On Thermal Relativity, Modified Hawking Radiation, and the Generalized Uncertainty Principle

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March 2019

Abstract

After a brief review of the thermal relativistic *corrections* to the Schwarzschild black hole entropy, it is shown how the Stefan-Boltzman law furnishes large modifications to the evaporation times of Planck-size mini-black holes, and which might furnish important clues to the nature of dark matter and dark energy since one of the novel consequences of thermal relativity is that black holes do *not* completely evaporate but leave a Planck size remnant. Equating the expression for the modified entropy (due to thermal relativity corrections) with Wald's entropy should in principle determine the functional form of the modified gravitational Lagrangian $\mathcal{L}(R_{abcd})$. We proceed to derive the generalized uncertainty relation which corresponds to the effective temperature $T_{eff} = T_H(1 - \frac{T_H^2}{T_P^2})^{-1/2}$ associated with thermal relativity and given in terms of the Hawking (T_H) and Planck (T_P) temperature, respectively. Such modified uncertainty relation agrees with the one provided by string theory up to first order in the expansion in powers of $\frac{(\delta p)^2}{M_P^2}$. Both lead to a minimal length (Planck size) uncertainty. Finally, an explicit analytical expression is found for the modifications to the purely thermal spectrum of Hawking radiation which could cast some light into the resolution of the black hole information paradox.

Keywords : Thermal Relativity; Gravity; Black Holes; Entropy; Generalized Uncertainty Principle; Born Reciprocity; Dark Matter; Finsler Geometry. PACS : 04.50.Kd; 11.10.Kk; 14.80.-j

Recently we derived the *exact* thermal relativistic *corrections* to the Schwarzschild, Reissner-Nordstrom, Kerr-Newman black hole entropies, and provided a detailed analysis of the many *novel* applications and consequences of thermal relativity to the physics of black holes, quantum gravity, minimal area, minimal mass, Yang-Mills mass gap, information paradox, arrow of time, dark matter, and dark energy [1]. The deep origins of the connection between Black Holes and Thermodynamics is still a mystery (to our knowledge). As pointed out by [8], the idea of describing classical thermodynamics using geometric approaches has a long history. Among various treatments, Weinhold [3] used the Hessian of internal energy to define a metric for thermodynamic fluctuations, Ruppeiner [4] used the Hessian of entropy for the same purpose. More recently, Quevedo [5] introduced a formalism called Geometrothermodynamics (GTD) which also introduces metric structures on the configuration space $\mathcal{E}$ of the thermodynamic equilibrium states spanned by all the extensive variables.

Another fact that was missing is that the above authors (to my knowledge) did not realize that their constructions are particular examples of the many important applications of Finsler geometry [6], to the field of Thermodynamics, contact geometry and a vast number of many other topics [7] . Zhao [8] was able to outline the essential principles of Thermal Relativity; i.e. invariance under the group $\mathcal{G}$ of general coordinate transformations on the thermodynamic configuration space, and *introduced* a metric with a *Lorentzian* signature on the space. The line element was identified as the square of the *proper entropy*. Thus the first and second law of thermodynamics admitted an invariant formulation under general coordinate transformations, which justified the foundations for the principle of Thermal Relativity.

In our case above, one may implement Zhao's formulation [8] of Thermal Relativity in the flat analog of Minkowski space as

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ - \ (dM)^2 \ \leftrightarrow \ (d\tau)^2 \ = \ (cdt)^2 \ - \ (dx)^2 \tag{1}$$

The maximal Planck temperature T_P plays the role of the speed of light, and $\mathbf{s}$ is the so-called *proper entropy* which is invariant under the thermodynamical version of Lorentz transformations [8]. Note the $\mathbf{s} \leftrightarrow \tau$ correspondence. Thus the flow of the proper entropy $\mathbf{s}$ is consistent with the arrow of time.

