Spin Electromagnetics and Spin-Vector-Potential-Coupling-Induced Force

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Abstract

By utilizing universal mathematical identities (UMI) and Coulomb's law, we establish Classical-Spin-Electromagnetics (C-Spin-EM). C-Spin-EM is self-consistence, powerful and fruitful, at classical level, in the perspective of fundamental physics: (1) universally explains and correlates family of Hall effects, zero longitudinal Hall coefficient/resistivity, Aharonov–Bohm effect, Extended Rashba SOC, and GMR/TMR. (2) predicts Spin-potential-coupling-induced force, which contributes to Aharonov–Bohm effect; (3) provides classical counterparts of Larmor-precession, Stark Effect, Landau–Lifshitz equation, Zeeman effect, and Aharonov–Casher effect; (4) propose that electric field induces spin precession. Combining UMI and C-Spin-EM shows that mathematical identities lead to physical dualities including duality between Electromagnetics and C-Spin-EM. We postulate a duality between Lagrangian-Lorentz force and Hamiltonian.

Key words: spin electromagnetics, spintronics, Rashba effect, anomalous Hall effect, spin Hall effect, topological insulator, GMR/TMR

Contents

## 1. Introduction

In the macroscopic world, Electromagnetics (**EM**) was established based on the experiments. In the microscopic world, phenomena related with spin, such as the family of Hall effects, topological insulators, Rashba SOC, and GMR/TMR, draw numerous activities.

We have questions: do those quantum phenomena have classical counterparts and/or origins. If the answer is yes, is there a theory providing a universal explanation and/or mechanism for those classical counterparts; what are further predictions for testing?

The classical correspondence of spin of e-particles is its angular momentum around its axis, denoted as $\mathbf{S_c}$. We postulate that, at classical level, the spin of e-particles induces new fields [1] like the velocity of e-particles induces magnetic field, and that field equations can be derived mathematically. Those field equations form Classical Spin-EM (abbreviated **C-Spin-EM**). Furthermore, we expect that the C-Spin-EM be closely related with EM, and that we can describe behaviors of e-particles with electric charge and spin by EM and C-Spin-EM universally.

**Motivation**: To address above questions motivates us to establish mathematically C-Spin-EM. We expect that C-Spin-EM serve as classical basis of Spintronics, as EM to Electronics.

## 2. Classical-Spin-Electromagnetics Derived from Coulomb's Law

Let's mathematically establish a theoretical framework, C-Spin-EM, for studying spin-related classic phenomena systematically.

### 2.1. Universal Mathematical Identities (UMI)

We need to find a vector analysis identity that connecting divergences/gradient of a vector and an axial vector, and curl of an induced axial vector. For this aim, the following mathematical identity is the most noteworthy,

$$\nabla\times(\mathbf{G}\times\mathbf{T}) = \mathbf{G}(\nabla\cdot\mathbf{T}) - \mathbf{T}(\nabla\cdot\mathbf{G}) + (\mathbf{T}\cdot\nabla)\mathbf{G} - (\mathbf{G}\cdot\nabla)\mathbf{T}, \tag{2.1}$$

By using another mathematical identity,

$$(\mathbf{T}\cdot\boldsymbol{\nabla})\mathbf{G} = \boldsymbol{\nabla}(\mathbf{G}\cdot\mathbf{T}) - (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{T} - \mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{T}) - \mathbf{T}\times(\boldsymbol{\nabla}\times\mathbf{G}),$$

Eq. (2.1) can be rewritten as an identity,

$$\nabla\times(\mathbf{G}\times\mathbf{T}) = \mathbf{G}(\nabla\cdot\mathbf{T}) - \mathbf{T}(\nabla\cdot\mathbf{G}) - \nabla(\mathbf{G}\cdot\mathbf{T}) + 2(\mathbf{T}\cdot\nabla)\mathbf{G} + \mathbf{G}\times(\nabla\times\mathbf{T}) + \mathbf{T}\times(\nabla\times\mathbf{G}). \tag{2.2}$$

Refer Eq. (2.1) and Eq. (2.2) as Universal Mathematical Identities (abbreviated **UMI**), which indicates that the vector calculus of two arbitrary vectors, **G** and **T**, induces inevitably an axial vector, $\mathbf{G}\times\mathbf{T}$. Terms, $(\nabla\cdot\mathbf{T})$ and $(\nabla\cdot\mathbf{G})$, represent inverse-square laws.

### 2.2. Definitions of Spin-electric Field and Spin-magnetic Field and Testing Experiment

Let's consider a spinning e-particle characterize by electric charge $\mathrm{Q_e}$ and spin $\mathbf{S_c}$. Let $\mathbf{G} = \mathbf{S_c}$ and $\mathbf{T}$ represents electric and magnetic field, **E** and **B**, respectively. Eq. (2.2) suggests to define spin-magnetic field $\mathbf{B_s}$ and spin-electric field $\mathbf{E_s}$, respectively,

$$\mathbf{B_s} \equiv \mathbf{S_c}\times\mathbf{E}, \tag{2.3}$$

$$\mathbf{E_s} \equiv -\mathbf{S_c}\times\mathbf{B}, \tag{2.4}$$

$\frac{\mathbf{B_s}}{\mathbf{E_s}} = -\frac{\mathbf{S_c}\times\mathbf{E}}{\mathbf{S_c}\times\mathbf{B}}$,

Subscript "s" indicates the quantity related to spin. The electric (magnetic) field can be either an externally applied electric (magnetic) field or a local electric (magnetic) field induced by nearby e-particles in the material. The $\mathbf{B_s}$ and $\mathbf{E_s}$ are dual to the magnetic $\mathbf{B}$ and electric $\mathbf{E}$ fields, respectively**.** The $\mathbf{B_s}$ and $\mathbf{E_s}$ mathematically satisfy,

$$\nabla\cdot\mathbf{B_s} = 0, \tag{2.5}$$

$$\nabla\cdot\mathbf{E_s} = 0. \tag{2.6}$$

We still keep those divergence terms in some of equations of C-Spin-EM, as well the magnetic monopole term $\nabla\cdot\mathbf{B}$, which shows the nature and the breaking mechanism of duality.

If there are "spin-charges" and "Spin-monopole", then we have (Appendix),

$$\nabla\cdot\mathbf{B_s} \neq 0, \tag{2.7}$$

$$\nabla\cdot\mathbf{E_s} \neq 0. \tag{2.8}$$

The definitions, Eq. (2.3) and Eq. (2.4), predict two kinds of phenomena.

Firstly, by interacting with an electric field $\mathbf{E}$ (magnetic field $\mathbf{B}$**),** the spin of e-particles induces an effective spin-magnetic field $\mathbf{B_s}$ (effective spin-electric field $\mathbf{E_s}$).

Secondly, the spin of an e-particle in an electric field $\mathbf{E}$ (magnetic field $\mathbf{B}$) will experience an effective spin-magnetic field $\mathbf{B_s}$ (effective spin-electric field $\mathbf{E_s}$).

**Remark**: the definitions of $\mathbf{B_s}$ and $\mathbf{E_s}$ are conceptually different from that of Spin EM established in quantum regime (abbreviated **Q-Spin-EM**) [2].

**Testing Experiment**: Testing Definitions of $\mathbf{B_s}$ and $\mathbf{E_s}$:

An e-particle 1, either without spin or with zero net spin outside a material, which induces an electric field that penetrates into the material (Fig.1).

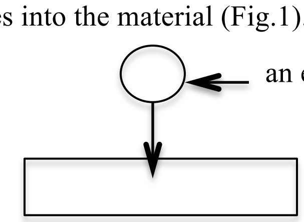


Fig.1

A spinning e-particle 2 inside the material will experience an electric field and an effective spin-magnetic field. Also the e-particle 2 induces an effective spin-magnetic field.

If the orientations of spins of e-particles inside the material are aligned, then the induced effective spin-magnetic field can be detected, which will justify Eq. (2.3) and Eq. (2.4).

