [8] for a brief overview of the main known computations of $<,>$.

We have the following definition, which is folklore, inspired by [10]:

Definition 1.1. *The soft liberation of a compact Lie group $G_N\subset U_N$ is the compact quantum group $G_N^+\subset U_N^+$ given by*

$$G_N^+=<G_N,S_N^+>$$

where S_N^+ is Wang's quantum permutation group, and where $<,>$ denotes the topological generation operation for the closed subgroups of U_N^+.

The terminology here comes from the fact that, as we will see soon, there is as well a second liberation operation, the "hard" one. For more on this, see [3].

As already mentioned, all this is inspired from [10]. To be more precise, we have the following result, which makes the link with the notion of liberation from there:

Theorem 1.2. *Assuming that we have an easy compact group $S_N\subset G_N\subset U_N$, its soft liberation $S_N^+\subset G_N^+\subset U_N^+$ is easy as well, the corresponding category of partitions being obtained from the one of G_N by removing all the crossing partitions.*

Proof. We use the well-known fact, from [16], that the operations $\cap$ and $<,>$ are dual to each other via the Tannakian correspondence $G\leftrightarrow H$, in the sense that:

$$C_{G\cap H}=<C_G,C_H>\quad,\quad C_{<G,H>}=C_G\cap C_H$$

With standard easy quantum group notations from [10], if we denote by $D\subset P$ the catehory of partitions producing G_N, we therefore have:

$$\begin{aligned}C_{G_N^+}&=C_{G_N}\cap C_{S_N^+}\\&=span(T_\pi|\pi\in D)\cap span(T_\pi|\pi\in NC)\\&=span(T_\pi|\pi\in D\cap NC)\end{aligned}$$

Thus G_N^+ is indeed easy, coming from the category $D\cap NC$. □

Here is as well a secondary statement, making the link with the recent work in [3]:

Proposition 1.3. *Assuming that we are in the case $H_N\subset G_N\subset U_N$, we have*

$$G_N^+=<G_N,H_N^+>$$

in analogy with the soft half-liberation formula $G_N^*=<G_N,H_N^*>$.

Proof. We have indeed, by using a number of standard facts on the topological generation operation $<,>$, for which we refer for instance to [8]:

$$\begin{aligned} G_N^+ &= < G_N, S_N^+ > \\ &= << G_N, H_N >, S_N^+ > \\ &= < G_N, < H_N, S_N^+ >> \\ &= < G_N, H_N^+ > \end{aligned}$$

As for the last assertion, this is rather a remark. □

Summarizing, our soft liberation theory extends the "easy" liberation theory from [10], and is compatible with the notion of soft half-liberation from [4].

2. Easy envelopes

Our purpose now is to discuss the soft liberation of the compact Lie groups $G \subset U_N$, in the non-easy case. As a first observation, we have the following result:

Proposition 2.1. *For any compact Lie group $G \subset U_N$ we have the formula*

$$G_N^+ =< G_N, S_N >^+$$

where $< G_N, S_N > \subset U_N$ is the compact group topologically generated by G_N, S_N.

Proof. We have indeed the following computation:

$$\begin{aligned} G_N^+ &= < G_N, S_N^+ > \\ &= << G_N, S_N >, S_N^+ > \\ &= < G_N, S_N >^+ \end{aligned}$$

As for the fact that our operation $<,>$ for quantum groups coincides with the usual topological generation operation $<,>$ in the classical case, here we refer to [16]. □

As a consequence of the above result, we can assume in practice that the compact group G_N to be liberated appears as an intermediate subgroup $S_N \subset G_N \subset U_N$.

On the other hand, it is folklore that such compact groups $S_N \subset G_N \subset U_N$ cannot really be "liberated", in some reasonable advanced sense, unless they are easy. However, this is just folklore knowledge, coming from the study of various examples, and on the impossibility of doing something with them, and not based on any general theorem.

Our notion of soft liberation allows a conceptual take on this old problem. In order to formulate our answer, we will need the following recent notion, from [2]:

Definition 2.2. *The easy envelope of a compact group $S_N \subset G_N \subset U_N$ is the smallest compact group $S_N \subset \widetilde{G_N} \subset U_N$ containing G_N, and which is easy.*

We refer to [2] for more details regarding this notion, and for some explicit computations, for basic examples of compact Lie groups. Here we will only need the following formula, which is actually the Tannakian definition of the easy envelope:

$$C_{\widetilde{G_N}} = span(T_\pi|\pi \in P, T_\pi \in C_{G_N})$$

We are ready know to formulate an answer to the above-mentioned folklore question regarding the liberation operation. Our answer is conjectural, as follows:

Conjecture 2.3 (Soft liberation conjecture, SLC)**.** *We have the formula*

$$G_N^+ = (\widetilde{G_N})^+$$

valid for any compact group $G_N \subset U_N$, where tilde denotes the easy envelope.