The left hand side of (1) yields, after recurring to the first law of Thermodynamics $TdS = dM \Rightarrow T = \frac{dM}{dS}$,

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ \left(\ 1 \ - \ \frac{T^2}{T_P^2} \ \right) \ \Rightarrow \ (d\mathbf{s}) \ = \ (T_P \ dS) \ \sqrt{\left(\ 1 \ - \ \frac{T^2}{T_P^2} \ \right)} \ =$$

$$T_P \ (\frac{dM}{T}) \ \sqrt{\left(\ 1 \ - \ \frac{T^2}{T_P^2} \ \right)} \ \Rightarrow \ dM \ = \ \frac{T}{T_P} \ \frac{1}{\sqrt{1 - \frac{T^2}{T_P^2}}} \ d\mathbf{s} \tag{2}$$

Eq-(1) allowed to *derive* the thermal relativistic corrections to the Black Hole Entropy [1] as follows. Given the thermal dilation factor one can always define an "effective" temperature by

$$T_{eff} \ = \ \frac{T}{\sqrt{1 - \frac{T^2}{T_P^2}}} \tag{3}$$

such that $dM = \gamma(T)T(d\mathbf{s}/T_P)$ becomes then the thermal relativistic analog of the Energy-Momentum relations $E = m_o c^2(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, $\vec{p} = m_o \vec{v}(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ in Special Relativity, in terms of the rest mass m_o, velocity v, and maximal speed of light c.

After renaming $\tilde{S} \equiv (\mathbf{s}/T_P)$, in terms of the *proper* entropy $\mathbf{s}$, the first law of black hole thermal-relativity dynamics $dM = \gamma(T_H)T_H d\tilde{S}$ yields the *corrected* entropy

$$\int_{\tilde{S}_o}^{\tilde{S}} d\tilde{S} \ = \ \tilde{S} - \tilde{S}_o \ = \ \int_{M_o}^{M} \frac{dM}{\gamma(T_H)T_H} \ = \ \int_{M_o}^{M} dM \ \frac{\sqrt{1 - (T_H^2/T_P^2)}}{T_H} \tag{4}$$

inserting the expression for the Hawking temperature $T_H(M) = (8\pi GM)^{-1}$ into eq-(4), and after setting $(T_P)^{-2} = (M_P)^{-2} = L_P^2 = G$, yields the following integral

$$\tilde{S} - \tilde{S}_o \ = \ \int_{M_o}^{M} dM \ (8\pi GM) \sqrt{1 - \frac{G}{(8\pi GM)^2}} \ = \ \int_{M_o}^{M} dM \ \sqrt{(8\pi GM)^2 \ - \ G} \tag{5}$$

The indefinite integral

$$\int dx \ \sqrt{a^2x^2 \ - \ b} \ = \ \frac{ax \ \sqrt{a^2x^2 \ - \ b}}{2a} \ - \ \frac{b}{2a} \ ln \left(\ a \ [\sqrt{a^2x^2 \ - \ b} \ + \ ax] \ \right) \tag{6}$$

permits to evaluate the definite integral in the right hand side of (5) between the *upper* limit M, and a *lower* limit M_o defined by $(8\pi GM_o)^2 - G = 0$, giving

$$\tilde{S} - \tilde{S}_o \ = \ \frac{A}{4G} \sqrt{1 - \frac{1}{16\pi}(\frac{A}{4G})^{-1}} - \frac{1}{16\pi} \ ln \left(\ 4\sqrt{\pi} \ (\frac{A}{4G})^{\frac{1}{2}} \ [\ 1 \ + \ \sqrt{1 - \frac{1}{16\pi}(\frac{A}{4G})^{-1}} \ ] \ \right) \tag{7}$$

after using the relation for the ordinary entropy in the Schwarzschild black hole (adopting the units $\hbar = c = k = 1$)

$$S = \frac{A}{4G} \ = \ 4\pi GM^2 \ \Rightarrow \ M \ = \ (\frac{A}{16\pi G^2})^{\frac{1}{2}} \tag{8}$$

and $(8\pi GM_o)^2 = G \Rightarrow 8\pi GM_o = \sqrt{G}$. The lower limit M_o of integration is required in eq-(5) to ensure the terms inside the square root are positive definite and the integral is real-valued. The value $\tilde{S}_o = \tilde{S}(A_o) = 0$ is the zero *modified* entropy associated to the minimal area $A_o = \frac{G}{4\pi} = \frac{L_P^2}{4\pi}$.