2.3. C-Spin-EM Derived from UMI and Coulomb Law

Substituting Eq. (2.3) and Eq. (2.4) into Eq. (2.2), respectively, gives Ampere-type equations for fields induced by spin of e-particles,

$$\nabla\times\mathbf{B_s} = \mathbf{S_c}(\nabla\cdot\mathbf{E}) - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) - \mathbf{E}[\nabla\cdot\mathbf{S_c}] + 2(\mathbf{E}\cdot\nabla)\mathbf{S_c} + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{S_c}) + \mathbf{S_c}\times(\boldsymbol{\nabla}\times\mathbf{E}), \tag{2.9}$$

$$\nabla\times\mathbf{E_s} = -\mathbf{S_c}(\nabla\cdot\mathbf{B}) + \nabla(\mathbf{S_c}\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{S_c}) - 2(\mathbf{B}\cdot\nabla)\mathbf{S_c} - \mathbf{B}\times(\nabla\times\mathbf{S_c}) - \mathbf{S_c}\times(\nabla\times\mathbf{B}). \tag{2.10}$$

Substituting Faraday's law and Ampere-Maxwell's law into Eq. (2.9) and Eq. (2.10) respectively, we obtain C-Spin-EM, which includes Ampere-Maxwell-type equation and Faraday-type equations,

$$\nabla\times\mathbf{B_s} = \mathbf{S_c}(\nabla\cdot\mathbf{E}) + \frac{\partial\mathbf{E_s}}{\partial t} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{S_c}) + 2(\mathbf{E}\cdot\nabla)\mathbf{S_c} + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{S_c}) +$$

$$+\mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \qquad (2.11)$$

$$\nabla\times\mathbf{E_s} = -\mathbf{S_c}(\nabla\cdot\mathbf{B}) - \frac{\partial\mathbf{B_s}}{\partial\mathbf{t}} + \nabla(\mathbf{S_c}\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{S_c}) - 2(\mathbf{B}\cdot\nabla)\mathbf{S_c} - \mathbf{B}\times(\nabla\times\mathbf{S_c}) -$$

$$-\mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \qquad (2.12)$$

Let's define the "spin-magnetic-current $\mathbf{j_{s-B}}$" and the "spin-electric-current $\mathbf{j_{s-E}}$", which induce the spin-magnetic field and the spin-electric field respectively, as

$$\mathbf{j_{s-B}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{E}) - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{S_c}) + 2(\mathbf{E}\cdot\nabla)\mathbf{S_c} + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{S_c}) +$$

$$+\mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \qquad (2.13)$$

$$\mathbf{j_{s-E}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{B}) - \nabla(\mathbf{S_c}\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{S_c}) + 2(\mathbf{B}\cdot\nabla)\mathbf{S_c} + \mathbf{B}\times(\nabla\times\mathbf{S_c}) +$$

$$+\mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \qquad (2.14)$$

Eq. (2.11) and Eq. (2.12) become respectively,

$$\nabla\times\mathbf{B_s} = \mathbf{j_{s-B}} + \frac{\partial\mathbf{E_s}}{\partial t}, \qquad (2.15)$$

$$\nabla\times\mathbf{E_s} = -\mathbf{j_{s-E}} - \frac{\partial\mathbf{B_s}}{\partial\mathbf{t}}. \qquad (2.16)$$

For the situations, in which spin is non-spatial-varying, i.e.,

$$(\mathbf{E}\cdot\nabla)\mathbf{S_c} = (\boldsymbol{\nabla}\times\mathbf{S_c}) = (\mathbf{B}\cdot\nabla)\mathbf{S_c} = (\nabla\cdot\mathbf{S_c}) = \mathbf{0},$$

Eq. (2.11) to Eq. (2.14) become respectively,

$$\nabla\times\mathbf{B_s} = \mathbf{S_c}(\nabla\cdot\mathbf{E}) + \frac{\partial\mathbf{E_s}}{\partial t} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) + \mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \qquad (2.17)$$

$$\nabla\times\mathbf{E_s} = -\mathbf{S_c}(\nabla\cdot\mathbf{B}) - \frac{\partial\mathbf{B_s}}{\partial\mathbf{t}} + \nabla(\mathbf{S_c}\cdot\mathbf{B}) - \mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \qquad (2.17)$$

$$\mathbf{j_{s-B}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{E}) - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}) + \mathbf{S_c}\times\{-\mathbf{v}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{B}\cdot\nabla)\mathbf{v}\}, \qquad (2.19)$$

$$\mathbf{j_{s-E}} \equiv \mathbf{S_c}(\nabla\cdot\mathbf{B}) - \nabla(\mathbf{S_c}\cdot\mathbf{B}) + \mathbf{S_c}\times\{\mathbf{v}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{v}) + (\mathbf{E}\cdot\nabla)\mathbf{v}\}. \qquad (2.20)$$

The terms, $\mathbf{E}(\nabla\cdot\mathbf{v})$ and $\mathbf{B}(\nabla\cdot\mathbf{v})$, describes stretching of the **E** and **B** fields due to source velocity compressibility, respectively; The terms, $(\mathbf{E}\cdot\nabla)\mathbf{v}$ and $(\mathbf{B}\cdot\nabla)\mathbf{v}$**,** describes the stretching or tilting of the **E** and **B** fields due to the velocity gradients, respectively.

For the situations of non-spatial-varying velocity**,**

$$(\nabla\cdot\mathbf{v}) = (\mathbf{B}\cdot\nabla)\mathbf{v} = (\mathbf{E}\cdot\nabla)\mathbf{v} = \mathbf{0},$$

C-Spin-EM, Eq. (2.17) to Eq. (2.20), are further simplified to,

$$\nabla\times\mathbf{B_s} = 4\pi q_e\mathbf{S_c} + \frac{\partial\mathbf{E_s}}{\partial t} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}), \qquad (2.21)$$

$$\nabla\times\mathbf{E_s} = -\frac{\partial\mathbf{B_s}}{\partial\mathbf{t}} + \nabla(\mathbf{S_c}\cdot\mathbf{B}) - 4\pi q_e\mathbf{S_c}\times\mathbf{v}. \qquad (2.22)$$

$$\mathbf{j_{s-B}} = 4\pi q_e\mathbf{S_c} - \boldsymbol{\nabla}(\mathbf{S_c}\cdot\mathbf{E}), \qquad (2.23)$$

$$\mathbf{j_{s-E}} = 4\pi q_e\mathbf{S_c}\times\mathbf{v} - \nabla(\mathbf{S_c}\cdot\mathbf{B}). \qquad (2.24)$$

The Coulomb's law, $\nabla\cdot\mathbf{E} = 4\pi q_e$, and $\nabla\cdot\mathbf{B} = \mathbf{0}$ have been used.

Eq. (2.21) and Eq. (2.22) show the following:

(1) The spin, $4\pi q_e\mathbf{S_c}$**,** and the time change of the $\mathbf{E_s}$ field, $\frac{\partial\mathbf{E_s}}{\partial t}$, are the spin-counterparts of

e-current and displacement-current respectively.

(2) The gradient of the spin-electric field coupling, $\nabla(\mathbf{S_c}\cdot\mathbf{E})$, induces the $\mathbf{B_s}$ field.

(3) The $\mathbf{E_s}$ field is induced by the time change of the $\mathbf{B_s}$ field, $\frac{\partial \mathbf{B_s}}{\partial t}$, as well $4\pi\frac{q_e}{m_e}\mathbf{S_c}\times\mathbf{p}$.

(4) The gradient of the spin-magnetic field coupling, $\nabla(\mathbf{S_c}\cdot\mathbf{B})$, induces the $\mathbf{E_s}$ field.

**Remark**: Eq. (2.21) shows that the spin $\mathbf{S_c}$ plays the role of "velocity" generating spin-magnetic field, which is conceptually different from that of Q-Spin-EM [2].

2.4. Duality between EM and C-Spin-EM

Eq. (1) derives EM [1]. As shown above, Eq. (2) derives C-Spin-EM. Eq. (1) and Eq. (2) are mathematical equivalent, i.e., mathematical duality. Thus mathematical duality between Eq. (1) and Eq. (2) leads to physical duality between EM and C-Spin-EM.

2.5. Equations of Continuity of Spin Currents

The $\mathbf{j_{s-B}}$ and $\mathbf{j_{s-E}}$ should satisfy the equation of continuity respectively. Taking divergence of Eq. (2.15) and Eq. (2.16) respectively, we obtain

$$\nabla\cdot(\nabla\times\mathbf{B_s}) = \nabla\cdot(\mathbf{j_{s-B}}) + \frac{\partial(\nabla\cdot\mathbf{E_s})}{\partial t},$$

$$\nabla\cdot(\nabla\times\mathbf{E_s}) = -\nabla\cdot(\mathbf{j_{s-E}}) - \frac{\partial(\nabla\cdot\mathbf{B_s})}{\partial t}.$$

We face two situations:

Firstly, if spin-charge and spin-monopole exist, then $\nabla\cdot\mathbf{E_s}\neq\mathbf{0}$, $\nabla\cdot\mathbf{B_s}\neq\mathbf{0}$. We have familiar form of Equations of Continuity of Spin Currents,

$$\nabla\cdot(\mathbf{j_{s-B}}) + \frac{\partial(\nabla\cdot\mathbf{E_s})}{\partial t} = 0,$$

$$\nabla\cdot(\mathbf{j_{s-E}}) + \frac{\partial(\nabla\cdot\mathbf{B_s})}{\partial t} = 0.$$

Secondly, if we don't have spin-charge and spin-monopoles, equations of continuity are

$$\nabla\cdot(\mathbf{j_{s-B}}) = 0, \tag{2.25}$$

$$\nabla\cdot(\mathbf{j_{s-E}}) = 0, \tag{2.26}$$

To obtain the familiar format of Equations of Continuity, let's take a different approach. Let's restudy situations, in which, e-particles carry both e-charge and spin, i.e., e-charge and spin are bound together always. Therefore the number density of e-charges is that of spin, namely, the time change and space varying of number density of e-charges are that of spin. Spin current is associated with e-current. The equation of continuity of e-currents is

$$\nabla\cdot\mathbf{j_{v-B}} + \frac{\partial\rho_e}{\partial t} = \mathbf{0}, \tag{2.27}$$

$$\mathbf{j_{v-B}} = 4\pi n q_e \mathbf{v}, \tag{2.28}$$

$$\rho_e = n q_e, \tag{2.29}$$

where "n" is the number density of e-charge, and thus of spin; $q_e$ is the e-charge of each individual e-particle.