As a first observation, this is a very general statement, not even assuming that we have an intermediate compact group $S_N \subset G_N \subset U_N$, as in the above discussion.

However, the passage to the $S_N \subset G_N \subset U_N$ case is straightforward, as follows:

Proposition 2.4. *Given a closed subgroup $G_N \subset U_N$, we have*

$$\widetilde{G_N} = \widetilde{< G_N, S_N >}$$

and so the SLC holds for G_N if and only if it holds for $< G_N, S_N >$.

Proof. Here the formula in the statement is standard, coming either from the algebraic or Tannakian definition of the easy envelope, and we refer here to [2]. As for the second assertion, this follows from it, and from Proposition 2.1 above. □

Thus, we can assume in what follows that we have $S_N \subset G_N \subset U_N$. In order to explain now what kind of difficulty is behind the SLC, let us begin with:

Proposition 2.5. *The SLC for a group $S_N \subset G_N \subset U_N$ is equivalent to*

$$\lambda_1 T_{\pi_1} + \ldots + \lambda_r T_{\pi_r} \in C_{G_N} \implies T_{\pi_1}, \ldots, T_{\pi_r} \in C_{G_N}$$

for any $r \in \mathbb{N}$, any partitions $\pi_1, \ldots, \pi_r \in NC$, and any scalars $\lambda_1, \ldots, \lambda_r \neq 0$.

Proof. Given a compact group $S_N \subset G_N \subset U_N$ we have inclusions as follows, and these inclusions are equalities if and only if the SLC holds for G_N:

$$G_N^+ \subset (\widetilde{G_N})^+ \quad : \quad C_{(\widetilde{G_N})^+} \subset C_{G_N^+}$$

According to the Tannakian formulation of the easy envelope construction, given after Definition 2.2 above, we have the following formula:

$$\begin{aligned} C_{(\widetilde{G_N})^+} &= C_{\widetilde{G_N}} \cap C_{S_N^+} \\ &= span(T_\pi|\pi \in P, T_\pi \in C_{G_N}) \cap span(T_\pi|\pi \in NC) \\ &= span(T_\pi|\pi \in NC, T_\pi \in C_{G_N}) \end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned} C_{G_N^+} &= C_{G_N} \cap C_{S_N^+} \\ &= C_{G_N} \cap span(T_\pi|\pi \in NC) \end{aligned}$$

Observe that this latter category contains indeed the former, as it should.

The SLC states that this inclusion must be an isomorphism, and this reads:

$$\sum_{\pi \in NC} \lambda_\pi T_\pi \in C_{G_N} \implies \left[T_\pi \in C_{G_N}, \forall \pi \in NC, \lambda_\pi \neq 0 \right]$$

But this is precisely the condition in the statement, and we are done. □

To the untrained eye, the verification of the condition in Proposition 2.5 might look like "standard combinatorics". However, and here comes our point, in compact quantum group theory such statements are quite common, with the difficulty being quite unpredictable, and ranging all over the spectrum, from trivial to undoable.

To be more precise here, the condition in Proposition 2.5 reminds various maximality statements for easy quantum groups, in Tannakian form. We refer to [2], [7] for a discussion of such statements, including some old conjectures, still open.

It is possible to prove the SLC in certain particular cases, by using various tricks from [2], [7]. However, regarding a proof in general, we have no idea here.

Regarding now the representation theory and probabilistic aspects of the soft liberation operation $G_N \to G_N^+$, the subject is very wide, and there are many things that can be said here. As a main statement on the subject, we have the following conjecture:

Conjecture 2.6 (Asymptotic liberation conjecture, ALC). *For a uniform compact group $A_N \subset G_N \subset U_N$, the asymptotic laws of truncated characters*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

with $t \in (0,1]$ form a truncated convolution semigroup for G_N, a truncated free convolution semigroup for G_N^+, and these semigroups are in Bercovici-Pata bijection [11].

In this statement the uniformity condition states that G_N must be part of a family $G = (G_N)$ with $N \in \mathbb{N}$, satisfying $G_{N-1} = G_N \cap U_{N-1}$ for any $N \in \mathbb{N}$. As with the SLC, it is possible to prove this in certain particular cases, usings various well-known facts and tricks. However, regarding a proof in general, we have no idea here.

Summarizing, our soft liberation theory extends [10], is compatible with the notion of soft half-liberation from [4], and conjecturally extends [10], at the probabilistic level.

