Source-Free Classical Electromagnetism, the Free-Photon Schrödinger Equation, and the Unphysical Conjugate-Pair Solutions of the Klein-Gordon and Dirac Equations

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Abstract

This tutorial begins with the relationship of source-free classical electromagnetism to ultra-relativistic free-photon quantum mechanics. The linear transformation of the source-free classical-electromagnetic real-valued transverse vector potential to its corresponding free-photon Schrödinger-equation complex-valued transverse vector wave function is obtained. It is then pointed out that despite the free-photon Klein-Gordon equation's being formally identical to the source-free classical-electromagnetic vector-potential wave equation, it yields not only free-photon Schrödinger-equation wave functions but also their complex conjugates, which don't satisfy the free-photon Schrödinger equation. This is a consequence of admitting complex-valued solutions of the Klein-Gordon equation—of course only its real-valued solutions apply to the classical vector potential. It is pointed out that solutions of the free-particle Dirac equation likewise occur in conjugate pairs, and that its Hamiltonian operator implies a variety of unphysical consequences, e.g., any Dirac free particle's speed is that of light times the square root of three.

Source-free classical electromagnetism's vector potential wave equation

The Maxwell field equations of source-free electromagnetism are,

$$\nabla\cdot\mathbf{E}=0,\qquad \nabla\times\mathbf{E}+(1/c)\dot{\mathbf{B}}=\mathbf{0},\qquad \nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{B}-(1/c)\dot{\mathbf{E}}=\mathbf{0}. \tag{1a}$$

If we express the transverse $\mathbf{B}$ and the transverse $\mathbf{E}$ in terms of a transverse vector potential $\mathbf{A}$ as follows,

$$\mathbf{B}=\nabla\times\mathbf{A},\qquad \mathbf{E}=-(1/c)\dot{\mathbf{A}},\qquad \nabla\cdot\mathbf{A}=0, \tag{1b}$$

then the *first three* source-free Maxwell field equations given by Eq. (1a) are all satisfied. The insertion of Eqs. (1b) into the *last* Eq. (1a) source-free Maxwell field equation then yields the following transverse vector-potential wave equation,

$$\big[(1/c)^2(\partial/\partial t)^2-\nabla^2\big]\mathbf{A}(\mathbf{r},\,t)=\mathbf{0},\ \text{ where } \nabla\cdot\mathbf{A}(\mathbf{r},\,t)=0. \tag{1c}$$

The free-photon Schrödinger equation

The relativistic Hamiltonian operator for a free particle of mass m is,

$$H=\big(m^2c^4+|c\,\mathbf{p}|^2\big)^{\frac{1}{2}}, \tag{2a}$$

which in the massless free-photon case reduces to,

$$H=|c\,\mathbf{p}|=c(\mathbf{p}\cdot\mathbf{p})^{\frac{1}{2}}. \tag{2b}$$

This Hamiltonian operator yields the free-photon velocity and speed,

$$\dot{\mathbf{r}}=(-i/\hbar)\big[\mathbf{r},\,c(\mathbf{p}\cdot\mathbf{p})^{\frac{1}{2}}\big]=\nabla_{\mathbf{p}}\big(c(\mathbf{p}\cdot\mathbf{p})^{\frac{1}{2}}\big)=c\,\mathbf{p}/(\mathbf{p}\cdot\mathbf{p})^{\frac{1}{2}}=c\,\mathbf{p}/|\mathbf{p}|\ \Rightarrow\ |\dot{\mathbf{r}}|=c, \tag{2c}$$

and also the free-photon Schrödinger equation,

$$i\hbar(\partial\boldsymbol{\Psi}/\partial t)=c(\mathbf{p}\cdot\mathbf{p})^{\frac{1}{2}}\boldsymbol{\Psi},\ \text{ where } \mathbf{p}\cdot\boldsymbol{\Psi}=0, \tag{2d}$$

because a free photon's wave function $\boldsymbol{\Psi}$ is a transverse vector entity.

