Mass and the fifth dimension

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Abstract

A correlation between mass and a compact 5th dimension is proposed. The 5th coordinate appears to represent the gravitational radius of the black hole. The source of curvature of the spacetime turns out to be an anisotropic null fluid with no energy density and isotropic pressure but nonzero energy flux and anisotropic pressures.

1 Introduction

Studying a 3-brane (3+1 dimensional subspace) embedded in five dimensions, Randall and Sundrum (RS) [1] have shown that the 4-dimensional Newtonian and Einsteinian gravity are reproduced to a reasonable precision (see also [2]). The 5th dimension may be infinite, without violating known tests of gravity. In addition, the RS recipe suggests the 4-dimensional Planck scale is comparable to the fundamental mass scale of the higher dimensional theory.

Recently, Zhang et al. [3], in a paper related to Kepler's harmonies and conformal symmetry, claimed that Kepler's rescalings become, when "lifted" to the 5-dimensional Bargmann gravitational wave space, an ordinary symmetry for motion along null geodesics, i.e. the lifts of Keplerian trajectories. Moreover, they showed that the lifted Kepler rescaling is a Chrono-Projective transformation (see also [4], where the authors emphasized that the classical motions of a particle in a Newtonian potential corresponds to null geodesics in a 5-dimensional Lorentzian manifold - the so-called "Bargmann structures").

Zhang et al. [3] also investigated a Ricci-flat metric with Lorenzian signature, carrying a covariant constant null vector, called the "vertical" vector $\xi = \partial_s$, where their s stands for the fifth coordinate. However, their metric is non-relativistic and therefore the hypersurfaces $s =$ const. do not give us the Schwarzschild (S) metric. Their Kepler orbits obey the Third Law and, while the planet goes around, the fifth (vertical) coordinate s changes and so the Keplerian trajectories become spirals.

We try in this paper to build a modified version of the Zhang et al. model without paying attention to the symmetry properties. In our view, the 5th coordinate has the meaning of the mass of the source of gravity. We investigate the stress tensor from which the metric with variable mass is rooted. It represents an anisotropic null fluid with no energy density, no isotropic pressures but nonzero energy flux. We are also interested to study the trajectories of a massive test particle with no acceleration in the t-, ϕ- and y-directions, where y refers to the 5th coordinate.

Throughout the paper we adopt the convention $c = 8\pi G = 1$, if not specified otherwise.

2 The five-dimensional line-element

Our starting spacetime outside the source mass is given by

$$ds^2 = -(1 - \frac{2m(y)}{r})dt^2 + dr^2 + 2dt\ dy + r^2 d\Omega^2, \tag{2.1}$$

where y is the 5th coordinate, $m(y)$ is the source mass and $d\Omega^2$ stands for the metric on the unit 2-sphere. Note that (2.1) acquires a Newtonian form of the S geometry, on the hypersurface $y =$ const.

We search for the properties of the spacetime (2.1) only in the region $r > 2m(y)$, even though the determinant $g = -r^4 sin^2\theta$ preserves its sign irrespective of the sign of g_{tt}. The Ricci and the Kretschmann scalars for the metric (2.1) appears as

$$R^a_a = \frac{2m''(y)}{r}, \qquad K = \frac{4m''^2(y)}{r^2} \tag{2.2}$$

$(a = t, r, \theta, \phi, y)$, with $m'(y) = dm/dy$. They diverge at $r = 0$, that is located outside our domain of investigation. For simplicity reasons, we choose $m(y) = y$. That means we identify the fifth coordinate with the gravitational radius of the source, considered as having spherical symmetry.

With the above choice, R^a_a and K become null and the energy-momentum tensor from the r.h.s. of Einstein's equations $G_{ab} = 8\pi T_{ab}$ which sources the metric (2.1), looks like

$$T^y_r = T^r_t = -\frac{m'(y)}{r^2} = -\frac{1}{r^2}. \tag{2.3}$$

All the other mixed components are vanishing. The only nonzero covariant components are $T_{rt} = T_{tr} = -1/r^2$; same for $T^{ry} = T^{yr} = -1/r^2$. We also notice that $T^a_a = 0$. In other words, T_{ab} corresponds to a null fluid.