3. Hard liberation

We have seen so far that the liberation procedure from [10], for the easy compact groups $S_N \subset G_N \subset U_N$, can be abstractly generalized by dropping the easiness assumption, simply by setting $G_N^+ =< G_N, S_N^+ >$, and that this latter construction lands in fact back into the easy case, via a conjecture which clarifies some folklore knowledge on the subject. All this is undoubtedly an advance, although abstract and "negative" in nature.

In what follows we discuss something which is definitely "positive", and technically very useful. Once again, our starting point will be some folklore. To be more precise, we would like to answer the following question, which has been on every lips since [10], and even since [9], or even before since [1], or even [30]: how come that that most obvious "liberation", namely $\mathbb{Z}^N \to F_N$, is not covered by the ongoing quantum group theory?

Traditionally, our answer to this question has always been "we're sorry", with no real idea on how to deal with this question. We will show here that, by suitably "harderning" the soft liberation operation, we will have a potential answer to this longstanding question. In addition, our answer will open up a whole new perspective on what can be done with the compact quantum groups, in relation with various analytic questions.

Let us begin with a key definition, coming from [35], as follows:

Definition 3.1. *Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, the corresponding abstract dual $T = \widehat{\Gamma}$, regarded as a compact quantum subgroup of U_N^+ via the matrix*

$$u = diag(g_1, \ldots, g_N)$$

over the algebra $C(T) = C^*(\Gamma)$, is called noncommutative torus.

All this comes of course from Pontrjagin duality, and we refer to [35] for details. We should perhaps mention here, in relation with an obvious amenability issue, that in what follows we identify closed subgroups $G_N \subset U_N^+$ once we have an isomorphism at the $*$-algebra level, mapping coordinates to coordinates. In this way, all the group algebras of a given group $\Gamma =< g_1, \ldots, g_N >$ give rise to the same torus, denoted $T = \widehat{\Gamma}$.

In relation now with our liberation questions, and with some half-liberation questions that we would like to discuss as well, let us introduce some special tori, as follows:

Definition 3.2. *We have the following examples of tori:*

1. *The real torus $T_N = \widehat{\mathbb{Z}_2^N}$, and its free version $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$.*
2. *The complex torus $\mathbb{T}_N = \widehat{\mathbb{Z}^N}$, and its free version $\mathbb{T}_N^+ = \widehat{\mathbb{Z}^{*N}}$.*

All this deserves some explanations. In the classical case, by Fourier transform, what we have here are the standard tori of $\mathbb{R}^N, \mathbb{C}^N$, the first one being also called cube:

$$T_N = \mathbb{Z}_2^N \quad , \quad \mathbb{T}_N = \mathbb{T}^N$$

In the free case now, there is nothing much to say, besides the fact that in the complex case, we obtain a free group dual. Thus, our tori here are as follows:

$$T_N^+ = \widehat{\mathbb{Z}_2^{*N}} \quad , \quad \mathbb{T}_N^+ = \widehat{F_N}$$

We can now introduce our hard liberation operations, as follows:

Definition 3.3. *The real and complex hard liberations of a compact group $G_N \subset U_N$ are*

$$G_N^{+r} = < G_N, T_N^+ > \quad , \quad G_N^{+c} = < G_N, \mathbb{T}_N^+ >$$

where $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$ and $\mathbb{T}_N^+ = \widehat{F_N}$ are the real and complex free tori.

As a first remark, we have the following result, which shows that we are on the track of solving the above-mentioned longstanding question:

Proposition 3.4. *The group-theoretic standard liberation operations, namely*

$$\mathbb{Z}_2^N \to \mathbb{Z}_2^{*N} \quad , \quad \mathbb{Z}^N \to F_N$$

can be interpreted, at the dual level, as being hard liberations.

Proof. According to our various notations and conventions, the operation $\mathbb{Z}_2^N \to \mathbb{Z}_2^{*N}$ corresponds, at the dual level, to the operation $T_N \to T_N^+$, and the operation $\mathbb{Z}^N \to F_N$ corresponds, at the dual level, to the operation $\mathbb{T}_N \to \mathbb{T}_N^+$. Thus, we have the result. $\square$

As a comment here, the use of complex hard liberations is unavoidable, $\mathbb{T}_N \to \mathbb{T}_N^+$ being not a real hard liberation. Indeed, the dual of $T_N^{+r} = < \mathbb{T}_N, T_N^+ >$ is not the free group F_N itself, because it is subject for instance to the relations $[g_i, g_j^2] = 1$ between generators.