One could then ask what is the *modified* gravitational action which corresponds to the corrected (proper) entropy found in eq-(7). Equating Wald's entropy (a Noether charge) [9]

$$S_{Wald} \sim \int \frac{\partial \mathcal{L}}{\partial R_{abcd}} \; n^{ab} \; n^{cd} \; d\Omega \tag{9}$$

with the expression for the modified entropy found in eq-(7) should in principle determine the functional form of modified gravitational Lagrangian $\mathcal{L}(R_{abcd})$ that would reproduce the entropy (7). The integral (9) is defined over the bifurcate horizon and n^{ab} are the binormals to the horizon.

Let us evaluate now the modifications to the black hole emission rate. Assuming the black hole radiates photons according to the Stefan-Boltzman law $P = A\sigma T^4$, the rate of mass loss through the horizon area $A = 4\pi r_s^2$ is

$$\frac{dM}{dt} \; = \; - \, A \, \sigma \, T^4, \;\; \sigma \; = \; \frac{\pi^2 k^4}{60 \hbar^3 c^2}, \;\; A \; = \; 4\pi (r_s)^2 \; = \; 4\pi (2GM)^2, \;\; T \; = \; T_H \; = \; \frac{1}{8\pi G M} \tag{10}$$

upon integrating eq-(10) yields the evaporation time

$$t \; = \; \frac{16 \pi^3 G^2}{\sigma} \; \frac{M^3}{3} \tag{11}$$

A solar-mass black hole's evaporation time is of the order of $G^2 M^3 = (\frac{M}{M_P})^3 t_P \sim 10^{62}$ years[1] which is much greater than the age of the universe.

The thermal relativistic corrections to the emission rate are simply obtained by replacing T in the Stefan-Boltzman law for the effective $T_{eff} = T(1 - \frac{T^2}{T_P^2})^{-1/2}$, and by setting the end point of evaporation to the minimal of mass $M_o \equiv \frac{M_P}{8\pi}$. The modified expression for the evaporation time becomes

$$\tilde t \; = \; \frac{16 \pi^3 G^2}{\sigma} \; \left(\; \frac{M^3}{3} \; - \; 2 M_o^2 M \; - \; \frac{M_o^4}{M} \; + \; \frac{8 M_o^3}{3} \; \right) \tag{12}$$

Taking the *ratio* of the expressions (11,12) gives

$$\frac{\tilde t}{t} \; = \; 1 \; - \; 6 (\frac{M_o}{M})^2 \; + \; 8 (\frac{M_o}{M})^3 \; - \; 3 (\frac{M_o}{M})^3 \tag{13}$$

from which one learns that for large masses $M >> M_o$, $\frac{\tilde t}{t} \simeq 1$ and the corrections are *negligible*. However for small masses $M \sim M_o$ (Planck size mini-black holes) the ratio is much smaller $\frac{\tilde t}{t} << 1$, consequently the mini-black holes evaporate much *faster* than before, and their lifetimes are much shorter.

This fact can have important consequences for Dark Matter. The possibility that the dark matter comprises primordial black holes (PBHs) has been considered by many [14]. While there exist various candidates, the nature of dark matter remains unresolved. It has been argued that the generalized uncertainty principle (GUP) may prevent a black hole from evaporating completely, and as a result there should exist a Planck-size black hole remnant at the end of its evaporation [15]. If a sufficient amount of small black holes can be produced in

[1] t_p is the Planck time

the early universe, then the resultant black hole remnants can be an interesting candidate for Dark Matter [14]. Because above we also have found a minimal black hole mass remnant of mass M_o, for this reason we shall analyze next the GUP and its connection to thermal relativity.