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In configuration representation $\mathbf{p} = -i\hbar\nabla$, so Eq. (2d) becomes,

$$\left[(i/c)(\partial/\partial t) - \left(-\nabla^2\right)^{\frac{1}{2}}\right]\mathbf{\Psi}(\mathbf{r},\,t) = \mathbf{0},\ \text{where}\ \nabla\cdot\mathbf{\Psi}(\mathbf{r},\,t) = 0. \tag{2e}$$

Since,

$$\left[(i/c)(\partial/\partial t) - \left(-\nabla^2\right)^{\frac{1}{2}}\right]\left[-(i/c)(\partial/\partial t) - \left(-\nabla^2\right)^{\frac{1}{2}}\right] = \left[(1/c)^2(\partial/\partial t)^2 - \nabla^2\right], \tag{2f}$$

we have the following homogeneous linear transformation of the source-free classical-electromagnetic real-valued transverse vector potential $\mathbf{A}(\mathbf{r},\,t)$ of Eq. (1c) to its corresponding free-photon Schrödinger-equation complex-valued transverse vector wave function $\mathbf{\Psi}(\mathbf{r},\,t)$ of Eq. (2e),

$$\mathbf{\Psi}(\mathbf{r},\,t) = \left[-(i/c)(\partial/\partial t) - \left(-\nabla^2\right)^{\frac{1}{2}}\right]\left(-\nabla^2\right)^{-\frac{1}{4}}\mathbf{A}(\mathbf{r},\,t)/(\hbar c)^{\frac{1}{2}}, \tag{2g}$$

where the two additional factors $(-\nabla^2)^{-\frac{1}{4}}$ and $(1/(\hbar c)^{\frac{1}{2}})$ which appear in Eq. (2g) but not in Eq. (2f) are present to reconcile the fact that the dimension of the source-free transverse vector potential $\mathbf{A}(\mathbf{r},\,t)$ is the square root of the quotient of energy divided by length, whereas the dimension of the free photon's transverse vector wave function $\mathbf{\Psi}(\mathbf{r},\,t)$ is of course the square root of inverse volume. Eqs. (1c), (2d), (2e), (2f) and (2g) show that source-free classical electromagnetism and ultra-relativistic free-photon Schrödinger-equation quantum mechanics are one and the same.

The Klein-Gordon equation's unphysical complex-conjugate solution pairs

If we multiply the free-photon Schrödinger equation of Eq. (2e) by the factor $[-(i/c)(\partial/\partial t) - (-\nabla^2)^{\frac{1}{2}}]$, we see from Eq. (2f) that the result is,

$$\left[(1/c)^2(\partial/\partial t)^2 - \nabla^2\right]\mathbf{\Psi}(\mathbf{r},\,t) = \mathbf{0},\ \text{where}\ \nabla\cdot\mathbf{\Psi}(\mathbf{r},\,t) = 0, \tag{3a}$$

namely the Klein-Gordon equation for $\mathbf{\Psi}(\mathbf{r},\,t)$. The Eq. (3a) Klein-Gordon equation is obviously satisfied by any solution of the Eq. (2e) free-photon Schrödinger equation, and it is equally obvious that it is *also* satisfied *by the complex-conjugate* of any such solution. Those *complex conjugates* $\mathbf{\Psi}^*(\mathbf{r},\,t)$ of solutions $\mathbf{\Psi}(\mathbf{r},\,t)$ of Eq. (2e) free-photon Schrödinger equation obviously satisfy *the entirely distinct complex-conjugate of the* Eq. (2e) *free-photon Schrödinger equation, namely*,

$$\left[-(i/c)(\partial/\partial t) - \left(-\nabla^2\right)^{\frac{1}{2}}\right]\mathbf{\Psi}^*(\mathbf{r},\,t) = \mathbf{0},\ \text{where}\ \nabla\cdot\mathbf{\Psi}^*(\mathbf{r},\,t) = 0, \tag{3b}$$

and therefore are incompatible with the Eq. (2e) *free-photon Schrödinger equation itself*. Thus the Klein-Gordon Eq. (3a) always has extraneous solutions which are complex conjugates of solutions of the free-photon Schrödinger equation, *and are incompatible with that equation.* Although the Eq. (1c) source-free classical-electromagnetic transverse vector potential wave equation *is identical in form to the Klein-Gordon* Eq. (3a), the fact that its solutions $\mathbf{A}(\mathbf{r},\,t)$ are rigidly stipulated to be *real-valued* obviously *completely prevents complex-conjugate solution pairs from being in play.*

Conjugate solution pairs *are* a prominent feature of the free-particle Dirac equation, however.