Let us discuss now the case of some basic compact Lie groups. With the convention that $C_N \subset U_N$ is the complex bistochastic group, we have:

Theorem 3.5. *The following conditions are equivalent:*

(1) $U_N^+ = < U_N, T_N^+ >$.
(2) $U_N^+ = < U_N, \mathbb{T}_N^+ >$.
(3) $U_N^+ = < O_N, \mathbb{T}_N^+ >$.
(4) $O_N^+ = < O_N, T_N^+ >$.

In addition, these conditions are equivalent to $U_N^+ = < C_N, \mathbb{T}_N^+ >$.

Proof. All these implications follow from some well-known facts, as follows:

(1) $\implies$ (2) This is trivial, coming from $T_N^+ \subset \mathbb{T}_N^+$.

(2) $\implies$ (3) By using the fact, from [7], that $\mathbb{T}O_N \subset U_N$ is maximal, we have:

$$\begin{aligned} U_N^+ &= < U_N, \mathbb{T}_N^+ > \\ &= << O_N, \mathbb{T}_N >, \mathbb{T}_N^+ > \\ &= < O_N, < \mathbb{T}_N, \mathbb{T}_N^+ >> \\ &= < O_N, \mathbb{T}_N^+ > \end{aligned}$$

(3) $\implies$ (4) This follows from the equality $PO_N^+ = PU_N^+$, from [1], by lifting.

(4) $\implies$ (1) By using the fact that U_N^+ is the soft liberation of U_N, we have:

$$\begin{aligned}
U_N^+ &= < U_N, O_N^+ > \\
&= < U_N, < O_N, T_N^+ >> \\
&= << U_N, O_N >, T_N^+ > \\
&= < U_N, T_N^+ >
\end{aligned}$$

Let us prove now that (2) is equivalent to $U_N^+ =< C_N, \mathbb{T}_N^+ >$. In one sense this is clear. In the other sense, we use the standard fact, from [19], that we have $U_N = \mathbb{T}_N C_N \mathbb{T}_N$. In particular we have $U_N =< C_N, \mathbb{T}_N >$, which leads to the following computation:

$$\begin{aligned}
U_N^+ &= < U_N, \mathbb{T}_N^+ > \\
&= << C_N, \mathbb{T}_N >, \mathbb{T}_N^+ > \\
&= < C_N, < \mathbb{T}_N, \mathbb{T}_N^+ >> \\
&= < C_N, \mathbb{T}_N^+ >
\end{aligned}$$

Thus, the five conditions in the statement are all equivalent. □

Our claim now is that the above conditions hold indeed. There are several potential proofs for this fact, as follows:

(1) Recurrence. The above formulae can be checked by recurrence on N, in the spirit of the computations in [12], [13], [16]. Chirvasitu et al. presently have a complete proof for this fact. We refer here to their future paper on the subject.
(2) Combinatorics. The above formulae can be all approached via Tannakian duality, the question being that of intersecting certain categories. This is reputed to be difficult. Some "warm-up" computations were formulated in [4], [5].
(3) Algebra. The last formula in the above statement, $U_N^+ =< C_N, \mathbb{T}_N^+ >$, can be regarded as being a free version of the weak Idel-Wolf theorem, $U_N =< C_N, \mathbb{T}_N >$. We believe that a potential proof can come along these lines.

In regards now with the conclusions, assuming that the above conditions hold indeed, the situation is as follows:

(1) O_N, U_N. Here everything is fine, with O_N^+ appearing via real hard liberation, and with U_N^+ appearing via real or complex hard liberation.
(2) B_N, C_N. Here our hard liberation methods do not lead to the quantum groups B_N^+, C_N^+ from [10], [24], [26]. We must blow up a Fourier torus here.
(3) S_N. Here the diagonal torus collapses, but according to [5], we can still obtain S_N^+, by blowing up all the standard tori, or perhaps just a part of them.
(4) H_N, K_N. Here we have same issues as with S_N, but once again by [5], the free versions H_N^+, K_N^+ are generated by their standard tori.

Summarizing, we are left with both computational and conceptual questions.

4. Open problems

The present paper is just a beginning, and there are many things to be done:

Problem 4.1. *General theory.*

One interesting problem here is that of understanding whether an arbitrary liberation $G_N \subset G_N^\times$ can be written as $G_N^\times =< G_N, I_N >$, with I_N being "minimal".

Problem 4.2. *Soft liberation.*

This amounts in proving the SLC and ALC. We refer to section 2 for this, and for various related questions, which are all interesting and technical.

Problem 4.3. *Hard liberation.*

There are many problems here, the general idea being that an arbitrary $G_N \subset U_N$ can be hard liberated by blowing up some of its standard tori, carefully chosen.