To get the equations of continuity of spin currents, we propose to attach spin to velocity and to convert e-charge density $\rho_e$ to spin density $\rho_s$. Eq. (2.23) and Eq. (2.27) to Eq. (2.29) give

$$\nabla\cdot\mathbf{j_{s-B}} + \frac{\partial\rho_s}{\partial t} = \mathbf{0}, \tag{2.30}$$

$$\mathbf{j}_{\mathbf{s-B}} = 4\pi n q_e \mathbf{S}_c \mathbf{v} - \nabla[(\mathbf{S_c v}) \cdot \mathbf{E}], \quad (2.31)$$

$$\rho_s = \mathbf{S}_c q_e n. \quad (2.32)$$

Note spin $\mathbf{S}_c$ has different orientations, the term, $4\pi n\mathbf{S}_c\mathbf{v}$, need to be expressed as a classical pseudo-tensor spin-magnetic-current,

$$j_{s-B,ij} = 4\pi q_e n v_i S_{cj} - \nabla_i\left[(\mathbf{S}_c\mathbf{v})_{jk} E_k\right], \quad (2.33)$$

$$\rho_{si} = S_{ci} q_e n. \quad (2.34)$$

The generally accepted definition of the spin current pseudo-tensor [3] is,

$$j_{ij} \sim \frac{1}{2}\{S_j v_i + v_i S_j\}. \quad (2.35)$$

**Remark:** the classical pseudo-tensor spin current represented by Eq. (2.33) is a classical counterpart of the spin current pseudo-tensor represented by Eq. (2.35).

## 2.6. Scalar and Vector Potentials

Defining spin-scalar-potential, $\varphi_s$, and spin-vector-potential, $\mathbf{A}_s$, as,

$$\mathbf{E_s} \equiv -\nabla\varphi_s - \frac{\partial \mathbf{A}_s}{\partial t}, \quad (2.36)$$

$$\mathbf{B_s} \equiv \nabla \times \mathbf{A}_s. \quad (2.37)$$

Under the gauge transformation,

$$\mathbf{A}_s \to \mathbf{A}_s + \nabla\Lambda_s, \;\; \varphi_s \to \varphi_s - \frac{\partial \Lambda_s}{\partial t}, \quad (2.38)$$

the spin-electric and spin-magnetic fields, $\mathbf{E_s}$ and $\mathbf{B_s}$, are invariant.

C-Spin-EM potentials can be written in terms of EM potentials. Combining Eq. (2.3), Eq. (2.4), Eq. (2.36) and Eq. (2.37), we obtain

$$\nabla \times \mathbf{A}_s = -\mathbf{S_c} \times \nabla\varphi - \mathbf{S_c} \times \frac{\partial \mathbf{A}}{\partial t}, \quad (2.39)$$

$$-\nabla\varphi_s - \frac{\partial \mathbf{A}_s}{\partial t} = -\mathbf{S_c} \times (\nabla \times \mathbf{A}). \quad (2.40)$$

## 2.7. Spin Wave

C-Spin-EM predicts classical spin waves described by,

$$\frac{\partial^2 \mathbf{B_s}}{\partial t^2} - \nabla^2 \mathbf{B_s} = \nabla \times \mathbf{j}_{\mathbf{s-B}} - \frac{\partial}{\partial t}\mathbf{j}_{\mathbf{s-E}}, \quad (2.41)$$

$$\frac{\partial^2 \mathbf{E_s}}{\partial t^2} - \nabla^2 \mathbf{E_s} = -\nabla \times \mathbf{j}_{\mathbf{s-E}} - \frac{\partial \mathbf{j}_{\mathbf{s-B}}}{\partial t}. \quad (2.42)$$

**Remark:** (1) By duality between EM and C-Spin-EM, spin waves can be quantized and the quanta are spin-one Bosons. (2) The propagation speed of spin wave is to be determined.

# 3. C-Spin-EM vs. Q-Spin-EM

Let's study the similarity and difference between C-Spin-EM and Q-Spin-EM. To convert to quantum theory, we need to introduce the concept of phase. Eq. (2.36) and Eq. (2.37) gives,

$$\oint \mathbf{E}_s \cdot dl = -\frac{\partial}{\partial t}\oint \mathbf{A}_s \cdot d\mathbf{l}, \quad (3.1)$$

$$\iint \mathbf{B_s} \cdot \mathbf{ds} = \oint \mathbf{A}_s \cdot d\mathbf{l}. \quad (3.2)$$

Let's define,

$$\oint \mathbf{A}_s \cdot d\mathbf{l} \equiv \Phi_s, \quad (3.3)$$

then have

$$\Phi_s = \iint \mathbf{B_s} \cdot \mathbf{ds}, \tag{3.4}$$

$$\dot{\Phi}_s = -\oint \mathbf{E}_s \cdot d\mathbf{l}. \tag{3.5}$$

Defining $\Phi_s$ as the phase, $e^{i\Phi_s}$, spin-electric field $\mathbf{E}_s$ and spin-magnetic field $\mathbf{B}_s$ of C-Sin-EM have the same form as that of Spin motive force $\mathbf{E}_{Qs}$ and Berry curvature $\mathbf{B}_{Qs}$ of Q-Spin-EM. However, the fundamental differences between C-Spin-EM and Q-Spin-EM are the definitions of spin-electric field and spin-magnetic field, as well the field equations.

There are analogies between quantities of EM and quantum anholonomy. Since the dualities between EM and C-Spin-EM, we propose that there are analogies between quantities of C-Spin-EM and quantum anholonomy, Table 1.

Table 1: Analogies

| Quantum Anholonomy | EM | C-Spin-EM |
|---|---|---|
| Berry connection | $\mathbf{A}$ | $\mathbf{A_s}$ |
| Berry curvature | $\mathbf{B}$ | $\mathbf{B}_s$ |
| Berry phase | Magnetic flux | Spin-magnetic flux $\Phi_s$ |

**4.** Lagrangian and Hamiltonian

For a non-relativistic non-spinning e-particle $Q_e$ in EM field, the classical Lagrangian and Hamiltonian are respectively,

$$\mathcal{L}_{reg} = \frac{1}{2}mv^2 + Q_e\mathbf{A}\cdot\mathbf{v} - Q_e\varphi,$$

$$H_{reg} = \frac{1}{2m}(p - Q_e\mathbf{A})^2 + Q_e\varphi. \tag{4.1}$$

For a spinning e-particle in C-Spin-EM fields, the Lagrangian should contain its rotation energy, $KE_{spin} = \frac{1}{2}I\omega^2$. Defining $a_{\mathcal{L}} \equiv \frac{I\omega^2}{(\mathbf{S}_c)^2}$, we have

$$KE_{spin} \equiv \frac{1}{2}a_{\mathcal{L}}(\mathbf{S}_c)^2, \tag{4.2}$$

Now, let's introduce Lagrangian,

$$\mathcal{L}_{spin} = \frac{1}{2}a_{\mathcal{L}}(\mathbf{S}_c)^2 + Q_e\mathbf{A_s}\cdot\mathbf{S_c} - Q_e\varphi_s. \tag{4.3}$$

Taking into account the interaction between the velocity and spin-vector-potential, and between the spin and vector potential, we obtain

$$\mathcal{L}_{inter} = Q_e\mathbf{A_s}\cdot\mathbf{v} + Q_e\mathbf{A}\cdot\mathbf{S_c}. \tag{4.4}$$

The total Lagrangian of a spinning e-particle in EM and C-Spin-EM fields is

$$\mathcal{L}_{total} = \frac{1}{2}mv^2 + Q_e\mathbf{A}\cdot\mathbf{v} - Q_e\varphi + \frac{1}{2}a_{\mathcal{L}}(\mathbf{S_c})^2 + Q_e\mathbf{A_s}\cdot\mathbf{S_c} - Q_e\varphi_s + \\ + Q_e\mathbf{A_s}\cdot\mathbf{v} + Q_e\mathbf{A}\cdot\mathbf{S_c}. \tag{4.5}$$

Now using spin as a "generalized velocity", substituting it into Hamiltonian,

$$H = \sum \dot{q}^i \frac{\partial\mathcal{L}_{total}}{\partial\dot{q}^i} - \mathcal{L}_{total} = v\frac{\partial\mathcal{L}_{total}}{\partial v} + \mathbf{S_c}\frac{\partial\mathcal{L}_{total}}{\partial\mathbf{S_c}} - \mathcal{L}_{total}, \tag{4.6}$$

we obtain the classical Hamiltonian for spinning e-particles,

$$H = \frac{(\mathbf{p} - Q_e\mathbf{A} - Q_e\mathbf{A_s})^2}{2m} + \frac{(\mathbf{p}_s - Q_e\mathbf{A}_s - Q_eA)^2}{2a_{\mathcal{L}}} + Q_e\varphi + Q_e\varphi_s, \tag{4.7}$$

which describes dynamics of spinning e-particles in both Extended EM and C-Spin-EM fields. Where the $\mathbf{p}_\mathrm{s}$ is a conjugate momentum corresponding to the "generalized velocity" $\mathbf{S_c}$,

$$\mathbf{p}_\mathrm{s} = \frac{\partial \mathcal{L}_\mathrm{total}}{\partial \mathbf{S_c}}. \quad (4.8)$$