Let us begin with the stringy uncertainty relation [10] in $\hbar = c = k = 1$ units

$$\delta x \; \delta p \;\geq\; \frac{1}{2} \;+\; \beta \; \frac{(\delta p)^2}{M_P^2}, \quad M_P = T_P \tag{14}$$

and follow the *heuristic* derivation of the modifications to the Hawking temperature described by [11]. The position uncertainty of photons emitted by the static spherically symmetric black hole is of the order of the Schwarzschild diameter (radius) $\delta x \sim 2r_s \sim 4GM$. The momentum uncertainty is represented by the characteristic energy of the emitted photons $\delta p \sim p = E$. According to the equipartition theorem the energy can be identified with the temperature, hence $\delta p \sim p = E = T$. If one sets the proportionality factor $\delta x \sim 2r_s$ as $\delta x = 2\pi r_s = 4\pi GM$, the stringy uncertainty relation (14) can be expressed in terms of the Hawking temperature $T_H = \frac{1}{8\pi GM}$ as follows

$$\frac{1}{2T_H} \;\geq\; \frac{1}{2T} \;+\; \beta \frac{T}{T_P^2} \tag{15}$$

Focusing only on the equal sign, the last equation yields T_H in terms of T. Inverting it gives T in terms of T_H

$$T \;=\; T(T_H) \;=\; \frac{T_P^2}{4\beta T_H} \; \left(1 \;-\; \sqrt{1 \;-\; 8\beta \; \frac{T_H^2}{T_P^2}} \right) \tag{16a}$$

Note that a *minus* sign must be chosen in front of the square root in (16a) otherwise $T \to \infty$ in the $\beta \to 0$ limit. Eq-(16a) in turn can be rewritten in terms of M by substituting $T_H = (8\pi GM)^{-1}$

$$T \;=\; T(M) \;=\; (8\pi GM) \; \frac{T_P^2}{4\beta} \; \left(1 \;-\; \sqrt{1 \;-\; 8\beta \; (\frac{1}{8\pi GMT_P})^2} \right) \tag{16b}$$

The expression for $T(T_H)$ in eqs-(16a,16b), and based on the generalized uncertainty principle inspired from string theory [10], is denoted by $T = T_{GUP}(T_H)$. The $\beta \to 0$ limit of eqs-(16a,16b) gives $T \to T_H = \frac{1}{8\pi GM}$ as expected.

After performing a Taylor expansion of the square root terms of the expression for $T = T_{GUP}(T_H)$ in eq- (16a) gives

$$T_{GUP} \;\sim\; \frac{T_P^2}{4\beta T_H} \; \left(1 \;-\; (1 - 4\beta \frac{T_H^2}{T_P^2} - \frac{1}{8}(8\beta)^2 \frac{T_H^4}{T_P^4} \;+\; \cdots \;) \right) \;=$$

$$T_H \;+\; 2\beta \; \frac{T_H^3}{T_P^2} \;+\; \cdots \tag{17}$$

whereas a Taylor expansion of the expression for the thermal relativistic effective temperature $T = T_{TR}(T_H) = T_H(1 - \frac{T_H^2}{T_P^2})^{-1/2}$ gives

$$T_{TR} \sim T_H + \frac{1}{2}\frac{T_H^3}{T_P^2} + \cdots \tag{18}$$

After comparing the first two terms of eqs-(17,18) we find an agreement when $\beta = \frac{1}{4}$. Therefore, the second order Taylor expansion of $T_{GUP}(T_H)$ agrees precisely with the first order Taylor expansion of $T_{TR}(T_H)$ when $\beta = \frac{1}{4}$.