The Dirac equation's conjugate solution pairs and unphysical particle speed

Up to this point we have been discussing free photons, which have zero mass and are described by transverse vector wave functions, but in this section it will be convenient to instead discuss relativistic free particles whose mass m isn't necessarily zero and whose wave-function characteristics are generic rather than particular, so we shall denote wave functions by using the generic symbol ψ. We therefore begin by considering the Schrödinger equation for the wave function ψ_S, which has the relativistic free-particle Hamiltonian operator $H = (m^2c^4 + |c\,\mathbf{p}|^2)^{\frac{1}{2}}$ given by Eq. (2a),

$$i\hbar(\partial\psi_S/\partial t) = \left(m^2c^4 + |c\,\mathbf{p}|^2\right)^{\frac{1}{2}}\psi_S. \tag{4a}$$

It is convenient to rewrite Eq. (4a) in a form which is similar to that of Eq. (2e),

$$\left[(i/c)(\partial/\partial t) - \left(|\mathbf{p}/\hbar|^2 + \mu^2\right)^{\frac{1}{2}}\right]\psi_S = 0,\ \text{where}\ \mu \stackrel{\text{def}}{=} (mc/\hbar). \tag{4b}$$

Since $\mathbf{p} = -i\hbar\nabla$ in configuration representation, $|\mathbf{p}/\hbar|^2$ is $-\nabla^2$ in that representation. Thus analogously to Eq. (2f) we have that,

$$\Big[-(i/c)(\partial/\partial t) - \big(|\mathbf{p}/\hbar|^2 + \mu^2\big)^{\frac{1}{2}}\Big]\Big[(i/c)(\partial/\partial t) - \big(|\mathbf{p}/\hbar|^2 + \mu^2\big)^{\frac{1}{2}}\Big] = \big[(1/c)^2(\partial/\partial t)^2 + |\mathbf{p}/\hbar|^2 + \mu^2\big], \tag{4c}$$

which implies that ψ_S satisfies the following Klein-Gordon equation that extends the Klein-Gordon equation of Eq. (3a) to free particles which have nonzero mass,

$$\big[(1/c)^2(\partial/\partial t)^2 + |\mathbf{p}/\hbar|^2 + \mu^2\big]\,\psi_S = 0. \tag{4d}$$

Unlike the Eq. (2a) relativistic free-particle $H = (m^2c^4 + |c\,\mathbf{p}|^2)^{\frac{1}{2}}$, the free-particle Dirac Hamiltonian,

$$H_D = c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2, \tag{5a}$$

deliberately eliminates the square root because its presence balks solution by separation of variables in cases such as that of the hydrogen atom where a central potential is added to the free-particle Hamiltonian. Simultaneously the algebraic properties of the coefficients $\vec{\alpha}$ and β of the Eq. (5a) free-particle Dirac Hamiltonian H_D are specifically chosen to make $(H_D)^2 = H^2 = |c\,\mathbf{p}|^2 + m^2c^4$, so that H_D implies exactly the same Eq. (4d) Klein-Gordon equation as is obtained from the Eq. (2a) $H = (m^2c^4 + |c\,\mathbf{p}|^2)^{\frac{1}{2}}$. The algebraic properties of $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β which make $(H_D)^2 = H^2 = |c\,\mathbf{p}|^2 + m^2c^4$ are,

$$(\alpha_1)^2 = (\alpha_2)^2 = (\alpha_3)^2 = \beta^2 = 1, \text{ and in addition } \alpha_1, \alpha_2, \alpha_3 \text{ and } \beta \text{ all mutually anticommute.} \tag{5b}$$

It is furthermore assumed that α_1, α_2, α_3 and β are all Hermitian so that H_D is Hermitian. These algebraic properties imply that the entity $(-i/2)(\vec{\alpha}\times\vec{\alpha})$ has all of the algebraic properties of the Pauli vector operator $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Since the above algebraic properties of $\vec{\alpha}$ and β can be realized by 4×4 matrices, Dirac wave functions are conventionally assumed to be four-component spinors.