3 Anisotropic fluid properties

Let us consider an observer in the spacetime (2.1) with the following velocity vector field

$$u^a = \left(\frac{1}{1-\sqrt{\frac{2y}{r}}}, 0, 0, 0, \sqrt{\frac{2y}{r}} \right), \quad u^a u_a = -1. \tag{3.1}$$

The 4-acceleration of the observer (3.1) is given by

$$a^b = u^a \nabla_a u^b = \left(0, \frac{y}{r^2 \left(1-\sqrt{\frac{2y}{r}}\right)^2}, 0, 0, 0 \right). \tag{3.2}$$

We see that, far from the horizon $r = 2y = 2m(y)$, namely for $2y << r$, the radial component of a^b acquires the Newtonian form m/r^2, with $m = \text{const}$. In contrast with $u^y = \sqrt{2y/r} \neq 0$, we have, however, $a^y = 0$. Another kinematical quantity of interest is the expansion of the worldlines (3.1)

$$\Theta = \nabla_a u^a = \frac{m'(y)}{r\sqrt{\frac{2m(y)}{r}}} = \frac{1}{\sqrt{2ry}}, \tag{3.3}$$

with $\dot{\Theta} \equiv u^a \nabla_a \Theta = -1/2ry < 0$.

We now write down the source tensor T_{ab} in the following general form [5]

$$T_{ab} = \rho u_a u_b + p h_{ab} + \Pi_{ab} + u_a q_b + u_b q_a, \tag{3.4}$$

where the energy density $\rho = T_{ab} u^a u^b$, the isotropic pressure $p = (1/3) h^{ab} T_{ab}$, the energy flux $q_a = -h_{ab} T^b_c u^c$ and the anisotropic pressure

$$\Pi_{ab} = \frac{1}{2}\left(h_{ac} h_{bd} T^{cd} + h_{bc} h_{ad} T^{cd} \right) - \frac{1}{3} h_{ab} h^{cd} T_{cd}, \tag{3.5}$$

with $h_{ab} = g_{ab} + u_a u_b$ and $h_{ab} u^b = \Pi_{ab} u^b = u^b q_b = \Pi_b{}^b = 0$. By means of the previous nonzero components of the stress tensor, we easily find that

$$q^b = \left(0, \frac{1}{r^2 \left(1-\sqrt{\frac{2y}{r}}\right)}, 0, 0, 0 \right), \tag{3.6}$$

with $\sqrt{q^b q_b} = q^r = 1/r^2(1-\sqrt{2y/r})$ and

$$\Pi_{tr} = \frac{\sqrt{\frac{2y}{r}}}{r^2\left(1-\sqrt{\frac{2y}{r}}\right)}, \quad \Pi_{ry} = -\frac{1}{r^2\left(1-\sqrt{\frac{2y}{r}}\right)^2} \tag{3.7}$$

as the only nonzero components of the anisotropic pressure. We also infered that $\rho = p = 0$. Let us observe that all the components of the anisotropic pressure and the energy flux diverge at the horizon $r = 2y$. We stress also that the metric determinant is regular at the same horizon.

4 Timelike trajectories with $a^t = a^\phi = a^y = 0$

Let us write the general expression of the covariant acceleration

$$a^b = \frac{dv^b}{d\sigma} + \Gamma^b_{ac} v^a v^c, \tag{4.1}$$

where σ is the affine parameter along trajectory and v^b - the tangent vector to the trajectory, with $v^b = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}, \dot{y})$, where a dot means the derivative w.r.t. the parameter σ. To avoid tedious calculations, we restrict to the equatorial plane $\theta = \pi/2$ and $\dot{r} = 0$ ($r = R =$const.) along trajectory. From (4.1) one obtains

$$a^t = \ddot{t} - \frac{1}{r}\dot{t}^2 = 0 \tag{4.2}$$

and

$$a^y = \ddot{y} - \frac{1}{r}(1 - \frac{2y}{r})\dot{t}^2 - \frac{2y}{r^2}\dot{t}\dot{r} + \frac{2}{r}\dot{t}\dot{y} = 0, \tag{4.3}$$

and

$$a^\phi = \ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2cot\theta\ \dot{\phi}\ \dot{\theta} = 0. \tag{4.4}$$

With $r = R$, we get from (4.2)

$$t(\sigma) = -Rln|1 - \frac{E\sigma}{R}|, \tag{4.5}$$

with initial conditions $t(0) = 0$ and $\dot{t}(0) = E$. When we put $\dot{r} = 0$ in (4.3) and using (4.2), we get

$$\frac{d}{d\sigma}\left[\left(1 - \frac{2y}{r}\right)\dot{t} - \dot{y}\right] = 0, \tag{4.6}$$

whence

$$\left(1 - \frac{2y}{r}\right)\dot{t} - \dot{y} = E, \tag{4.7}$$

where E is the same constant of integration, related to the energy per unit mass ($\mu = 1$) of the test particle, measured by an observer at rest at infinity [6]. Eq. (4.4) leads to $\ddot{\phi} = 0$, or $\dot{\phi} = L/R^2$, where L (the angular momentum per unit mass) is again a constant of integration. The trajectories being circular ($\dot{r} = 0$) we have $E^2 = 1 + L^2/R^2$.