However, since there is no exact agreement in the higher order terms of the expansions of T_{GUP} and T_{TR} one can still show that the full thermal relativistic expression $T_{TR}(T_H) = T_H(1 - \frac{T_H^2}{T_P^2})^{-1/2}$ can be derived exactly from the modified uncertainty relation

$$\delta x\ \delta p \ \geq \ \frac{1}{2}\sqrt{1 + \frac{(\delta p)^2}{M_P^2}}, \quad T_P = M_P \tag{19}$$

when $\delta p \sim p = E = T$, and $M_P = T_P$, eq-(19) leads to

$$\delta x \ \geq \ \frac{1}{2}\sqrt{\frac{1}{(\delta p)^2} + \frac{1}{T_P^2}} \ = \ \frac{1}{2}\sqrt{\frac{1}{T^2} + \frac{1}{T_P^2}} \tag{20}$$

given $\delta x = 2\pi r_s = 4\pi G M = \frac{1}{2T_H}$, eq-(20) yields

$$\frac{1}{T_H} \ \geq \ \sqrt{\frac{1}{T^2} + \frac{1}{T_P^2}} \ \Rightarrow \ T \ \geq \ \frac{T_H}{\sqrt{1 - \frac{T_H^2}{T_P^2}}} \tag{21}$$

and one recovers the thermal relativistic expression for the modified Hawking temperature $T = T_H\gamma(T_H)$. Concluding, the modified uncertainty relation (19) is the one which is associated with the modified temperature $T_H \rightarrow T_H\gamma(T_H)$ consistent with thermal relativity. Note that the first two terms of the Taylor expansion of the right hand side in eq-(19) yields the initial stringy uncertainty relation (14) for $\beta = \frac{1}{4}$ as found earlier. Hence, the thermal relativity theory singles out the modified uncertainty relation (19) from a number of many other plausible choices.

Having

$$(\delta p)^2 \ \equiv \ < (\hat{p} \ - \ < \hat{p} >)^2 > \ = \ < \hat{p}^2 > \ - \ (< \hat{p} >)^2 \ \Rightarrow$$
$$< \hat{p}^2 > \ \geq \ (\delta p)^2 \tag{22}$$

and the other inequalities

$$< \hat{p}^{2k} > \ \geq \ (< \hat{p}^2 >)^k \ \geq \ (\delta p)^{2k}, \quad k = 1, 2, 3, \cdots \tag{23}$$

a modified Weyl-Heisenberg algebra given by

$$[\hat{x},\ \hat{p}] \ = \ i\ (1 \ - \ \frac{(\hat{p})^2}{M_P^2})^{-\frac{1}{2}}, \ \ \hbar = 1. \tag{24}$$

and which has the *same* functional form as the thermal relativistic dilation factor, leads to the modified uncertainty relation of the form

$$\delta x\ \delta p \ \geq \ \frac{1}{2}\ \frac{1}{\sqrt{1 - \frac{(\delta p)^2}{M_P^2}}}, \ \ 0 \leq \frac{(\delta p)^2}{M_P^2} \leq 1 \tag{25}$$

Eq-(25) is a result of the following relations

$$\delta x\ \delta p \ \geq \ \frac{1}{2}\ | < \ [\hat{x}, \hat{p}] \ > | \ = \ \frac{1}{2}\ < \ (1 \ - \ \frac{(\hat{p})^2}{M_P^2})^{-\frac{1}{2}} \ > \ \geq \ \frac{1}{2}\ \frac{1}{\sqrt{1 - \frac{(\delta p)^2}{M_P^2}}} \ \geq$$

$$\frac{1}{2}\ \sqrt{1 + \frac{(\delta p)^2}{M_P^2}}, \ \ 0 \leq \frac{(\delta p)^2}{M_P^2} \leq 1 \tag{26}$$

provided that $\frac{(\delta p)^2}{M_P^2} \leq 1$ since the radius of convergence of the binomial/Taylor series of eqs-(25,26) is 1. A binomial (Taylor) expansion of (24) yields positive coefficients for all its terms, and allows to take expectation values of each one of the operator-valued terms appearing in the expansion. The last term in eq-(26) is just the expression associated to the effective temperature (21) resulting from thermal relativity.