Next we will study the effects of the following terms of Eq. (4.7),

$$\frac{(\mathbf{p}-\mathrm{Q_e}\mathbf{A}-\mathrm{Q_e}\mathbf{A_s})^2}{2m} \approx \frac{(\mathbf{p})^2}{2m} - \frac{\mathrm{Q_e}\mathbf{p}\cdot\mathbf{A}}{m} - \frac{\mathrm{Q_e}\mathbf{p}\cdot\mathbf{A_s}}{m}, \quad (4.9)$$

$$\frac{(\mathbf{p}_\mathrm{s}-\mathrm{Q_e}\mathbf{A}_\mathrm{s}-\mathrm{Q_e}\mathrm{A})^2}{2\mathrm{a}_\mathcal{L}} \approx \frac{(\mathbf{p}_\mathrm{s})^2}{2\mathrm{a}_\mathcal{L}} - \frac{\mathrm{Q_e}\mathbf{p}_\mathrm{s}\cdot\mathbf{A}_\mathrm{s}}{\mathrm{a}_\mathcal{L}} - \frac{\mathrm{Q_e}\mathbf{p}_\mathrm{s}\cdot\mathrm{A}}{\mathrm{a}_\mathcal{L}}. \quad (4.10)$$

where non-linear terms have been ignored.

In the following applications, both uniform magnetic field **B** and uniform spin-magnetic field $\mathbf{B}_\mathrm{s}$ are in z-direction, vector potential **A** and spin-vector-potential $\mathbf{A}_\mathrm{s}$ have similar form,

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \quad (4.11)$$

$$\mathbf{A}_\mathrm{s} = -\mathbf{B}_\mathrm{s}\times\mathbf{r} = -(\mathbf{S_c}\times\mathbf{E})\times\mathbf{r}. \quad (4.12)$$

Eq. (34.12) shows the relation between spin-vector potential and spin-magnetic field induced by spin, which has the same form as that induced by magnetic momentum [4].

**Remark:** With Hamiltonian of Eq. (4.7), C-spin-EM can be converted to its quantum version. The Hamiltonian not only provides classical counterparts/origins of several quantum phenomena, but also predicts several classical effects that may be converted to quantum effects.

5. Effects of Hamiltonian

5.1.Extended-Rashba-SOC-1: Spin-Zeeman Effect as Testing Experiment

Rashba SOC is a fundamental effect. Let's extend Rashba SOC. Substituting Eq. (4.12) into the third term of Eq. (4.9), we obtain,

$$-\frac{\mathrm{Q_e}\mathbf{p}\cdot\mathbf{A}_\mathrm{s}}{m} = \frac{\mathrm{Q_e}}{m}\mathbf{p}\cdot(\mathbf{B}_\mathrm{s}\times\mathbf{r}) = \frac{\mathrm{Q_e}}{m}\mathbf{B}_\mathrm{s}\cdot\mathbf{L}. \quad (5.1)$$

With the definitions of spin-magnetic field, let's re-write Eq. (5.1), denote as $\mathrm{H_{SOC-1}}$,

$$\mathrm{H_{SOC-1}} = \frac{\mathrm{Q_e}}{m}\mathbf{B}_\mathrm{s}\cdot\mathbf{L} = \frac{\mathrm{Q_e}}{m}\mathbf{E}\cdot(\mathbf{L}\times\mathbf{S_c}). \quad (5.2)$$

Comparing with Rashba SOC, $\mathrm{H_{Rashba}} = \alpha_R\mathbf{E}\cdot(\boldsymbol{\sigma}\times\mathbf{p})$, we refer Eq. (5.2) as Extended-Rashba-SOC-1.

**Remark**: The **p** represents a linear motion; $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ represents an orbiting motion; thus the term, $\mathbf{L}\times\mathbf{S}_\mathrm{c}$, represents indeed a Spin-Orbit-coupling. Actually, when Rashba SOC is applied to several situations, the momentum **p** is replaced by angular momentum **L** [5].

We will show that the spin-magnetic force causes $\mathrm{H_{SOC}}$.

Zeeman Effect,

$$\mathrm{H_{Zeeman}} = -\gamma\mathbf{L}\cdot\mathbf{B}$$

has important applications. The second term of Hamiltonian, Eq. (4.52), causes the regular Zeeman effect. The third term causes an additional shift, denoted as spin-Zeeman effect,

$$\mathrm{H_{spin-Z}} = -\frac{\mathrm{Q_e}\mathbf{p}\cdot\mathbf{A}_\mathrm{s}}{m} = \frac{\mathrm{Q_e}}{m}\mathbf{B}_\mathrm{s}\cdot\mathbf{L}, \quad (5.3)$$

which represents the interaction between a $\mathbf{B}_\mathrm{s}$ field and orbiting motion.

**Testing Experiment:** The spin-Zeeman effect is identical to Extended-Rashba-SOC-1, which provides a test that by measuring the spin-Zeeman shift one can test Extended-Rashba-SOC-1.

5.2.Extended-Rashba-SOC-2

Substituting Eq. (4.12) into the second term of Eq. (4.10), we obtain Extended-Rashba-SOC-2, denote as $\mathrm{H}_{\mathrm{SOC-2}}$,

$$\mathrm{H}_{\mathrm{SOC-2}} = -\frac{\mathrm{Q_e}\mathbf{p}_\mathrm{s}\cdot\mathbf{A}_\mathrm{s}}{\mathrm{a}_\mathcal{L}} = \frac{\mathrm{Q_e}}{\mathrm{a}_\mathcal{L}}\mathbf{p}_\mathrm{s}\cdot(\mathbf{B}_\mathrm{s}\times\mathbf{r}) = \frac{\mathrm{Q_e}}{\mathrm{a}_\mathcal{L}}\mathbf{E}\cdot(\mathbf{L}_\mathrm{s}\times\mathbf{S}_\mathbf{c}), \tag{5.4}$$

where $\mathbf{L}_\mathrm{s}$ is defined as

$$\mathbf{L}_\mathrm{s} \equiv \mathbf{r}\times\mathbf{p}_\mathrm{s}, \tag{5.5}$$

called “conjugate angular momentum” corresponding to conjugate momentum $\mathbf{p}_\mathrm{s}$.

5.3. Extended-Rashba-SOC-3

An orbiting spinning particle has angular momentum $\mathbf{L}$ and conjugate angular momentum $\mathbf{L}_\mathrm{s}$ that contains spin $\mathbf{S}_\mathbf{c}$. To derive a total angular momentum in C-Spin-EM, combining Eq. (5.2) and Eq. (5.4), we define a total angular momentum $\mathbf{J}$ and Hamiltonian as

$$\mathbf{J} \equiv \frac{\mathrm{Q_e}}{m}\mathbf{L} + \frac{\mathrm{Q_e}}{\mathrm{a}_\mathcal{L}}\mathbf{L}_\mathrm{s}. \tag{5.6}$$

$$\mathrm{H}_{\mathrm{SOC-3}} \equiv -\mathbf{B}_\mathrm{s}\cdot\mathbf{J} = -(\mathbf{S}_\mathbf{c}\times\mathbf{E})\cdot\mathbf{J} = \mathbf{E}\cdot(\mathbf{S}_\mathbf{c}\times\mathbf{J}). \tag{5.7}$$

We refer it as Extended-Rashba-SOC-3.

5.4. Conjugate Angular Momentum-Magnetic Field Coupling

Combining Eq. (4.54) and the term, $\frac{\mathrm{Q_e}\mathbf{p_s}\cdot\mathbf{A}}{\mathrm{a}_\mathcal{L}}$, of Eq. (4.53), we obtain Hamiltonian for Conjugate Angular Momentum-Magnetic Field Coupling, denoted as $\mathrm{H}_{\mathbf{L}_\mathrm{s}-\mathbf{B}}$,

$$\mathrm{H}_{\mathbf{L}_\mathrm{s}-\mathbf{B}} = \frac{\mathrm{Q_e}\mathbf{p_s}\cdot\mathbf{A}}{\mathbf{a}_\mathcal{L}} = \frac{\mathrm{Q_e}\mathbf{p_s}\cdot(\mathbf{B}\times\mathbf{r})}{2\mathrm{a}_\mathcal{L}} = \frac{\mathrm{Q_e}\mathbf{B}\cdot(\mathbf{r}\times\mathbf{p_s})}{2\mathrm{a}_\mathcal{L}} = \frac{\mathrm{Q_e}}{2\mathrm{a}_\mathcal{L}}\mathbf{B}\cdot\mathbf{L}_\mathrm{s}. \tag{5.8}$$

5.5. Total angular Momentum-Magnetic Field Coupling

Combining $\mathrm{H}_{\mathbf{L}_\mathrm{s}-\mathbf{B}}$, the second term of Eq. (4.9), Eq. (4.11), and Eq. (5.6), we obtain the Total angular Momentum-Magnetic Field Coupling,

$$\mathrm{H}_{\mathrm{B-total}} = \frac{1}{2}\mathbf{B}\cdot\mathbf{J}. \tag{5.9}$$