In the discussion underneath Eq. (3a) it was pointed out that in configuration representation the complex conjugate of any solution of the Klein-Gordon equation as well satisfies the Klein-Gordon equation; this is clearly *also* the case for the *extended* Eq. (4d) version of the Klein-Gordon equation, since $|\mathbf{p}/\hbar|^2 = -\nabla^2$ in configuration representation. It was also pointed out in connection with Eq. (3b) that these complex conjugates of solutions of the Klein-Gordon equation very frequently *aren't* in fact solutions of the Schrödinger equation which is the precursor of the Klein-Gordon equation; the Klein-Gordon equation has the *defect* of being beset *by a plethora of extraneous solutions* that arise *from repetitious application of operators*, such as that illustrated in Eq. (4c). We now show that *the algebraic properties of* α_1, α_2, α_3 *and* β very similarly cause *any solution* of the free-particle Dirac equation *to be beset by a conjugate partner*.

The free-particle Dirac equation,

$$i\hbar(\partial\psi_D/\partial t) = \big(c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2\big)\,\psi_D. \tag{6a}$$

is conveniently reexpressed in a form similar to that of Eq. (4b),

$$[(i/c)(\partial/\partial t) - (\vec{\alpha}\cdot(\mathbf{p}/\hbar) + \beta\mu)]\psi_D = 0. \tag{6b}$$

Since $\vec{\alpha}$, $\mathbf{p}$ and β are all Hermitian operators, the Hermitian conjugate of Eq. (6b) is,

$$\psi_D^\dagger[-(i/c)(\partial/\partial t) - (\vec{\alpha}\cdot(\mathbf{p}/\hbar) + \beta\mu)] = 0. \tag{6c}$$

We now define the operator α_5 as,

$$\alpha_5 \stackrel{\text{def}}{=} \alpha_1\alpha_2\alpha_3\beta, \tag{6d}$$

and note that α_5 is Hermitian because, since α_1, α_2, α_3 and β are Hermitian,

$$\alpha_5^\dagger = \beta\alpha_3\alpha_2\alpha_1 = -\alpha_3\alpha_2\alpha_1\beta = -\alpha_2\alpha_1\alpha_3\beta = \alpha_1\alpha_2\alpha_3\beta = \alpha_5. \tag{6e}$$

We also note that $(\alpha_5)^2 = 1$ because,

$$(\alpha_5)^2 = \alpha_5\alpha_5^\dagger = \alpha_1\alpha_2\alpha_3\beta\beta\alpha_3\alpha_2\alpha_1 = 1. \tag{6f}$$

Finally, it is obvious that α_5 *anticommutes* with α_1, α_2, α_3 and β, e.g.,

$$\alpha_5\beta = \alpha_1\alpha_2\alpha_3\beta\beta = \alpha_1\alpha_2\alpha_3 \ \text{ and } \ \beta\alpha_5 = \beta\alpha_1\alpha_2\alpha_3\beta = -\alpha_1\alpha_2\alpha_3\beta\beta = -\alpha_1\alpha_2\alpha_3 \ \text{ so } \ \alpha_5\beta + \beta\alpha_5 = 0. \quad \text{(6g)}$$

We now multiply Eq. (6c) on the right by α_5 to produce,

$$\psi_D^\dagger[-(i/c)(\partial/\partial t) - (\vec{\alpha}\cdot(\mathbf{p}/\hbar) + \beta\mu)]\alpha_5 = 0. \quad \text{(6h)}$$

Using the fact that α_5 *anticommutes* with $\vec{\alpha}$ and β allows us to reexpress Eq. (6h) as,

$$\psi_D^\dagger\alpha_5[-(i/c)(\partial/\partial t) + (\vec{\alpha}\cdot(\mathbf{p}/\hbar) + \beta\mu)] = 0. \quad \text{(6i)}$$

Multiplying Eq. (6i) by (-1) produces,

$$(\psi_D^\dagger\alpha_5)[(i/c)(\partial/\partial t) - (\vec{\alpha}\cdot(\mathbf{p}/\hbar) + \beta\mu)] = 0. \quad \text{(6j)}$$

Comparison of Eq. (6j) with Eq. (6b) shows that *if* ψ_D *satisfies the free-particle Dirac equation*, then the particular *conjugate* of ψ_D which is given by $(\psi_D^\dagger\alpha_5)$ *also satisfies the free-particle Dirac equation*. Note that $(\psi_D^\dagger\alpha_5)$ *is indeed a particular conjugate of* ψ_D *because*,

$$\alpha_5(\psi_D^\dagger\alpha_5)^\dagger = \alpha_5(\alpha_5\psi_D) = (\alpha_5)^2\psi_D = \psi_D, \quad \text{(6k)}$$

where we have used the facts that α_5 is Hermitian and its square is unity.