One can straightforwardly verify the inequalities (22,23) in the quantum harmonic oscillator case via the use of the ladder operators ($\hbar = 1$)

$$\hat{a} \ = \ \sqrt{\frac{m\omega}{2}}\ (\hat{x} \ + \ \frac{i}{m\omega}\ \hat{p}); \ \ \hat{a}^\dagger \ = \ \sqrt{\frac{m\omega}{2}}\ (\hat{x} \ - \ \frac{i}{m\omega}\ \hat{p}) \tag{27}$$

acting on the quantum states

$$\hat{a}^\dagger\ |n> \ \ = \ \sqrt{n+1}\ |n+1>, \ \ \hat{a}\ |n> \ \ = \ \sqrt{n}\ |n-1> \tag{28}$$

simply by writing the momentum operator as

$$\hat{p} \ = \ i\ \sqrt{\frac{m\omega}{2}}\ (\hat{a}^\dagger \ - \hat{a}\ ), \ \ [\hat{a},\ \hat{a}^\dagger] \ = \ 1 \tag{29}$$

and evaluating the expectation values of eqs-(22,23) with respect to the quantum harmonic oscillator states $|n>$.

The modified uncertainty relation (25) yields $\delta x = \infty$ at $\delta p = 0$, and $\delta p = M_p$. The minimum length uncertainty $(\delta x)_{min} = L_P$ coincides precisely with the Planck length, and occurs at $\delta p = \frac{M_P}{\sqrt{2}}$. This behavior should be contrasted with the stringy uncertainty relation (14) which yields $\delta x = \infty$ at $\delta p = 0$, and $\delta p = \infty$. Whereas the minimal length uncertainty is $(\delta x)_{min} = L_P\sqrt{2\beta}$ ($L_P = 1/M_P$) at $\delta p = \frac{M_P}{\sqrt{2\beta}}$. As the string's energy increases it can probe smaller and smaller distances. When the energy reaches values of the order of the Planck's energy, and higher, the string then begins to grow in size, instead of probing smaller scales than the Planck length.

On the other hand, the modified uncertainty relation (19), leads to a minimal length uncertainty $(\delta x)_{min} = \frac{L_p}{2}$ of the order of the Planck length L_P when $\delta p \rightarrow \infty$; i.e it takes an infinite momentum to reach the Planck scale, this is consistent with Scale Relativity [18] (based on fractals) and Doubly Special Relativity [19] (based on κ-deformed Poincare algebra).

If one sets $\beta < 0$ in the stringy uncertainty relation (14), there is no longer a minimum length uncertainty. It was argued by [11] that the issue of the sign of β has not been solved completely. In particular, it was found by [12] that $\beta < 0$ is the only choice compatible with the Chandrasekhar limit, otherwise arbitrarily large white dwarfs would exist conflicting astrophysical observations. An alternative method that achieves the same effect was found later in [13] by including a cosmological constant term in the GUP (known as "extended GUP" in the literature). In this way, the introduction of an arbitrarily small but nonzero cosmological constant can restore the Chandrasekhar limit. One should emphasize that the modified uncertainty relations (19,25) have *not* been used to our knowledge in the study of the Chandrasekhar limit.

It is important to remark also that the "extended GUP" is in fact *consistent* with Born's reciprocal relativity theory [1]. Under Born's reciprocity $x \leftrightarrow p$, the simplest modified uncertainty relation respecting the principle of Born's reciprocity is of the form

$$\delta x \ \delta p \ \geq \ \frac{1}{2} \ + \ \beta_1 \ \frac{(\delta p)^2}{M_P^2} \ + \ \beta_2 \ \frac{(\delta x)^2}{R_H^2} \tag{30}$$

where the Hubble radius R_H is introduced in eq-(30) as some sort of infrared cutoff (upper length scale), and $L_P = \frac{1}{M_P}$ can be seen as an ultraviolet cutoff (lower length scale). The thermal relativity principle can also be applied to eq-(30) by working in the full thermodynamical *phase* space, see [1] for more details.