5.6. Spin-Aharonov–Bohm Effect and Testing Experiment

The regular Hamiltonian, Eq. (4.1), causes the phase shift of Aharonov–Bohm effect. The first term of Eq. (4.7) predicts, in addition to the regular Aharonov–Bohm effect, an effect that the spin-vector-potential $\mathbf{A}_\mathrm{s}$ induces a phase shift, $\Delta\varphi_{\mathrm{spin}}$,

$$\Delta\varphi_{\mathrm{spin}} \sim \frac{\mathrm{Q_e}}{\hbar}\oint \mathrm{A}_\mathrm{s}\cdot d\mathbf{r}, \tag{5.10}$$

$$\oint \mathrm{A}_\mathrm{s}\cdot d\mathbf{r} = \iint \mathbf{B}_\mathbf{s}\cdot d\mathbf{s} = \iint(\mathbf{S}_\mathbf{c}\times\mathbf{E})\cdot d\mathbf{s} = \iint\left\{\mathbf{S}_\mathbf{c}\times\left(-\boldsymbol{\nabla}\varphi_\mathbf{e} - \frac{\partial\mathrm{A}}{\partial\mathrm{t}}\right)\right\}\cdot d\mathbf{s}. \tag{5.11}$$

which we denote as the Spin-Aharonov–Bohm effect, which is caused by the interaction between e-particles’ spin, gradient of electric scalar potential, $\boldsymbol{\nabla}\varphi_\mathbf{e}$, and time changing of magnetic vector potential, $\frac{\partial \mathbf{A}}{\partial t}$.

When a spinning e-particle travelling along the same path P in a region with non-zero **A**, $\nabla\varphi_\mathbf{e}$ and $\frac{\partial \mathbf{A}}{\partial t}$, acquires a total phase shift, $\Delta\varphi_{\mathrm{total}}$, which extends Aharonov–Bohm effect,

$$\Delta\varphi_{\mathrm{total}} = \Delta\varphi_{\mathrm{AB}} + \Delta\varphi_{\mathrm{spin}} = \frac{Q_e}{\hbar}\oint \mathbf{A}\cdot d\mathbf{r} + \frac{Q_e}{\hbar}\oint \mathbf{A}_s\cdot d\mathbf{r}. \tag{5.12}$$

**Testing Experiment**: In the Aharonov–Bohm double-slit experiment, the change of the vector potential **A** with time will cause an additional shift.

6. Spin-Lorentz-type Force

Based on the duality between Extended EM and C-Spin-EM, we postulate that there is a duality between Lorentz force and spin-Lorentz-type force, i.e., under the transformation,

$$\mathbf{E} \leftrightarrow \mathbf{E_s},\ \ \mathbf{B} \leftrightarrow \mathbf{B_s}$$

Lorentz force $\mathbf{F_L}$,

$$\mathbf{F_L} = Q_e\mathbf{E} + Q_e\mathbf{v}\times\mathbf{B},$$

converts to Spin-Lorentz-type force $\mathbf{F}_{\mathrm{SL}}$, and vice versa,

$$\mathbf{F}_{\mathrm{SL}} = m\frac{dv}{dt} = Q_e\mathbf{E_s} + Q_e\mathbf{v}\times\mathbf{B_s}. \tag{6.1}$$

The "**v**" is the velocity of a test e-particle. We refer "$Q_e\mathbf{E_s}$" as spin-electric force, "$Q_e\mathbf{v}\times\mathbf{B_s}$" as spin-magnetic force. A moving spinning e-particle $Q_e$ experiences both Lorentz force and Spin-Lorentz-type forces, denoted as Total-Lorentz-type force $\mathbf{F}_{\mathrm{TL}}$,

$$\mathbf{F}_{\mathrm{TL}} = \mathbf{F_L} + \mathbf{F_{sL}} = Q_e\mathbf{E} + Q_e\mathbf{v}\times\mathbf{B} + Q_e\mathbf{E_s} + Q_e\mathbf{v}\times\mathbf{B_s} \tag{6.2}$$

Using definitions of $\mathbf{E_s}$ and $\mathbf{B_s}$ in terms of electric **E** and magnetic **B** fields, we obtain,

$$\mathbf{F}_{\mathrm{TL}} = Q_e\mathbf{E} + Q_e\mathbf{v}\times\mathbf{B} - Q_e[a\mathbf{S_c}\times\mathbf{B}] + Q_e\mathbf{v}\times[a\mathbf{S_c}\times\mathbf{E}]. \tag{6.3}$$

The "a" is a coefficient, such that $Q_e[(a\mathbf{S_c})\times\mathbf{B}]$ and $Q_e\mathbf{v}\times[(a\mathbf{S_c})\times\mathbf{E}]$ have the unit of force. Here after, absorbing "a" into $\mathbf{S_c}$.

7. Extended Landau–Lifshitz and Landau–Lifshitz-Gilbert Equations

Moreover, base on the duality, under the transformation,

$$\mathbf{v} \leftrightarrow \mathbf{S_c},\ \ \mathbf{E} \leftrightarrow \mathbf{E_s},\ \ \mathbf{B} \leftrightarrow \mathbf{B_s},$$

Lorentz force equation converts to a Landau–Lifshitz-type equation,

$$m\frac{dS_c}{dt} = Q_e\mathbf{E_s} + Q_e\mathbf{S_c}\times\mathbf{B_s} = -Q_e\mathbf{S_c}\times\mathbf{B} + Q_e\mathbf{S_c}\times(\mathbf{S_c}\times\mathbf{E}). \tag{7.1}$$

Which predicts that not only magnetic field but also electric field induces spin precession, which is a counterpart of gyroscope precession in gravitational field. Combining Eq. (7.1) with LL and LLG equations respectively, we obtain Extended LL and Extended LLG equations,

$$\frac{d\mathbf{M}}{dt} = -\gamma\mathbf{M}\times\mathbf{H}_{\mathrm{eff}} - \lambda\mathbf{M}\times(\mathbf{M}\times\mathbf{H}_{\mathrm{eff}}) + \alpha\mathbf{M}\times(\mathbf{M}\times\mathbf{D}_{\mathrm{eff}}), \tag{7.2}$$

$$\frac{d\mathbf{M}}{dt} = -\gamma\mathbf{M}\times\mathbf{H}_{\mathrm{eff}} + \beta\mathbf{M}\times\frac{d\mathbf{M}}{dt} + \alpha\mathbf{M}\times(\mathbf{M}\times\mathbf{D}_{\mathrm{eff}}). \tag{7.3}$$

Where the spin has been replaced by the magnetization **M**, and $\mathbf{B} \rightarrow \mathbf{H}_{\mathrm{eff}}$, $\mathbf{E} \rightarrow \mathbf{D}_{\mathrm{eff}}$; the "α" is coefficient.

8. Effects of Spin-Lorentz-type Force

8.1. Extended-Rashba SOC-1

The work of Spin-Lorentz-type force, Eq. (6.1), causes Extended-Rashba SOC-1, as

$$\mathbf{r}\cdot\mathbf{F_{SL}} = -2\mathrm{Q_e}\mathbf{S_{c\pm}}\cdot\mathbf{A} + \frac{\mathrm{Q_e}}{m}\mathbf{E}\cdot(\mathbf{L}\times\mathbf{S_c}). \quad (8.1)$$

where Eq. (2.3), Eq. (2.4) and Eq. (4.11) have been utilized.

**Remark:** The first term represents the spin-potential coupling, or the vector potential does a work. The second term is identical to Extended-Rashba SOC-1 of Eq. (5.2) derived from Hamiltonian. Thus the force, $\mathbf{F_{SL}}$, causes effect of Hamiltonian, and is underlying mechanism of Rashba SOC.

8.2. Dual-Hall Effect/Topological Insulator and Testing Experiment

The term, "$\mathrm{Q_e}\mathbf{S_c}\times\mathbf{B}$", of Eq. (6.3) causes a new effect that a magnetic field **B** in z-direction acting on the spin of e-particles, even the centers of the spinning e-particles are originally at rest, drives e-particles to move, which causes e-particles accumulation at the opposite surrounding edges, which in turn causes transverse electric fields, $\mathrm{E_x}$ and $\mathrm{E_y}$. Assuming the magnetic field **B** is in z-direction, at equilibrium, $\mathrm{v_x} = \mathrm{v_y} = 0$, we have,

$$\mathrm{E_x} = \mathrm{S_{cy\pm}}\mathrm{B_z}, \quad (8.2)$$

$$\mathrm{E_y} = -\mathrm{S_{cx\pm}}\mathrm{B_z}. \quad (8.3)$$

Comparing with Hall transverse electric field, $\mathrm{E_y} = \mathrm{v_x}\mathrm{B_z}$, we refer this effect as the Dual-Hall Effect, i.e., under the transformation,

$$\mathrm{V_x} \leftrightarrow \mathrm{S_{cx\pm}},\ \mathrm{v_x} \leftrightarrow \mathrm{S_{cy\pm}}$$

the regular Hall transverse electric field $\mathrm{E_y}$ converts to the transverse Dual-Hall electric field, $-\mathrm{E_y}$ and $\mathrm{E_x}$, respectively, and vice versa.

**Testing Experiment**: Place a sheet of material in a magnetic field in z-direction, measuring transverse electric fields without applying an external electric field.