Thus the Dirac equation is much more akin to the Klein-Gordon Eq. (4d) than it is to the Schrödinger Eq. (4b); *no analogous solution conjugation theorem applies to the Schrödinger* Eqs. (4b) *and* (4a). In fact, it can *in addition* be shown that the Dirac Hamiltonian $H_D = c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2$ of Eq. (5a) *egregiously violates tenets of special relativity which are upheld by the Hamiltonian* $H = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}}$ *of* Eq. (2a) *that is at the heart of the Schrödinger* Eqs. (4a) *and* (4b). Analogous to the free-photon velocity and speed obtained in Eq. (2c) from the specialized free-photon Hamiltonian given by Eq. (2b), we now apply the relativistic Hamiltonian of Eq. (2a) for free particles of mass $m \geq 0$ to obtain their relativistic velocity and speed,

$$\begin{gathered}\dot{\mathbf{r}} = (-i/\hbar)\Big[\mathbf{r},\, \big(m^2c^4 + |c\mathbf{p}|^2\big)^{\frac{1}{2}}\Big] = c\nabla_{\mathbf{p}}\big((mc)^2 + (\mathbf{p}\cdot\mathbf{p})\big)^{\frac{1}{2}} = \\ c\,\mathbf{p}/\big((mc)^2 + |\mathbf{p}|^2\big)^{\frac{1}{2}} \ \Rightarrow \ |\dot{\mathbf{r}}| = c|\mathbf{p}|/\big((mc)^2 + |\mathbf{p}|^2\big)^{\frac{1}{2}} \leq c.\end{gathered} \quad \text{(7a)}$$

Thus the relativistic free-particle Hamiltonian $H = (m^2c^4 + |c\,\mathbf{p}|^2)^{\frac{1}{2}}$ of Eq. (2a) yields that the free particle's velocity $\dot{\mathbf{r}}$ is always parallel to its momentum $\mathbf{p}$, and its speed $|\dot{\mathbf{r}}|$ never exceeds c. The Dirac Hamiltonian $H_D = c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2$ of Eq. (5a) however yields,

$$\begin{gathered}\dot{\mathbf{r}} = (-i/\hbar)\left[\mathbf{r},\, c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2\right] = \nabla_{\mathbf{p}}\big(c\vec{\alpha}\cdot\mathbf{p} + \beta mc^2\big) = c\vec{\alpha} \ \Rightarrow \\ |\dot{\mathbf{r}}| = c|\vec{\alpha}| = c\sqrt{(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2} = c\sqrt{1+1+1} = c\sqrt{3} = 1.732c > c,\end{gathered} \quad \text{(7b)}$$

which *egregiously contradicts* the relativistic tenets that a free particle's velocity $\dot{\mathbf{r}}$ is always parallel to its momentum $\mathbf{p}$, and that its speed $|\dot{\mathbf{r}}|$ never exceeds c. In fact, the Dirac Hamilton result that $\dot{\mathbf{r}} = c\alpha$ goes *beyond* merely contradicting special relativity; it implies that the three observable components $(\dot{\mathbf{r}})_1 = c\alpha_1$, $(\dot{\mathbf{r}})_2 = c\alpha_2$ and $(\dot{\mathbf{r}})_3 = c\alpha_3$ of a free particle's velocity *anticommute with each other even in the limit that* $\hbar \to 0$, which flatly contradicts the correspondence-limit requirement of quantum mechanics that observables must all commute with each other in the limit that $\hbar \to 0$. Violation of the quantum mechanics correspondence principle by the free-particle Dirac Hamiltonian H_D is part and parcel of the spontaneous acceleration of Dirac free particles which is known as *zitterbewegung*. The magnitude of this *zitterbewegung* spontaneous acceleration is of order $(mc^3/\hbar)$ for a zero-momentum Dirac free particle of mass m, which for an electron works out to approximately $10^{28}g$, where g equals 9.8 meters per second squared, the acceleration of gravity at the earth's surface. Of course the spontaneous *zitterbewegung* acceleration magnitude $(mc^3/\hbar)$ goes to *infinity* in the classical correspondence limit $\hbar \to 0$.