Problem 4.4. *Intermediate liberations.*

We have here several interesting soft and hard intermediate liberation questions, using input from the recent work in [22], [25].

Problem 4.5. *Free probability.*

We have here, as big open problem, the unification of the Bercovici-Pata bijection with the Meixner/free Meixner correspondence, by using the hard liberation.

Problem 4.6. *Random matrices.*

There are probably some interesting random matrix questions as well, in relation with Wigner's paper [34], and with the subsequent work on the subject.

References

[1] T. Banica, The free unitary compact quantum group, *Comm. Math. Phys.* **190** (1997), 143–172.
[2] T. Banica, Homogeneous quantum groups and their easiness level, *Kyoto J. Math.*, to appear.
[3] T. Banica, Quantum groups under very strong axioms, *Bull. Pol. Acad. Sci. Math.*, to appear.
[4] T. Banica, A note on the half-liberation operation, preprint 2019.
[5] T. Banica, Compact quantum groups generated by their tori, preprint 2019.
[6] T. Banica, J. Bichon and B. Collins, The hyperoctahedral quantum group, *J. Ramanujan Math. Soc.* **22** (2007), 345–384.
[7] T. Banica, J. Bichon, B. Collins and S. Curran, A maximality result for orthogonal quantum groups, *Comm. Algebra* **41** (2013), 656–665.
[8] T. Banica and A. Chirvasitu, Modelling questions for transitive quantum groups, preprint 2017.
[9] T. Banica and B. Collins, Integration over compact quantum groups, *Publ. Res. Inst. Math. Sci.* **43** (2007), 277–302.
[10] T. Banica and R. Speicher, Liberation of orthogonal Lie groups, *Adv. Math.* **222** (2009), 1461–1501.
[11] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory, *Ann. of Math.* **149** (1999), 1023–1060.

[12] M. Brannan, A. Chirvasitu and A. Freslon, Topological generation and matrix models for quantum reflection groups, preprint 2018.

[13] M. Brannan, B. Collins and R. Vergnioux, The Connes embedding property for quantum group von Neumann algebras, *Trans. Amer. Math. Soc.* **369** (2017), 3799–3819.

[14] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. of Math.* **38** (1937), 857–872.

[15] L. Brown, Ext of certain free product C*-algebras, *J. Operator Theory* **6** (1981), 135–141.

[16] A. Chirvasitu, Residually finite quantum group algebras, *J. Funct. Anal.* **268** (2015), 3508–3533.

[17] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic groups, *Comm. Math. Phys.* **264** (2006), 773–795.

[18] A. Connes, Noncommutative geometry, Academic Press (1994).

[19] M. Idel and M.M. Wolf, Sinkhorn normal form for unitary matrices, *Linear Algebra Appl.* **471** (2015), 76–84.

[20] V.F.R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. Math.* **137** (1989), 311–334.

[21] F. Klein, Vergleichende Betrachtungen über neuere geometrische Forschungen, *Math. Ann.* **43** (1893), 63–100.

[22] A. Mang and M. Weber, Categories of two-colored pair partitions, part II: Categories indexed by semigroups, preprint 2019.

[23] K. McClanahan, C*-algebras generated by elements of a unitary matrix, *J. Funct. Anal.* **107** (1992), 439–457.

[24] S. Raum, Isomorphisms and fusion rules of orthogonal free quantum groups and their complexifications, *Proc. Amer. Math. Soc.* **140** (2012), 3207–3218.

[25] S. Raum and M. Weber, The full classification of orthogonal easy quantum groups, *Comm. Math. Phys.* **341** (2016), 751–779.

[26] P. Tarrago and M. Weber, Unitary easy quantum groups: the free case and the group case, *Int. Math. Res. Not.* **18** (2017), 5710–5750.

[27] N.H. Temperley and E.H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem, *Proc. Roy. Soc. London* **322** (1971), 251–280.

[28] A. Van Daele and S. Wang, Universal quantum groups, *Internat. J. Math.* **7** (1996), 255–263.

[29] D. Voiculescu, Addition of certain noncommuting random variables, *J. Funct. Anal.* **66** (1986), 323–346.

[30] S. Wang, Free products of compact quantum groups, *Comm. Math. Phys.* **167** (1995), 671–692.

[31] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.

[32] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, *J. Math. Phys.* **19** (1978), 999–1001.

[33] H. Weyl, The classical groups: their invariants and representations, Princeton (1939).

[34] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* **62** (1955), 548–564.

[35] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[36] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups, *Invent. Math.* **93** (1988), 35–76.

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