**Remark:** The fundamental differences are: (1) In Hall effect, the motion of e-particles is required, while not required for Dual-Hall effect; (2) In Hall effect, the transverse electric field points to one direction, say either $+\mathrm{y}$ or $-\mathrm{y}$, while in Dual-Hall effect, transverse electric fields are in two directions, $\pm\mathrm{x}$ and $\pm\mathrm{y}$, depend on the orientations of spin, which causes topological insulator.

8.3. Extended-Hall Effect/Topological Insulator

Eq. (6.3) shows that the spin-magnetic force, $\mathrm{Q_e}\{\mathbf{v}\times(\mathbf{S_c}\times\mathbf{E})\}$, deflects the trajectory of moving spinning e-particles, which causes the buildup of e-particles on opposite surrounding edge-surfaces. The buildup induces transverse electric fields balancing the spin-magnetic force.

For classical spin there is no restriction on orientation of spin. Eq. (6.3) indicates that the electric field has not effect on spins that are in the same direction; thus, one can say that there is no spin in the applied electric field/current direction, or one only needs to consider the spins with orientations perpendicular to the direction of applied electric field. However, in our case, there is longitudinal electric field in x-direction, and transverse electric fields in both y- and z-directions, thus we need to consider spins in all x-, y-, and z-directions. We still use the term "spin" to represent the intrinsic angular momentum including those in the same direction of movement of e-particles. When convert to quantum, the concepts of Chirality and Helicity appear.

The spin-Lorentz-type force is the spin's orientation-dependent. Let's consider random equal distribution of spin. For simplicity, denote spins with orientations along positive/negative x-axis, y-axis, and z-axis, as, respectively, $S_{cx\pm}$, $S_{cy\pm}$ and $S_{cz\pm}$. The positive/negative sings "$\pm$" refer to spin-up/spin-down, $S_{cx+}/S_{cx-}$, $S_{cy+}/S_{cy-}$, $S_{cz+}/S_{cz-}$, in that direction, respectively.

In regular Hall experiment, both an external magnetic field **B** and an external electric field **E** are applied simultaneously. Now we study a 3D material placed in both a **B** field (z-axis) and an **E** field (x-axis) that drives a longitudinal current density $j_x$ flowing along x-axis (Fig. 2).

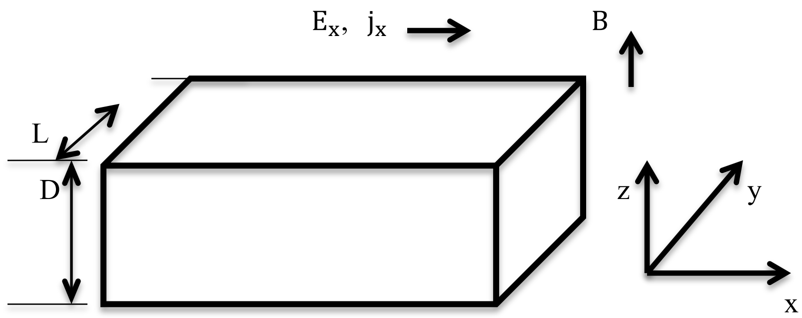


Fig. 2

Spin $\mathbf{S_c}$ can be either in $\pm$x direction, or $\pm$y direction, or $\pm$z direction, denoted as 3-current model. Thus, an e-current in one direction contains e-particles with spins in all of 3 different orientations. In equilibrium, $v_y = v_z = 0$, we obtain, respectively,

$$v_x = \frac{\tau Q_e}{m}\{E_x - S_{cy\pm}B_z\}, \quad (8.4)$$

$$E_x = \frac{m v_x}{Q_e \tau} + S_{cy\pm}B_z, \quad (8.5)$$

$$E_y = \frac{v_x B_z - S_{cx\pm}B_z - v_x S_{cy\pm}E_x}{(1 - v_x S_{cx\pm})}, \quad (8.6)$$

$$E_z = -\frac{v_x(S_{cz\pm}E_x)}{(1 - v_x S_{cx\pm})}. \quad (8.7)$$

The induced transverse electric fields depend on orientations of spins. The spinning e-particles are driven to surrounding edge/surfaces at $\pm$y and $\pm$z directions symmetrically. The accumulations make edge/surfaces having better conductivity than bulk.

We use the same definitions of Hall coefficients $R_{c-ij}$ and resistivity $\rho_{c-ij}$, and obtain

$$R_{ext-xx} \equiv \frac{E_x}{j_x B_z} = \frac{m}{n\tau Q_e^2 B_z} + \frac{m S_{cy\pm}}{n\tau Q_e^2(E_x - S_{cy\pm}B_z)}, \quad (8.8)$$

$$\rho_{ext-xx} \equiv R_{ext-xx}B_z = \frac{m}{n\tau Q_e^2} + \frac{m S_{cy\pm}B_z}{n\tau Q_e^2(E_x - S_{cy\pm}B_z)}, \quad (8.9)$$

$$R_{ext-yx} \equiv \frac{E_y}{j_x B_z} = \frac{1}{nQ_e(1 - v_x S_{cx\pm})}\left\{1 - \frac{S_{cx\pm}}{v_x} - S_{cy\pm}\left(\frac{E_x}{B_z}\right)\right\}, \quad (8.10)$$

$$\rho_{ext-xy} \equiv R_{ext-yx}B_z = \frac{B_z}{nQ_e(1 - v_x S_{cx\pm})}\left\{1 - \frac{S_{cx\pm}}{v_x} - S_{cy\pm}\left(\frac{E_x}{B_z}\right)\right\}, \quad (8.11)$$

$$R_{ext-zx} \equiv \frac{E_z}{j_x B_z} = -\frac{(S_{cz\pm})}{nQ_e(1 - v_x S_{cx\pm})}\left(\frac{E_x}{B_z}\right), \quad (8.12)$$

$$\rho_{ext-xz} \equiv R_{ext-zx}B_z = -\frac{(S_{cz\pm})E_x}{nQ_e(1-v_xS_{cx\pm})}. \quad (8.13)$$

We refer the effect described by Eq. (8.4) to Eq. (8.13) as *Extended-Hall effect/Topological insulator,* which is caused by Total-Lorentz-type force.

**Remark:** The term, "$\mathbf{v}\times(\mathbf{S}_c\times\mathbf{E})$", is the classical origin of that no magnetic field required in Spin-Hall effect of quantum**,** but an electric field, either an external or a local, is required.

8.4. Extended-Hall effect having Zero Longitudinal Hall Coefficient/Resistivity

For strong magnetic field, $E_x \ll S_{cy\pm}B_z$, Eq. (8.8) to Eq. (8.11) reduce to,

$$R_{ext-xx} = \frac{m}{n\tau Q_e^2 B_z} - \frac{m}{nQ_e^2\tau B_z\left\{1-\frac{E_x}{S_{cy\pm}B_z}\right\}} \approx 0, \quad (8.14)$$

$$\rho_{ext-xx} = \frac{m}{n\tau Q_e^2} - \frac{m}{nQ_e^2\tau\left\{1-\frac{E_x}{S_{cy\pm}B_z}\right\}} \approx 0, \quad (8.15)$$

$$R_{ext-yx} \approx \frac{1}{nQ_e(1-v_xS_{cx\pm})}\left\{1 + \frac{m}{\tau Q_e B_z}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - S_{cy\pm}\left(\frac{E_x}{B_z}\right)\right\}, \quad (8.16)$$

$$\rho_{ext-xy} \approx \frac{B_z}{nQ_e(1-v_xS_{cx\pm})} + \frac{m}{n\tau Q_e^2(1-v_xS_{cx\pm})}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{E_xS_{cy\pm}}{nQ_e(1-v_xS_{cx\pm})}. \quad (8.17)$$

For $1 \gg v_xS_{cx\pm}$, we obtain

$$R_{ext-yx} \approx \frac{1}{nQ_e} + \frac{m}{n\tau Q_e^2 B_z}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{S_{cy\pm}}{nQ_e}\left(\frac{E_x}{B_z}\right), \quad (8.18)$$

$$\rho_{ext-xy} \approx \frac{B_z}{nQ_e} + \frac{m}{n\tau Q_e^2}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{E_xS_{cy\pm}}{nQ_e}. \quad (8.19)$$

$$R_{ext-zx} \approx -\frac{(S_{cz\pm})}{nQ_e}\left(\frac{E_x}{B_z}\right), \quad (8.20)$$

$$\rho_{ext-xz} \approx -\frac{E_xS_{cz\pm}}{nQ_e}. \quad (8.21)$$

**Remarks**: (1) When spin-electric field, $\mathbf{E}_s = \mathbf{S}_c\times\mathbf{B}$**,** dominants the longitudinal current, there are zero longitudinal, $R_{ext-xx} \approx 0$ and $\rho_{ext-xx} \approx 0$, which is a classical counterpart of Hall conductance quantization in edge state, $R_{H-xx} = 0$ and $\rho_{H-xx} = 0$. (2) For strong longitudinal electric field, we have $\rho_{ext-xx} \neq 0$.