To finalize, the non-thermal distribution spectrum due to thermal relativity is given by

$$N \ = \ \frac{1}{e^{\frac{E}{T_H\gamma(T_H)}} - 1} \ = \ \frac{1}{e^{\frac{E}{T_H}} - 1} \ \left(\frac{e^{\frac{E}{T_H}} - 1}{e^{\frac{E}{T_H\gamma(T_H)}} - 1} \right) \ \equiv \ f \ \frac{1}{e^{\frac{E}{T_H}} - 1} \tag{31}$$

where the deviation from the purely thermal spectrum is encoded in the multiplicative factor f. Given $A = \frac{E}{T_H}$, $B = \frac{E}{T_H\gamma(T_H)}$, one has

$$\frac{1}{e^B - 1} \ = \ \frac{1}{e^A - 1} \ \frac{e^A - 1}{e^B - 1} \ = \ \frac{1}{e^A - 1} \ \left(1 \ + \ \frac{e^A - e^B}{e^B - 1} \right) \tag{32}$$

The following fraction can be expanded as

$$\frac{e^A - e^B}{e^B - 1} \ = \ \frac{e^A}{e^B - 1} \ (1 - e^{B-A}) \ \sim \ (A - B)\frac{e^A}{e^B - 1} \ + \ \cdots \tag{33}$$

Eqs-(32,33) allow us then to evaluate the multiplicative factor f

$$f \sim 1 + \frac{1}{2} \frac{E}{T_H} \frac{T_H^2}{T_P^2} \frac{e^{\frac{E}{T_H}}}{e^{\frac{E}{T_H}} - 1} + \cdots \tag{34}$$

where the higher order corrections to the factor f are of the form

$$\left(\frac{1}{2} \frac{E}{T_H} \frac{T_H^2}{T_P^2} \frac{e^{\frac{E}{T_H}}}{e^{\frac{E}{T_H}} - 1} \right)^n, \quad n \geq 2 \tag{35}$$

In the thermal non-relativistic limit $T_P \rightarrow \infty$ one recovers $f \rightarrow 1$ as expected. The facts that thermal relativity leads to a Planck-size black hole remnant and to modifications to the thermal spectrum could cast some light into the resolution of the black hole information paradox (loss of unitarity).

We conclude by reflecting on our proposal towards a Space-Time-Matter Unification program where matter can be converted into spacetime quanta, and vice versa [1]. Our minimal mass M_o of the order of the Planck mass corresponding to a Planck-size black hole, and whose horizon has Planck-sized area, could be viewed as spacetime quanta ("atoms" of spacetime). This proposal must *not* be confused with the view by [16] of classical background geometries as quantum Bose-condensates with large occupation numbers of soft gravitons, such that a black hole is a leaky bound-state in form of a cold Bose-condensate of N weakly-interacting soft gravitons (very low energy) of wave-length $\sqrt{N} L_P$, and of quantum interaction strength $1/N$. Nor with the view that the event horizon of a black hole is a quantum phase transition of the vacuum of spacetime analogous to the liquid-vapor critical point of a Bose fluid [17].

There is a fundamental difference between quantization *in* spacetime versus quantization *of* spacetime. The generalized uncertainty principle (GUP) and corpuscular gravity within the context of quantum Bose-condensates was recently studied by [11] in order to explain the GUP-induced shift of the Hawking temperature found in eqs-(16a, 16b). Based on our findings in this letter, it is warranted to explore all these topics deeper. In particular, to investigate the modified uncertainty relations (19,25) in the study of the Chandrasekhar limit, and in the more technical front, if thermal relativity theory can accommodate the thermodynamics of black holes with modified dispersion relations and Perelman entropies for Finsler-Lagrange-Hamilton Spaces [20].

Acknowledgements

We thank M. Bowers for very kind assistance.

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