To find $y(\sigma)$ on the trajectory, we write (2.1) as

$$(1 - \frac{2y}{R})\dot{t}^2 - 2\dot{t}\dot{y} - \frac{L^2}{R^2} = 1. \tag{4.8}$$

Taking the derivative of (4.5) w.r.t. σ and using (4.2) and (4.6), one obtains

$$\ddot{y} = \frac{1}{R} + \frac{L^2}{R^3} = \frac{E^2}{R}, \tag{4.9}$$

whence

$$y(\sigma) = \frac{E^2}{2R}\sigma^2, \tag{4.10}$$

with $y(0) = \dot{y}(0) = 0$ [1]

From (4.5) and (4.8) we find now the components of v^a

$$v^a = \left(\frac{E}{1 - \frac{E\sigma}{R}}, 0, 0, \frac{L}{R^2}, \frac{E^2\sigma}{R} \right). \tag{4.11}$$

One could check that the condition $v^a v_a = -1$ is fulfilled. We reverse now Eq. (4.5) to get

$$\sigma(t) = \frac{R}{E}\left(1 - e^{-\frac{t}{R}}\right). \tag{4.12}$$

Thanks to (4.10), one finds the time dependence of y and ϕ

$$y(t) = m(t) = \frac{R}{2}\left(1 - e^{-\frac{t}{R}}\right)^2, \quad \phi(t) = \frac{L}{ER}\left(1 - e^{-\frac{t}{R}}\right), \tag{4.13}$$

with $\phi(0) = 0$ and $0 \leq t < \infty$.

It is worth noting that $m_{max} = R/2$ (when R becomes the gravitational radius) and $\phi_{max} = L/ER < 2\pi$. $m(t)$ is a monotonically increasing function and has an inflexion point at $t_i = Rln2$, when the mass takes the value $R/8$. The situation $t << R$ gives us $m(t) \approx t/2$. This resembles the behavior of the "interior volume" of a black hole [7, 8], $V(v) = 3\sqrt{3}\pi M^2 v$, where V is a maximum volume when the time v is large enough w.r.t. the black hole mass M. The volume V is very large, as Christodoulou and Rovelli have shown in their paper. If we introduce all fundamental constants in our expression, one obtains $m(t) \approx c^3 t/2G$, which is also very large, even when $t << R$, because R might be a macroscopic radius ($m(t)$ varies, of course, within the 5th dimension).

An observation is in order here. Although (4.2) - (4.4) are satisfied, the above trajectories are not geodesics as one could see from the expression of the radial acceleration

$$a^r = \ddot{r} + \frac{y}{r^2}\dot{t}^2 - r\dot{\phi}^2. \tag{4.14}$$

One notices that, when $\dot{r} = 0$, $a^r = 0$ is not obeyed. That may be also seen from (2.1) written under the form

$$\dot{r}^2 - \left(1 - \frac{2y}{r}\right)\dot{t}^2 + 2\dot{t}\dot{y} + r^2\dot{\phi}^2 = -1. \tag{4.15}$$

[1] The boundary condition $y(0) = m(0) = 0$ keeps track of the fact that on the surface $y = 0$ we have empty space, i.e. Minkowski spacetime.

If one considers

$$V(r) = -\left(1 - \frac{2y}{r}\right)\dot{t}^2 + 2\dot{t}\dot{y} + r^2\dot{\phi}^2 \tag{4.16}$$

as a function of r, the conditions $\dot{r} = 0$ and $\partial V/\partial r = 0$ leads to

$$\frac{y}{R^2}\dot{t}^2 - R\dot{\phi}^2 = 0, \tag{4.17}$$

which is not fulfilled when (4.5) and $\phi(\sigma) = L\sigma/R^2$ are used. We have again $a^r \neq 0$, so that there are no circular geodesics. We notice also from (4.13) that, when the time t varies from zero to infinity, ϕ varies from zero to L/ER, y varies from zero to $R/2$ and $y(\phi) = (E^2R^3/2L^2)\phi^2$ represents a spiral.

To summarize, we proposed in this short paper a different interpretation to the Zhang et al. [3] model of the 5-dimensional Bargmann spacetime (their Eq. II.1). However, our main geometry (2.1) is slightly different, with the time-time metric coefficient depending on the 5th coordinate which has the meaning of the mass of the source and equals its half gravitational radius. The source of curvature of the spacetime (2.1) is an anisotropic null fluid with zero energy density and isotropic pressure but nonzero energy flux and anisotropic pressures. We studied also the trajectories of test particles with no acceleration on t-, y- and ϕ-directions but nonzero radial acceleration, though the radial velocity $\dot{r}$ was chosen to vanish.

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