8.5. Extended-Hall effect Contributing to GMR/TMR Effect

Utilizing Eq. (8.9) for the 3D model in Fig. 2, the relative resistance change is calculated as，

$$\delta\rho_{ext-xx} = \frac{\rho_{ext-xx}(\mathbf{B})-\rho_{ext-xx}(0)}{\rho_{ext-xx}(0)} = \frac{S_{cy\pm}B_z}{E_x-S_{cy\pm}B_z}. \quad (8.22)$$

Starting from zero magnetic field and increases it, $\delta\rho_{ext-xx}$ increases. There is a turning point,

$$E_x = S_{cy\pm}B_z. \quad (8.23)$$

After the turning point, the **B** field continuously increases, we have $E_x < S_{cy\pm}B_z$, thus

$$\rho_{ext-xx}(\mathbf{B}) < \rho_{ext-xx}(0), \quad (8.24)$$

which implies that an external magnetic field **B**, starting at certain strength, decreases the magnetoresistance. When a situation

$$E_x \ll |S_{cy\pm} B_z|, \tag{8.25}$$

is reached, Eq. (8.221) gives

$$\delta\rho_{ext-xx} \approx -1, \tag{8.26}$$

which implies a Giant/TMR magnetoresistance vanishes,

$$\rho_{ext-xx}(\mathbf{B}) \approx 0, \tag{8.27}$$

which agrees with zero longitudinal Hall resistance of Eq. (8.15) and of the quantum Hall effect.

Note, there is always resistance from the insulator layer in GMR/TMR, the net magnetoresistance is equal to that of insulator and, thus is a non-zero constant.

**Remark:** This derivation contributes a mechanism of GMR/TMR in addition to spin scattering.

8.6. Extended-Hall Effect Contributing to Spin Hall Effect

With absence of a magnetic field, there are still transverse dual electric fields, $E_y$ and $E_z$,

$$E_y = -\frac{v_x S_{cy\pm} E_x}{(1 - v_x S_{cx\pm})} = -\frac{\tau Q_e S_{cy\pm}}{m(1 - v_x S_{cx\pm})} E_x^2, \tag{8.28}$$

$$E_z = -\frac{v_x S_{cz\pm} E_x}{(1 - v_x S_{cx\pm})} = -\frac{\tau Q_e S_{cz\pm}}{m(1 - v_x S_{cx\pm})} E_x^2. \tag{8.29}$$

**Remark**: the spin-magnetic force, $Q_e\{\mathbf{v}\times(\mathbf{S_c}\times\mathbf{E})\}$, contributes to spin Hall effect. In the absence of magnetic field: (1) transverse $E_y$ and $E_z$ are dual/symmetry; an external magnetic field **B** breaks the duality/symmetry; (2) transverse $E_y$ and $E_z$ fields are proportional to the square of longitudinal electric field, which agrees with experiments.

8.7. Extended-Hall effect Contributing to Anomalous-Hall Effect

Let's compare with the empirical equation of Anomalous Hall effect,

$\rho_{Anom-xy} = R_{H-yx-0} B_z + R_s M(T, B)$.

Substituting Eq. (4.77) into Eq. (4.91), we obtain,

$$\rho_{ext-xy} \approx \frac{B_z}{nQ_e} + \frac{m}{n\tau Q_e^2}\left(\frac{S_{cx\pm}}{S_{cy\pm}}\right) - \frac{m}{n\tau Q_e^2}\left(\frac{j_x S_{cy\pm}}{nQ_e}\right) - \frac{S_{cy\pm}^2}{nQ_e} B_z. \tag{8.30}$$

The third term shows that Extended-Hall resistivity is dependent on the product of current and spin/magnetization, thus, contributes to $R_s M(T, B)$.

**Remark:** The third term, $\frac{m}{n\tau Q_e^2}\left(\frac{j_x S_{cy\pm}}{nQ_e}\right) = \rho_{Hall-xx}\left(\frac{j_x S_{cy\pm}}{nQ_e}\right)$, of Eq. (8.30) contributes to both spin Hall effect and Anomalous Hall effect. We show that, from the perspective of C-Spin-EM, GMR/TMR and family of Hall effects belong to the same category, i.e., they are, at least partially, caused by Spin-Lorentz-type force. C-Spin-EM indeed provides universal classical models for several fundamental quantum phenomena.

8.8. Temperature Dependence of Extended-Hall Effect

For describing the temperature-dependent behaviors of Extended-Hall effect, we utilize, for simplicity, a model of thermal velocity, $\mathrm{v_x} = \frac{\sqrt{3kT}}{\sqrt{m}}$, and obtain temperature-dependent Extended-Hall coefficient/resistivity,

$$\mathrm{R_{ext-xx}} = \frac{m}{n\tau \mathrm{Q_e^2 B_z}} + \frac{\sqrt{m}}{n\mathrm{Q_e}}\left(\frac{1}{\sqrt{3\mathrm{kT}}}\right)\mathrm{S_{cy\pm}}, \tag{8.31}$$

$$\rho_{\mathrm{ext-xx}} = \frac{\mathrm{m}}{\mathrm{n}\tau \mathrm{Q_e^2}} + \frac{\sqrt{m}}{n\mathrm{Q_e}}\left(\frac{\mathrm{B_z}}{\sqrt{3\mathrm{kT}}}\right)\mathrm{S_{cy\pm}}, \tag{8.32}$$

$$\mathrm{R_{ext-yx}} = \frac{1}{n\mathrm{Q_e}\left(1-\frac{\sqrt{3kT}}{\sqrt{m}}\mathrm{S_{cx\pm}}\right)}\left\{1 - \frac{\mathrm{S_{cx\pm}}\sqrt{m}}{\sqrt{3kT}} - \mathrm{S_{cy\pm}}\left(\frac{\mathrm{E_x}}{\mathrm{B_z}}\right)\right\}, \tag{8.33}$$

$$\rho_{\mathrm{ext-xy}} = \frac{1}{n\mathrm{Q_e}\left(1-\frac{\sqrt{3kT}}{\sqrt{m}}\mathrm{S_{cx\pm}}\right)}\left\{\mathrm{B_z} - \sqrt{m}\mathrm{S_{cx\pm}}\left(\frac{\mathrm{B_z}}{\sqrt{3kT}}\right) - \mathrm{S_{cy\pm}E_x}\right\}, \tag{8.34}$$

$$\mathrm{R_{ext-zx}} = -\frac{(\mathrm{S_{cz\pm}})}{n\mathrm{Q_e}\left(1-\frac{\sqrt{3kT}}{\sqrt{m}}\mathrm{S_{cx\pm}}\right)}\left(\frac{\mathrm{E_x}}{\mathrm{B_z}}\right), \tag{8.35}$$

$$\rho_{\mathrm{ext-xz}} = -\frac{(\mathrm{S_{cz\pm}})\mathrm{E_x}}{n\mathrm{Q_e}\left(1-\frac{\sqrt{3kT}}{\sqrt{m}}\mathrm{S_{cx\pm}}\right)}. \tag{8.36}$$

Remark: The second term of Eq. (8.34), $\sqrt{m}\mathrm{S_{cx\pm}}\left(\frac{\mathrm{B_z}}{\sqrt{3kT}}\right)$, shows that the Dual-Hall effect depends on the **B** field and temperature; The first term and third term of Eq. (8.34), $\mathrm{B_z}$ and $\mathrm{S_{cy\pm}E_x}$, shows weak-temperature-dependent.

9. Lagrangian-Lorentz-type Force

Starting with Lagrangian's equation, $\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \mathcal{L}_{\mathrm{total}}}{\partial \mathrm{v}} = \frac{\partial \mathcal{L}_{\mathrm{total}}}{\partial \mathrm{r}}$, where $\mathcal{L}_{\mathrm{total}}$ is given by Eq. (4.5),

$$\mathcal{L}_{\mathrm{total}} = \frac{1}{2}\mathrm{mv}^2 + \mathrm{Q_e}\mathbf{A}\cdot\mathbf{v} - \mathrm{Q_e}\varphi + \frac{1}{2}\mathrm{a}_{\mathcal{L}}(\mathbf{S_c})^2 + \mathrm{Q_e}\mathbf{A_s}\cdot\mathbf{S_c} - \mathrm{Q_e}\varphi_{\mathrm{s}} + \mathrm{Q_e}\mathbf{A_s}\cdot\mathbf{v} + \mathrm{Q_e}\mathbf{A}\cdot\mathbf{S_c},$$

we derive a force, denoted it as the *Lagrangian-Lorentz-type Force* $\mathbf{F}_{\mathrm{LL}}$,

$$\frac{1}{\mathrm{Q_e}}\mathbf{F}_{\mathrm{LL}} = \mathbf{E} + \mathbf{v}\times\mathbf{B} + \mathbf{v}\times(\mathbf{S_c}\times\mathbf{E}) + \mathbf{S_c}\times(\mathbf{S_c}\times\mathbf{E}) + (\mathbf{S_c}\cdot\nabla)\mathbf{A_s} + (\mathbf{S_c}\cdot\nabla)\mathbf{A}. \tag{9.1}$$

**Remark**: the last two terms of Eq. (9.1) predict a totally new concept of force, named as, a Spin-Potential-Coupling-Induced force.

10. Spin-Potential-Coupling-Induced Force Contributing to Aharonov–Bohm Effect

Let's consider a beam of spinning e-particles shooting at a solenoid. Experiencing the spin-potential-coupling force, the beam splits into opposite direction, due to the orientations of spins described by Eq. (10.1), and go around the solenoid, which contributes to Aharonov–Bohm Effect (Fig.5) [6].

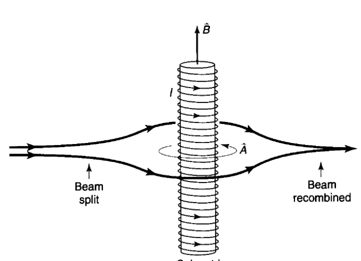


Fig. 5

**Remark:** Hamiltonian, Eq. (4.1), derives the A–B Effect. Now Lagrangian-Lorentz-type force contributes to the A-B effect, which implies that force is, at least partially, cause of the A-B effect of Hamiltonian.

## 11. Effects of Lagrangian-Lorentz-type Force

### 11.1. Lagrangian-Hall-type Effect/Topological Insulator

In regular Hall experiment, an external magnetic field **B** and an external electric field **E** are applied simultaneously. Let's study effects of the Lagrangian-Lorentz-type force. Considering a 3D material with width L in y-direction and thickness D in z direction, placed in a **B** field (z-axis) and an **E** field (x-axis), the longitudinal current density $j_x$ along x-axis, same as Fig. 2.

For simplicity, ignoring the potential-dependent force, and let $S_{cx\pm} \equiv a$, $S_{cy\pm} \equiv b$, $S_{cz\pm} \equiv c$. At equilibrium, $v_y = v_z = 0$, the Lagrangian-Lorentz-type force gives the electric fields,

$$\frac{mv_x}{Q_e\tau} = E_x(1 - bb - cc),$$

$$E_x = \frac{mv_x}{Q_e\tau(1-bb-cc)}, \tag{12.1}$$

$$E_y = \frac{v_xB_z - bv_xE_x}{(1-av_x-aa-cc)}, \tag{12.2}$$

$$E_z = -\frac{cv_xE_x}{(1-av_x-aa-bb)}. \tag{12.3}$$

The Extended Hall parameters are,

(1) Extended Hall coefficients:

$$R_{E-H-xx} \equiv \frac{\mathbf{E}_x}{j_xB_z} = \frac{m}{n\tau Q_e^2B_z(1-bb-cc)}, \tag{12.4}$$

$$R_{E-H-yx} \equiv \frac{\mathbf{E}_y}{j_xB_z} = \frac{B_z - bE_x}{nQ_eB_z(1-av_x-aa-cc)}, \tag{12.5}$$

$$R_{E-H-zx} \equiv \frac{\mathbf{E}_z}{j_xB_z} = -\frac{cE_x}{nQ_eB_z(1-av_x-aa-bb)}. \tag{12.6}$$

(2) Extended Hall conductivity,

$$\sigma_{SH-xx} \equiv \frac{j_x}{\mathbf{E}_x} = \frac{nQ_e^2\tau}{m} - \frac{nQ_e^2\tau}{m}S_{cy\pm}^2 - \frac{nQ_e^2\tau}{m}S_{cz\pm}^2, \tag{12.7}$$

$$\sigma_{E-H-xy} \equiv \frac{j_x}{\mathbf{E}_y} = \frac{nQ_e}{B_z-bE_x} - \frac{nQ_ev_x}{B_z-bE_x}S_{cx\pm} - \frac{nQ_e}{B_z-bE_x}S_{cx\pm}^2 - \frac{nQ_e}{B_z-bE_x}S_{cz\pm}^2, \tag{12.8}$$

$$\sigma_{E-H-xz} \equiv \frac{j_x}{\mathbf{E}_z} = -\frac{nQ_e}{cE_x} + \frac{nQ_ev_x}{cE_x}S_{cx\pm} + \frac{nQ_e}{cE_x}S_{cx\pm}^2 + \frac{nQ_e}{cE_x}S_{cy\pm}^2. \tag{12.9}$$

We refer the effects described by Eq. (12.1) to Eq. (12.9) as the "Lagrangian-Hall-type Effect".

The Lagrangian-Lorentz-type Force induces Lagrangian-Hall-type effect, as Lorentz force induces Hall effect.

### 11.2. Spin-Larmor-type Precession

Lagrangian-Lorentz-type force $\mathbf{F_{LL}} = Q_e\mathbf{S_c}\times(\mathbf{S_c}\times\mathbf{E})$ induces Spin-Larmor-type precession,

$$\frac{d\mathbf{S_c}}{dt} = \mathbf{\Omega}\times\mathbf{S_c}, \tag{12.10}$$

$$\mathbf{\Omega} \equiv Q_e(\mathbf{S_c}\times\mathbf{E}). \tag{12.11}$$

12.Effects of Lorentz Force on Spin

12.1. Classical Origin of Aharonov–Casher effect

We suggest that Lorentz force acts on spin as,

$$\mathbf{S_c}\times\mathbf{F_L} = \mathrm{Q_e}\mathbf{S_c}\times\mathbf{E} + \frac{\mathrm{Q_e}}{\mathrm{m_e}}\mathbf{S_c}\times(\mathbf{p}\times\mathbf{B}). \tag{13.1}$$

The first term provides a classical origin of the Aharonov–Casher effect.

12.2.Spin-Stark Effect; Magnetic-Rashba-type SOC

We can find how Lorentz force affects spin differently,

$$\mathbf{S_c}\cdot\mathbf{F_L} = \mathrm{Q_e}\mathbf{S_c}\cdot\mathbf{E} + \frac{\mathrm{Q_e}}{\mathrm{m_e}}\mathbf{S_c}\cdot(\mathbf{p}\times\mathbf{B}) = \mathrm{Q_e}\mathbf{S_c}\cdot\mathbf{E} + \frac{\mathrm{Q_e}}{\mathrm{m_e}}\mathbf{B}\cdot(\mathbf{S_c}\times\mathbf{p}). \tag{13.2}$$

The $\mathrm{Q_e}\mathbf{S_c}\cdot\mathbf{E}$ causes spin-Stark effect. The $\frac{\mathrm{Q_e}}{\mathrm{m_e}}\mathbf{B}\cdot(\mathbf{S_c}\times\mathbf{p})$ is a magnetic-Rashba-type SOC.

13. Summary and Discussion

Euclidean geometry is the first self-consistent mathematical systems established based on few axioms, and deriving other theorems from axioms.

Based on mathematical vector identities, we establish self-consistent UMFE that universally describes classical physical fields induced respectively by velocity and classical spin of a source.

Combining UMI and Coulomb's law, we derived C-Spin-EM in the perspective of fundamental physics. We argue that the self-consistence, powerfulness and fruitfulness of C-Spin-EM support itself. The benefits and effects/predictions of C-Spin-EM achieved are the following.

(1) connects explicitly with EM;
(2) derives spin wave;
(3) predicts Spin-Lorentz-type force and Lagrangian-Lorentz-type force;
(4) predicts Spin-Potential-Coupling-Induced force, which, if detected, would be a revolutionary concept, indicates that not only field strength but also potential act as force, and contributes to Aharonov–Bohm Effect;
(5) shows that Spin-Lorentz-type and Lagrangian-Lorentz-type force cause, for 3D model, Dual-Hall Effect, Extended-Hall Effect, Temperature Dependence of Extended-Hall Effect, Extended-Rashba SOC, Lagrangian-Hall effect;
(1) shows that Extended-Hall effect, at classical level, contributes universally to zero longitudinal Hall coefficient/resistivity, GMR/TMR, Anomalous Hall effect, Spin Hall effect, and topological insulator.
(6) predicts a Landau–Lifshitz-type equation as a supplement of Spin-Lorentz-type force;
(7) suggests several new effects, such as Spin-Aharonov–Bohm Effect, spin-Aharonov–Casher effect, Spin-Larmor Precession and Spin-Stark Effect;
(8) Proposes several experiments to test proposed effects, such as, definition of spin-electric and spin-magnetic fields, Spin-Aharonov–Bohm effect, Dual-Hall Effect/Topological Insulator, whether $\rho_{\mathrm{ext-xx}}(\mathbf{B}) \approx 0$ of GMR/TMR, Zeeman effect/Extended-Rashba SOC.

We argue that the self-consistency, powerfulness and fruitfulness are evidences supporting C-Spin-EM.

The Lagrangian-Lorentz-type force and the Hamiltonian are derived from the same Lagrangian. We suggest that there is a duality between the Lagrangian-Lorentz-type force and the Hamiltonian, i.e., an effect of Hamiltonian corresponds to an effect of Lagrangian-Lorentz-type force, and vice versa, which is heuristic for exploring new effects.

The Lagrangian-Lorentz-type force causes the several effects of Hamiltonian。

The extent of validity of C-Spin-EM is its extent to correctly predict and agree with experimental results.

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**Appendix :** Inver-square Law for Spin

Postulate a physically hypothetical $\mathrm{Q_{spin-momopole}}$ (abbreviated $\mathrm{Q_{s-m}}$), call “spin-charge”, which is a spin counterpart of e-particle, and satisfies an inverse-square law,

$$\nabla \cdot \mathbf{E_s} = \mathrm{Q_{s-m}}.$$